Minimax-regret treatment choice with missing outcome data

Charles F. Manski*

Department of Economics and Institute for Policy Research, Northwestern University, 2003 Sheridan Road, Evanston, IL 60208, USA

Available online 21 July 2006

Abstract

I use the minimax-regret criterion to study choice between two treatments when some outcomes in the study population are unobservable and the distribution of missing data is unknown. I first assume that observable features of the study population are known and derive the treatment rule that minimizes maximum regret over all possible distributions of missing data. When no treatment is dominant, this rule allocates positive fractions of persons to both treatments. I then assume that the data are a random sample of the study population and show that in some instances, treatment rules that estimate certain point-identified population means by sample averages are finite-sample minimax regret.

© 2006 Elsevier B.V. All rights reserved.

JEL classification: C44

Keywords: Statistical decision theory; Partial identification; Social choice

1. Introduction

This paper continues the normative investigation of treatment choice begun in Manski (2000, 2004). Here, as there, a utilitarian social planner must choose treatments for a heterogeneous population. The planner observes outcomes in a study population whose treatments have previously been selected. The feasible treatment rules are functions that map observed covariates of population members and the available data on outcomes into a treatment allocation.

*Tel.: +1 847 491 8223; fax: +1 847 491 7001.
E-mail address: cfmanski@northwestern.edu.
Manski (2000) showed in general terms that partial identification of treatment response makes treatment choice a problem of decision making under ambiguity. I did not recommend a particular approach to decision making and did not address the problem of treatment choice with sample data. Manski (2004) used the minimax-regret criterion to study treatment choice with data from a classical randomized experiment. Classical experiments point-identify the distribution of treatment response, but the planner must decide how to perform statistical inference with sample data.

Here I use the minimax-regret criterion to study treatment choice when some outcomes in the study population are unobservable and the distribution of missing data is unknown. The data may be a random sample of persons whose treatments were selected in an unknown way; then the planner neither observes the outcomes of non-selected treatments nor knows the distribution of such outcomes. Or the data may be the findings of a randomized experiment in which some subjects drop out for unknown reasons before their outcomes can be measured. In these settings, the planner must address an identification problem and perform statistical inference as well.

Section 2 sets forth basic concepts. Section 3 shows how the identification problem per se affects treatment choice. Assuming that there are two treatments and that the planner knows all observable features of the study population, I derive the treatment rule that minimizes maximum regret over all possible distributions of missing data. The minimax-regret rule allocates all observationally identical persons to one treatment if this treatment is dominant; that is, optimal under all distributions of missing data. If no treatment is dominant, the minimax-regret rule allocates positive fractions of such persons to both treatments. The minimax-regret rule is asymptotically implementable if the planner does not know the observable features of the study population but has random-sample data on these features.

Section 4 describes the finite-sample minimax-regret criterion based on the Wald (1950) development of statistical decision theory, which focuses on the expected performance of a statistical decision function when applied across independent data samples. Section 5 considers a setting in which the available data are a random sample of a study population in which treatments were selected in an unknown way. I show that in some instances of this setting, treatment rules that estimate certain point-identified population means by sample averages are finite-sample minimax regret. Curiously, large samples have no advantage relative to small ones in these instances. This is so because the finite-sample minimax-regret criterion based on Wald assumes risk neutrality and because the optimal treatment rule is linear in the relevant population means.

Before commencing the analysis, I reiterate the observation of Manski (2004) that the strength of this or any other application of decision theory is also its vulnerability. The strength of this paper is that it takes an explicit stand on the problem of treatment choice with missing outcome data and, in return, delivers specific conclusions about what constitutes a good treatment rule. The vulnerability is that reasonable persons may prefer other formulations of the problem and, hence, may conclude that the findings reported here do not meet their needs. A planner may or may not aim to maximize a utilitarian social welfare function. He may or may not want to use the minimax-regret criterion to choose a treatment rule. He may or may not want to address the problem of finite-sample inference in the manner of Wald. Perspectives on what constitutes the problem of interest may vary in these and other respects as well.
2. Basic concepts

The basic concepts and notation are the same as in Manski (2000, 2004). The planner’s problem is to choose treatments from a finite set $T$ of mutually exclusive and exhaustive treatments. Much of this paper supposes that there are two treatments, in which case $T = \{a, b\}$. Each member $j$ of the treatment population, denoted $J^*$, has a response function $y_j(\cdot): T \rightarrow Y$ mapping treatments $t \in T$ into outcomes $y_j(t) \in Y$. The population is a probability space $(J, \Omega, P)$, and the probability distribution $P[y(\cdot)]$ of the random function $y(\cdot): T \rightarrow Y$ describes treatment response across the population. The population is “large”, in the sense that $J^*$ is uncountable and $P(j) = 0$, $j \in J^*$.

The planner, who must assign a treatment to each member of $J^*$, observes covariates $x_j \in X$ for each person $j$; thus, $x: J^* \rightarrow X$ is the random variable mapping persons into their covariates. Given that the planner observes $(x_j, j \in J)$, he knows the covariate distribution $P(x)$. The covariate space $X$ is finite and $P(x = \xi) > 0$, $\xi \in X$.

A feasible treatment rule assigns all persons with the same observed covariates to one treatment or, more generally, randomly allocates such persons across the treatments. Let $Z$ denote the space of functions that map $T \times X$ into the unit interval and that satisfy the adding-up conditions: $z(\cdot, \cdot) \in Z \Rightarrow \sum_{t \in T} z(t, \xi) = 1$, $\xi \in X$. The feasible treatment rules are the elements of $Z$.

The welfare from assigning treatment $t$ to person $j$ is $u_j(t) = y_j(t) + c(t, x_j)$, where $c(t, x)$ is the real-valued cost of assigning treatment $t$ to a person with covariates $x$ and $y(t)$ is the real-valued benefit of this treatment. The planner wants to maximize population mean welfare. For each treatment rule $z$, the mean welfare that would be realized with choice of rule $z$ is

$$U(z, P) \equiv \sum_{\xi \in X} P(x = \xi) \sum_{t \in T} z(t, \xi) \cdot E[u(t)|x = \xi]. \quad (1)$$

The planner wants to solve the problem $\max_{z \in Z} U(z, P)$. The maximum is achieved by a rule that allocates all persons with covariates $\xi$ to a treatment solving the problem

$$\max_{t \in T} E[u(t)|x = \xi]. \quad (2)$$

The population welfare achieved by an optimal rule is

$$U^*(P) \equiv \sum_{\xi \in X} P(x = \xi)\{\max_{t \in T} E[u(t)|x = \xi]\}. \quad (3)$$

A planner who knows the treatment-response distributions $P[y(t)|x]$, $t \in T$ can choose an optimal treatment rule. My concern is a planner who does not know the response distributions but who can observe a study population in which treatments have been selected and outcomes realized. I suppose that the study population, denoted $J$, is identical in distribution to the treatment population $J^*$. Thus, $J$ is a probability space whose probability measure $P$ is the same as that of $J^*$. The only difference between $J$ and $J^*$ is that some status quo treatment rule has already been applied and outcomes experienced in the former population, whereas a treatment rule is yet to be chosen in the latter.

The specific problem addressed in this paper is treatment choice when some outcomes in the study population are not observable and the distribution of missing data is unknown. Suppose that, for each treatment $t$ and covariate value $\xi$, the outcome $y(t)$ is only
observable for persons who belong to some sub-population \( J_{t, \xi} \) of the study population \( J \). By the Law of Total Probability,

\[
P[y(t)|x = \xi] = P[y(t)|x = \xi, J_{t, \xi}] \cdot P(J_{t, \xi}|x = \xi)
\]

\[
+ P[y(t)|x = \xi, \text{not } J_{t, \xi}].
\]

\[
P(\text{not } J_{t, \xi}|x = \xi).
\]

(4)

The probability \( P(J_{t, \xi}|x = \xi) \) of observing outcome \( y(t) \) and the distribution \( P[y(t)|x = \xi, J_{t, \xi}] \) of observable outcomes can be learned empirically, but the distribution \( P[y(t)|x = \xi, \text{not } J_{t, \xi}] \) of unobservable outcomes cannot. I consider treatment choice when the planner has no information about \( P[y(t)|x = \xi, \text{not } J_{t, \xi}] \).

The scenario described here applies to two familiar situations in empirical analysis of treatment response. First, it applies to randomized experiments in which some subjects drop out before their outcomes can be measured, and nothing is known about why this occurs. Then \( P(J_{t, \xi}|x = \xi) \) is the probability that a person with covariates \( \xi \) who is assigned to treatment \( t \) does not drop out. Second, it applies when the treatments of the study population were selected in an unknown way. Then \( P(J_{t, \xi}|x = \xi) \) is the joint probability that a person with covariates \( \xi \) was given treatment \( t \) and that this person’s realized outcome was measured.

3. The identification problem and treatment choice

Section 3.1 derives the minimax-regret treatment rule when the planner knows the empirically learnable quantities \( \{ P[y(t)|x = \xi, J_{t, \xi}], P(J_{t, \xi}|x = \xi) \} \), \( (t, \xi) \in T \times X \). Section 3.2 shows that this rule can be estimated consistently from random sample data. Section 3.3 gives an empirical illustration.

3.1. Minimax regret over all distributions of missing data

In general, the minimax-regret criterion for treatment choice is

\[
\inf_{z \in Z} \sup_{\gamma \in \Gamma} U^*(P_\gamma) - U(z, P_\gamma).
\]

Here \( \Gamma \) denotes the space of feasible states of nature, \( (P_\gamma, \gamma \in \Gamma) \) is the set of feasible values for \( P \), and \( U^*(P_\gamma) \) is the optimal welfare that would be achievable if it were known that \( P = P_\gamma \). The quantity

\[
U^*(P_\gamma) - U(z, P_\gamma) = \sum_{\xi \in X} P(x = \xi) \left\{ \max_{t \in T} E_\gamma[u(t)|x = \xi] \right\}
\]

\[
- \sum_{t \in T} z(t, \xi) \cdot E_\gamma[u(t)|x = \xi]\}
\]

(6)

is the regret of rule \( z \) in state of nature \( \gamma \). Thus, regret is the loss in welfare that results from not knowing the true state of nature.

It generally is complex to solve problem (5), but a relatively simple result emerges when there are two treatments and the feasible states of nature index all possible values for the
distribution of missing data. Proposition 1 derives the minimax-regret treatment rule in this case.

**Proposition 1.** Let \( T = \{a, b\} \). Let \( \{P[y(t)|x = \xi, J_{t\xi}]\}, P(J_{t\xi}|x = \xi); (t, \xi) \in T \times X \) be known. Let \( u_{0,t\xi} = \inf_{\gamma \in Y} u(y, t, \xi) \) and \( u_{1,t\xi} = \sup_{\gamma \in Y} u(y, t, \xi) \) be finite. Let \( e_{t\xi} \equiv E[u(t)|x = \xi, J_{t\xi}] \) and \( p_{t\xi} \equiv P(J_{t\xi}|x = \xi) \). Then the minimax-regret rule is

\[
z^*(b, \xi) = 1 \quad \text{if} \quad (e_{a\xi} - u_{1a\xi})p_{a\xi} + (u_{0b\xi} - e_{b\xi})p_{b\xi} + (u_{1a\xi} - u_{0b\xi}) < 0,
\]

\[
= 0 \quad \text{if} \quad (e_{b\xi} - u_{1b\xi})p_{b\xi} + (u_{0a\xi} - e_{a\xi})p_{a\xi} + (u_{1b\xi} - u_{0a\xi}) < 0,
\]

\[
= \frac{(e_{b\xi} - u_{1b\xi})p_{b\xi} + (u_{0a\xi} - e_{a\xi})p_{a\xi} + (u_{1b\xi} - u_{0a\xi})}{(u_{0b\xi} - u_{1b\xi})p_{b\xi} + (u_{0a\xi} - u_{1a\xi})p_{a\xi} + (u_{1b\xi} - u_{0b\xi}) + (u_{1a\xi} - u_{0a\xi})}
\]

otherwise.

(7)

**Proof.** By the Law of Iterated Expectations,

\[
E[u(t)|x = \xi] = e_{t\xi}p_{t\xi} + E[u(t)|x = \xi, \text{ not } J_{t\xi}] \cdot (1 - p_{t\xi}).
\]

All quantities on the right side are known except for \( E[u(t)|x = \xi, \text{ not } J_{t\xi}] \), which can take any value in the interval \([u_{0,t\xi}, u_{1,t\xi}]\). Hence, the regret of rule \( z \) in state of nature \( \gamma \) is

\[
U^*(P_{\gamma}) - U(z, P_{\gamma}) = \sum_{\xi \in X} P(x = \xi) \{ \max_{t \in T} [e_{t\xi}p_{t\xi} + \kappa_{J_{t\xi}}(1 - p_{t\xi})] - \sum_{t \in T} z(t, \xi) \cdot [e_{t\xi}p_{t\xi} + \kappa_{J_{t\xi}}(1 - p_{t\xi})] \},
\]

where \( \kappa_{J_{t\xi}} \in [u_{0,t\xi}, u_{1,t\xi}] \) is a possible value of the unknown \( E[u(t)|x = \xi, \text{ not } J_{t\xi}] \). Maximum regret is

\[
\sum_{\xi \in X} P(x = \xi) \max_{\kappa_{J_{t\xi}} \in [u_{0,t\xi}, u_{1,t\xi}]} \{ \max_{t \in T} [e_{t\xi}p_{t\xi} + \kappa_{J_{t\xi}}(1 - p_{t\xi})] - \sum_{t \in T} z(t, \xi) \cdot [e_{t\xi}p_{t\xi} + \kappa_{J_{t\xi}}(1 - p_{t\xi})] \}.
\]

Let \( C(t, \xi) = (\kappa_{J_{t\xi}}, s \in T) \in [u_{0,t\xi}, u_{1,t\xi}], \ s \in T \) s.t. \( e_{t\xi}p_{t\xi} + \kappa_{J_{t\xi}}(1 - p_{t\xi}) \geq e_{s\xi}p_{s\xi} + \kappa_{J_{s\xi}}(1 - p_{s\xi}), \ s \in T \). Thus, \( C(t, \xi) \) is the region in which treatment \( t \) is optimal for persons with covariates \( \xi \). Within this region, regret is maximized by setting \( \kappa_{J_{t\xi}} = u_{1\xi} \xi \) and \( \kappa_{J_{s\xi}} = u_{0\xi} \xi, \ s \neq t \). Thus,

\[
\max_{(\kappa_{J_{t\xi}}, s \in T) \in C(t, \xi)} \{ \max_{s \in T} [e_{s\xi}p_{s\xi} + \kappa_{J_{s\xi}}(1 - p_{s\xi})] - \sum_{s \in T} z(s, \xi) \cdot [e_{s\xi}p_{s\xi} + \kappa_{J_{s\xi}}(1 - p_{s\xi})] \} = [1 - z(t, \xi)] \cdot [e_{t\xi}p_{t\xi} + u_{1\xi}t(1 - p_{t\xi})] - \sum_{s \neq t} z(s, \xi) \cdot [e_{s\xi}p_{s\xi} + u_{0\xi}s(1 - p_{s\xi})].
\]

Hence,

\[
\max_{\kappa_{J_{s\xi}} \in [u_{0,t\xi}, u_{1,t\xi}], t \in T} \{ \max_{t \in T} [e_{t\xi}p_{t\xi} + \kappa_{J_{t\xi}}(1 - p_{t\xi})] - \sum_{t \in T} z(t, \xi) \cdot [e_{t\xi}p_{t\xi} + \kappa_{J_{t\xi}}(1 - p_{t\xi})] \} = \max_{t \in T} \{ [1 - z(t, \xi)] \cdot [e_{t\xi}p_{t\xi} + u_{1\xi}t(1 - p_{t\xi})] - \sum_{s \neq t} z(s, \xi) \cdot [e_{s\xi}p_{s\xi} + u_{0\xi}s(1 - p_{s\xi})] \}.
\]
It follows that, for each \( \xi \in X \), the minimax-regret rule solves the problem

\[
\min_{(x_t, t) \in S^T} \max_{t \in T} \left\{ (1 - x_t) \cdot \left[ e_{b_t} p_{b_t} + u_{1b_t}(1 - p_{b_t}) \right] - \sum_{s \neq t} x_s \cdot \left[ e_{a_t} p_{a_t} + u_{0a_t}(1 - p_{a_t}) \right] \right\},
\]

where \( S^T \) denotes the unit simplex in \( R^T \).

Now suppose that there are two treatments, \( t = a \) and \( t = b \). Let \( z = z(b, \xi) \). Then the above simplifies to

\[
\min_{x \in [0, 1]} \max_{t \in \{a, b\}} \left\{ (1 - x) \cdot \left[ e_{b_t} p_{b_t} + u_{1b_t}(1 - p_{b_t}) \right] - (1 - x) \cdot \left[ e_{a_t} p_{a_t} + u_{0a_t}(1 - p_{a_t}) \right] \right\}
\]

\[
\begin{align*}
&= \min_{x \in [0, 1]} \max_{t \in \{a, b\}} \left\{ (1 - x) \cdot \left[ (e_{b_t} - u_{1b_t}) p_{b_t} + (u_{0a_t} - e_{a_t}) p_{a_t} + (u_{1b_t} - u_{0a_t}) \right] \right\} \\
&= \min_{x \in [0, 1]} \max_{t \in \{a, b\}} \left\{ (e_{a_t} - u_{1a_t}) p_{a_t} + (u_{0b_t} - e_{b_t}) p_{b_t} + (u_{1a_t} - u_{0b_t}) \right\} \end{align*}
\]

The minimum occurs at \( x = 1 \) if \( (e_{a_t} - u_{1a_t}) p_{a_t} + (u_{0b_t} - e_{b_t}) p_{b_t} + (u_{1a_t} - u_{0b_t}) < 0 \) and at \( x = 0 \) if \( (e_{b_t} - u_{1b_t}) p_{b_t} + (u_{0a_t} - e_{a_t}) p_{a_t} + (u_{1b_t} - u_{0a_t}) < 0 \). Otherwise, the minimum solves the equation

\[
(1 - x) \cdot \left[ (e_{b_t} - u_{1b_t}) p_{b_t} + (u_{0a_t} - e_{a_t}) p_{a_t} + (u_{1b_t} - u_{0a_t}) \right] = x \cdot \left[ (e_{a_t} - u_{1a_t}) p_{a_t} + (u_{0b_t} - e_{b_t}) p_{b_t} + (u_{1a_t} - u_{0b_t}) \right].
\]

Then

\[
x = \frac{(e_{b_t} - u_{1b_t}) p_{b_t} + (u_{0a_t} - e_{a_t}) p_{a_t} + (u_{1b_t} - u_{0a_t})}{(u_{0b_t} - u_{1a_t}) p_{b_t} + (u_{0a_t} - u_{1a_t}) p_{a_t} + (u_{1b_t} - u_{0b_t}) + (u_{1a_t} - u_{0b_t})}. \quad \square.
\]

Proposition 1 shows that the minimax-regret rule assigns all persons with covariates \( \xi \) to treatment \( b \) if and only if this treatment dominates the alternative; that is, if and only if \( b \) yields higher mean welfare than under all possible distributions of the missing data. It is easy to verify that \( b \) dominates \( a \) if and only if \( (e_{a_t} - u_{1a_t}) p_{a_t} + (u_{0b_t} - e_{b_t}) p_{b_t} + (u_{1a_t} - u_{0b_t}) < 0 \). Similarly, the minimax-regret rule assigns all persons with covariates \( \xi \) to treatment \( a \) if and only if that treatment dominates \( b \). When neither treatment dominates, the minimax-regret rule randomly assigns positive fractions of observationally identical persons to both treatments. The fraction allocated to treatment \( b \) increases with \( e_{b_t} \), which measures the observable success of treatment \( b \), and decreases with \( e_{a_t} \), which measures the observable success of treatment \( a \).

Suppose that \( u_{0a_t} = u_{0b_t} = u_{1a_t} = u_{1b_t} \), and \( p_{b_t} \leq 1 \). Inspection of (7) shows that neither treatment dominates when these conditions hold, whatever values \( e_{a_t} \) and \( e_{b_t} \) may take. The minimax-regret rule is the fractional treatment allocation

\[
z^*(b, \xi) = \frac{e_{b_t} p_{b_t} + u_{1b_t}(1 - p_{b_t}) - e_{a_t} p_{a_t} - u_{0a_t}(1 - p_{a_t})}{(u_{1b_t} - u_{0a_t})(2 - p_{b_t})}.
\]

I henceforth call \( p_{b_t} \) the observability score for persons with covariates \( \xi \) and I say that missing data are prevalent if \( p_{b_t} \leq 1 \). Missing data are necessarily prevalent if treatments in the study population were selected in an unknown way. In this situation, \( p_{a_t} \) and \( p_{b_t} \) can be no larger than the probabilities that treatments \( a \) and \( b \) are chosen; hence, \( p_{a_t} + p_{b_t} \leq 1 \).

The fact that the minimax-regret rule yields a fractional treatment allocation in the absence of dominance makes this rule contrast sharply with Bayes rules and the maximin rule. The latter rules generically assign all observationally identical persons to the same
treatment. For each \( \xi \in X \), a Bayes rule solves the problem \( \max_{t \in T} \int [e_{t \xi} p_{t \xi} + \kappa_{t \xi} (1 - p_{t \xi})] \, d\pi(y) \), where \( \pi(y) \) is a subjective probability distribution over the feasible states of nature \( \Gamma \). The maximin rule solves \( \max_{t \in T} e_{t \xi} p_{t \xi} + \omega_{t \xi} (1 - p_{t \xi}) \).

In some settings, legal requirements for “equal treatment of equals” may prevent planners from implementing the minimax-regret rule or other treatment rules that randomly allocate observationally identical persons across different treatments. However, such rules are entirely sensible from the utilitarian perspective. Proposition 1 gives a principled argument for diversification when making treatment choices with missing outcome data.

### 3.2. Asymptotic minimax regret

Computation of rule \( z^* \) requires knowledge of \( [(p_{t \xi}, e_{t \xi}), (t, \xi) \in T \times X] \). In practice, a planner may not know these quantities but may be able to estimate them from sample data on the study population. Let \( N \) denote sample size and let \( [(p_{Nt \xi}, e_{Nt \xi}), (t, \xi) \in T \times X] \) be the sample estimates. Then the planner can estimate \( z^*(\cdot, \cdot) \) by

\[
\begin{align*}
z_N(b, \xi) &= 1 \text{ if } (e_{Nt \xi} - u_{1a \xi})p_{Na \xi} + (u_{0b \xi} - e_{Nb \xi})p_{Nb \xi} + (u_{1a \xi} - u_{0b \xi}) < 0, \\
&= 0 \text{ if } (e_{Nt \xi} - u_{1b \xi})p_{Nb \xi} + (u_{0a \xi} - e_{Na \xi})p_{Na \xi} + (u_{1b \xi} - u_{0a \xi}) < 0, \\
&= \frac{(e_{Nb \xi} - u_{1b \xi})p_{Nb \xi} + (u_{0a \xi} - e_{Na \xi})p_{Na \xi} + (u_{1b \xi} - u_{0a \xi})}{(u_{0b \xi} - u_{1b \xi})p_{Nb \xi} + (u_{0a \xi} - u_{1a \xi})p_{Na \xi} + (u_{1b \xi} - u_{0b \xi}) + (u_{1a \xi} - u_{0a \xi})}.
\end{align*}
\]

The available data may be \( N \) randomly drawn members of a study population whose treatments were selected in an unknown way. Then \( p_{Nt \xi} \) is the sample frequency of cases in which a person with covariates \( \xi \) received treatment \( t \) and reported his outcome \( y(t) \), and \( e_{Nt \xi} \) is the sample average of these reported outcomes. Or the data may be the findings of an experiment in which \( N_a \) randomly drawn members of the study population were assigned to treatment \( a \) and \( N_b \) to treatment \( b \); thus, \( N = (N_a, N_b) \) here. Then \( p_{Nt \xi} \) is the frequency of cases in which \( y(t) \) is observed among those subjects who have covariates \( \xi \) and who were assigned to treatment \( t \). Again, \( e_{Nt \xi} \) is the sample average of these observed outcomes.

Under these and various other sampling processes,

\[
\lim_{N \to \infty} \left[ (p_{Nt \xi}, e_{Nt \xi}), (t, \xi) \in T \times X \right] = \left[ (p_{t \xi}, e_{t \xi}), (t, \xi) \in T \times X \right], \quad \text{a.s.}
\]

This, plus continuity of \( z^*(\cdot, \cdot) \) as a function of \( [(p_{t \xi}, e_{t \xi}), (t, \xi) \in T \times X] \) implies that

\[
\lim_{N \to \infty} z_N(\cdot, \cdot) = z^*(\cdot, \cdot), \quad \text{a.s.}
\]

Thus, sample data on the study population enable asymptotic implementation of the minimax-regret rule.

### 3.3. Illustration: sentencing and recidivism of juvenile offenders

To illustrate application of the minimax-regret rule, consider a judge who must choose sentences for convicted juvenile offenders. The criminal justice system produces ample data on the sentences that past offenders received and the outcomes they subsequently experienced, but little information about the way sentences were determined. Hence, sentencing exemplifies a treatment choice problem in which a planner can observe the
outcomes of a study population but does not know how treatments were chosen in that population.

Manski and Nagin (1998) analyzed data on the sentencing and recidivism of male offenders under age 16 living in the state of Utah. We compared recidivism under the two main sentencing options available to judges: confinement in residential facilities \( t = b \) and sentences that do not involve confinement \( t = a \). The outcome of interest was taken to be a binary measure of recidivism, with \( y = 1 \) if an offender is not convicted of a subsequent crime in the 2-year period following sentencing, and \( y = 0 \) otherwise. The present discussion supposes that this binary outcome measures welfare; thus, \( u[y(t), t, \xi] = y(t) \).

The distribution of treatments and outcomes in the study population was

\[
P(s = b) = 0.11, \quad P(y = 1|s = b) = 0.23, \quad P(y = 1|s = a) = 0.41,
\]

where \( s \) denotes the treatment received by a member of the study population. The problem is to use this empirical evidence to choose treatments for an analogous population of offenders who have not yet been sentenced. Proposition 1 gives the minimax-regret rule. In this illustration, \( e_b = 0.23, p_b = 0.11, e_a = 0.41, p_a = 0.89 \), and \( \{u_{0|b} = 0, u_{1|b} = 1\} \), \( t = a, b \). Hence, the minimax-regret rule assigns 0.55 of all offenders to treatment \( b \) and 0.45 to treatment \( a \).

4. Basic concepts continued: finite-sample minimax regret

Section 3 followed the standard two-step econometric approach to empirical inference. One first studies identification under the assumption that all observable population quantities are known. One then uses asymptotic theory to justify estimation of observable population quantities by sample analogs. This approach has been productive and convenient. Nevertheless, it is well appreciated that asymptotic theory can be an imperfect guide to finite-sample statistical inference.

The Wald (1950) development of statistical decision theory addresses the problem of finite-sample inference directly, without explicit reference to identification and without recourse to the large-sample approximations of asymptotic theory. Manski (2004) used the Wald framework and, in particular, the Savage (1951) minimax-regret criterion to analyze treatment choice when sample data are generated by a classical randomized experiment. A classical experiment has no missing outcome data.

To apply the Wald and Savage ideas when there are missing outcome data requires the same basic concepts as in Manski (2004). Here, as there, a statistical treatment rule is a function that maps the available data into a treatment allocation. Let \( Q \) denote the sampling process generating the data and let \( \Psi \) denote the sample space; that is, is the set of data samples that may be drawn under \( Q \). Let \( Z \) henceforth denote the space of functions that map \( T \times X \times \Psi \) into the unit interval and that satisfy the adding-up conditions: \( z \in Z \Rightarrow \sum_{t \in T} z(t, \xi, \psi) = 1, \forall (\xi, \psi) \in X \times \Psi \). Then each function \( z \in Z \) is a feasible statistical treatment rule.

Repeated engagement of the sampling process to draw independent samples makes population mean welfare a random variable. Wald recommended evaluation of a statistical decision function (here a statistical treatment rule) by its expected performance across repeated samples. Let \( I \) index the set of feasible states of nature; thus, \( \{(P, Q, \gamma), \gamma \in I\} \) is the set of \( (P, Q) \) pairs that the planner deems possible. In state of nature \( \gamma \), the expected
welfare yielded by rule \( z \) in repeated samples is

\[
W(z, P_\gamma, Q_\gamma) = \sum_{\xi \in X} \sum_{t \in T} P(x = \xi) \sum_{i \in T} E_{ij}[z(t, \xi, \psi)] \cdot E_{ij}[u(t|x = \xi)],
\]

(12)

where \( E_{ij}[z(t, \xi, \psi)] = \int z(t, \xi, \psi) dQ_\gamma(\psi) \) is the expected (across samples) fraction of persons with covariates \( \xi \) who are assigned to treatment \( t \). The finite-sample minimax-regret criterion suggested by Savage is

\[
\inf_{z \in Z} \sup_{\gamma \in \Gamma} U^*(P_\gamma) - W(z, P_\gamma, Q_\gamma).
\]

(13)

The finite-sample minimax-sample-regret criterion defined in (12), (13) is similar to the criterion previously defined in (5), (6), but it is important to recognize the differences. First, the space \( \Gamma \) of states of nature is larger here than it was there: the unknown quantities now include \( \{P(y(t)|x = \xi, \not J_{\xi t}\}, P(y(t)|x = \xi, J_{\xi t}\}, P(J_{\xi t}|x = \xi); (t, \xi) \in T \times X\) whereas they previously were only \( \{P(y(t)|x = \xi, \not J_{\xi t}\}; (t, \xi) \in T \times X\). Second, the space \( Z \) of treatment rules here is larger than it was there, the treatment allocation now varying with the sample data as well as with a person’s covariates. Third, \( E_{ij}[z(t, \xi, \psi)] \) here takes the place of \( z(t, \xi) \) there; observe that \( E_{ij}[z(t, \xi, \psi)] \) varies with the state of nature \( \gamma \), whereas \( z(t, \xi) \) does not.

5. Finite-sample minimax regret with no knowledge of treatment selection in the study population

It is generally complex to solve problem (13). I focus on a case that yields a curiously simple and instructive result, given in Proposition 2.

**Proposition 2.** Let \( T = \{a, b\} \). Let \( u_{0\xi} \equiv u_{0a\xi} = u_{0b\xi} \) and \( u_{1\xi} \equiv u_{1a\xi} = u_{1b\xi} \). Let nothing be known about treatment selection in the study population and let all realized outcomes in this population be observable. Consider any sampling process that, for \( \xi \in X \), randomly draws \( N_{\xi} > 0 \) members of the study population with covariates \( \xi \). Then

\[
z'_{\nu}(b, \xi, \psi) = \frac{e_{N\xi}P_{N\xi} + u_{1\xi}P_{N\xi} - e_{Na\xi}P_{Na\xi} - u_{0\xi}P_{Na\xi}}{u_{1\xi} - u_{0\xi}}
\]

(14)
is a finite-sample minimax-regret treatment rule.

**Proof.** Fix \( \xi \in X \). The observability score \( p_{\xi} = 1 \) under the maintained assumptions. It follows from this and from (8) that the minimax-regret rule given knowledge of \( (e_{a\xi}; p_{a\xi}) \) and \( (e_{b\xi}, p_{b\xi}) \) is

\[
z'_{\nu}(b, \xi) = \frac{e_{b\xi}P_{b\xi} + u_{1\xi}P_{b\xi} - e_{a\xi}P_{a\xi} - u_{0\xi}P_{a\xi}}{u_{1\xi} - u_{0\xi}}.
\]

(15)

Rule (15) is not computable when \( (e_{a\xi}, p_{a\xi}) \) and \( (e_{b\xi}, p_{b\xi}) \) are not known but rule (14), which replaces the unknown quantities with their sample analogs, is well-defined and computable when \( N_{\xi} > 0 \).

The numerator of \( z'_{\nu}(b, \xi) \) can be written as a sum of expected values. For each member \( j \) of the study population, let \( s_j(t) = 1 \) if treatment \( t \) is observable and \( s_j(t) = 0 \).
otherwise. Then

\[
e_{b\xi}p_{b\xi} + u_{1\xi}p_{a\xi} - e_{a\xi}p_{a\xi} - u_{0\xi}p_{b\xi} = E[u(b) \cdot s(b)|x = \xi] + u_{1\xi}E[s(a)|x = \xi] - E[u(a) \cdot s(a)|x = \xi] - u_{0\xi}E[s(b)|x = \xi].
\]

The numerator of \(z_N^*(b, \xi, \psi)\) is the corresponding sum of sample averages, namely

\[
e_{N\xi}p_{N\xi} + u_{1\xi}p_{N\xi} - e_{N\xi}p_{N\xi} - u_{0\xi}p_{N\xi} = E_N[u(b) \cdot s(b)|x = \xi] + u_{1\xi}E_N[s(a)|x = \xi] - E_N[u(a) \cdot s(a)|x = \xi] - u_{0\xi}E_N[s(b)|x = \xi].
\]

Hence, \(E[z_N^*(b, \xi, \psi)] = z^*(b, \xi)\). This holds whatever values \((e_{a\xi, p_{a\xi}})\) and \((e_{b\xi, p_{b\xi}})\) may take.

The above shows that, whatever values \((e_{a\xi, p_{a\xi}})\) and \((e_{b\xi, p_{b\xi}})\) may take, the finite-sample maximum regret achieved by rule \(z_N^*\) equals the maximum regret achieved by rule \(z^*\). The latter rule minimizes maximum regret. Hence, the former rule minimizes finite-sample maximum regret. □

The unbiasedness of \(z_N^*\) is a delicate property whose proof uses all of the assumptions maintained in the proposition. Consider the assumption that \(N_{\xi} > 0, \xi \in X\). This ensures that rule \(z_N^*\) always is well-defined, a pre-condition for it to have an expected value. The assumption is not satisfied by simple random sampling, which produces samples having empty covariate cells with positive probability. It is satisfied by the variant of simple random sampling that does not fix the total sample size but rather continues to draw members of the study population until the sample contains at least one person with each covariate value. It is also satisfied by stratified random sampling processes that fix \((N_{\xi} > 0, \xi \in X)\) and randomly draw members of each stratum.

Consider the assumptions that nothing is known about treatment selection in the study population, that all realized outcomes are observable, and that \((u_{0a\xi} = u_{0b\xi}, u_{1a\xi} = u_{1b\xi})\), \(\xi \in X\). These assumptions imply that the minimax-regret rule has the special form (15), which is a sum of expected values, rather than the general form (7), which lacks this property. The first two assumptions also imply that \(p_{Na\xi} + p_{Nh\xi} = 1\) in all samples, which ensures that \(z_N^*(b, \xi, \psi) \in [0, 1]\).

Proposition 2 shows that \(z_N^*\) is a finite-sample minimax-regret rule, not the unique such rule. Any unbiased estimate of the minimax-regret rule is finite-sample minimax regret. A curious implication is that, under the assumptions of Proposition 2, there is no advantage to large sample size from the finite-sample minimax-regret perspective – observing one randomly drawn person with covariates \(\xi\) is as good as observing all such persons. The reason is that Wald’s statistical decision theory supposes that the decision maker is risk neutral, concerned only with the expected performance of statistical decision functions across repeated samples.

I am not aware of other applications of the finite-sample minimax-regret criterion in which there exist unbiased estimates of the minimax-regret rule. In particular, unbiased estimates do not exist when the data are from a classical randomized experiment. In this case, the minimax-regret rule (7) assigns all observationally identical persons to the optimal treatment. A finite-sample estimate of this rule is unbiased only if it makes the correct assignment with probability one. No treatment rule using random sample data can be this effective.
Acknowledgment

This research was supported in part by National Science Foundation Grant SES-0314312. I am grateful to two referees for their comments.

References