On Identification of a Beckerian Marriage Matching Model

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Abstract

In this note, we study the identification of structural parameters in a simple marriage matching model with assortative matching and transferable utility. We show that an intuitive identification strategy runs into an indeterminacy problem. We then propose a solution to the problem based on random matching.

Keywords. Assortative Matching, Identification, Random Matching, Becker marriage model.

JEL codes. C51, C78

Consider a simple two-sided marriage matching model where each man (woman) is characterized by \( \tilde{X} \equiv (X, \epsilon) \) (\( \tilde{Y} \equiv (Y, \eta) \)), where \( X \) and \( Y \) are observables and \( \epsilon \) and \( \eta \) are unobservables. Further assume that each man’s (woman’s) \( \epsilon \) (resp. \( \eta \)) is drawn from the distribution \( F_{\epsilon}(\cdot, \beta_\epsilon) \) (resp. \( F_{\eta}(\cdot, \beta_\eta) \)), which is known up to the finite-dimensional parameters \( \beta_\epsilon \) (\( \beta_\eta \)). Assume that \( X \) and \( \epsilon \) are independent and so are \( Y \) and \( \eta \). The characteristics are independently drawn across individuals.

Assume that the marriage market is a transferable utility two-sided matching market. The total surplus which a man with characteristics \((X, \epsilon)\) and a woman with characteristics \((Y, \eta)\) obtain from being matched is given by the surplus function \( S(X, \epsilon, Y, \eta) \). Following Becker (1973), we assume that each agent’s observed and unobserved characteristics affect the surplus function via a single-index; that is to say, \( S(X, \epsilon, Y, \eta, \theta) = S(f(X, \epsilon, \theta), g(Y, \eta, \theta)) \), for scalar-valued

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mappings \( f(\cdot) \) and \( g(\cdot) \) and a parameter \( \theta \). In what follows, we will refer to \( U := f(X, \epsilon, \beta_\epsilon) \) and \( V := g(Y, \eta, \beta_\eta) \) as the quality indices of men and women, and denote generic values of these elements by \( u \) and \( v \), respectively, and the surplus by \( S(u, v) \). Notice that, so far, our set up is similar to that of the transferable utility model studied in Chiappori et. al. (2012), although we do not assume that \( X \) and \( \epsilon \) (\( Y \) and \( \eta \)) are separable in the single index function.

Moreover, in this note we restrict attention to surplus functions which are supermodular in the indices \( u \) and \( v \). As is well known, supermodularity of the surplus function implies that the optimal matching is assortative (see Section 1 of the Appendix of Becker (1973)), with higher quality women (as measured by their indices \( V \)) being matched to men with higher values for their indices \( U \). That is, letting \( F_U \) denote the CDF of the men’s quality index \( u \), and analogously \( F_V \), we have that the optimal matching will satisfy that, for each \( \forall \tau \in [0,1] \) the optimal matching will associate a man with quality index \( u = F_U^{-1}(\tau) \) to a woman with index \( v = F_V^{-1}(\tau) \).

Suppose that we observe the joint distribution of the observable characteristics: \( F_{X,Y} \) of married couples in the data, and would like to learn about \( \theta, \beta_\epsilon \) and \( \beta_\eta \) based on \( F_{X,Y} \). The next section of this note describes a natural identification strategy. But this natural strategy runs into a non-uniqueness problem which is discussed in detail in Section 3. In Section 4, we propose a solution to this problem, based on the idea of “random matching”.

## 1 A Natural Identification Strategy

A natural identification strategy is that of “matching moments,” which is described now. First, given the independence between \( X \) and \( \epsilon \), as well as the independence between \( Y \) and \( \eta \), we know the joint distribution of \((X, \epsilon)\) and that of \((Y, \eta)\), which are

\[
F_{X,\epsilon}(x,\epsilon) = F_X(x)F_\epsilon(\epsilon) \quad \text{and} \quad F_{Y,\eta}(y,\eta) = F_Y(y)F_\eta(\eta),
\]

where \( F_X \) and \( F_Y \) are marginal distributions derived from the joint distribution \( F_{X,Y} \). For fixed value of \( \beta_\epsilon, \beta_\eta \) and \( \theta \), the distributions \( F_{X,\epsilon} \) and \( F_{Y,\eta} \) imply a unique marginal distribution for the

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1In Becker’s original formulation, these single-indices are interpreted as the time inputs that a husband or wife contribute towards household production. See also Roth and Sotomayor (1990).

2We also focus on a specific problem in the identification of structural model parameters that Chiappori et. al. (2012) do not discuss in detail.
single indices $U$ and $V$, which can be denoted as

$$F_U(u, \beta_\epsilon, \theta)$$ and $$F_V(v, \beta_\eta, \theta).$$

(2)

The single index assumption above and the supermodularity of the surplus function together imply an optimal matching rule characterized by assortative matching:

**Definition 1** (Single Index Assortative Matching Rule). A man with $U = u$ is matched with a woman with $V = F_V^{-1}(\tau, \beta_\eta, \theta)$ where $\tau = F_U(u, \beta_\epsilon, \theta)$.

An intuitive identification strategy for the model parameters $\beta_\epsilon, \beta_\eta, \theta$ can be formulated, provided that the following additional assumption is satisfied:

**Assumption 1.** The marginal distributions $F_{X,\epsilon}$ and $F_{Y,\eta}$ and the single index assortative matching rule in Definition 1 implies a unique joint distribution of $X$ and $Y$ in married couples:

$$F_{X,Y}^*(x,y, \beta_\epsilon, \beta_\eta, \theta).$$

Under Assumption 1, one can form the following moment matching system of equations:

$$F_{X,Y}(x,y) = F_{X,Y}^*(x,y, \beta_\epsilon, \beta_\eta, \theta), \; \forall x,y.$$ (3)

From this system of equations, the parameters of interest: $\beta_\epsilon, \beta_\eta$ and $\theta$ can be identified (either to a point or to a subset of the parameter space).

Assumption 1 seems innocuous enough, given the restrictions implied by the assortative matching rule. However, we show in the next section that Assumption 1 generally fails without additional assumptions.

## 2 Indeterminancy of the Matching Rule

In this section we show how Assumption 1 fails, that is, how the single index assortative matching rule fails to determine a unique joint distribution of the observable characteristics for married couples. We start with two simple motivating examples.

**Bernoulli Example.** Suppose that $X, \epsilon, Y, \eta$ are independent from each other and are all identically distributed as $Bernoulli(0.5)$. We assume that the index functions are additive:

$$U := X + \epsilon \quad \text{and} \quad V := Y + \eta.$$
For a supermodular surplus function $S(U, V)$, the optimal assortative matching is characterized by:

\[
(X, \epsilon) \iff (Y, \eta):
\]

\[
(1, 1) \iff (1, 1)
\]

\[
(1, 0) \iff (1, 0)
\]

\[
(0, 1) \iff (0, 1)
\]

\[
(0, 0) \iff (0, 0)
\]

In an optimal match, any woman with characteristics $(0, 1)$ or $(1, 0)$ can match with any man with characteristics $(0, 1)$ or $(1, 0)$. This leads to indeterminacy of the joint distribution of $(X, Y)$ in the matched population.

**Bivariate Normal Example.** Assume that $(X, Y, \epsilon, \eta)$ are all mutually independent standard normal random variables. The joint distribution of $(X, \epsilon)$ is $N(0, I_2)$; so is the distribution of $(Y, \eta)$. As before, we consider the additive index case where $U = X + \epsilon$, and $V = Y + \eta$.

In this case, an assortative matching will match men and women according to the equality

\[ X + \epsilon = Y + \eta. \]

This implies a continuum of joint distributions for $(X, Y)$. For instance, this match is consistent with the matching

\[ \{(X, Y, \epsilon, \eta) : X = \eta, Y = \epsilon\} \]

in which case $(X, Y) \sim N(0, I_2)$, $(\epsilon, \eta) \sim N(0, I_2)$. It is also consistent with the matching

\[ \{(X, Y, \epsilon, \eta) : X = Y, \eta = \epsilon\} \]

$(X, Y) \sim N(0, [1, 1; 1, 1])$. In fact, it is consistent with anything between these polar examples as well.

**General Case.** As we can see from the examples, the key to the indeterminacy of $F_{X,Y}^*$ is that different combinations of $X$ and $\epsilon$ can yield the same value of $U \equiv g(X, \epsilon)$, and different combinations of $Y$ and $\eta$ can yield the same value of $V \equiv g(Y, \eta)$. In other words, a man with $U = u$ is indifferent to women with the same $V = F_V^{-1}(F_U(u, \beta_\epsilon, \theta), \beta_\eta, \theta)$ but different combinations of $(Y, \eta)$; and vice versa.
Mathematically, the indeterminacy problem can be described as follows: The single index assortative matching implies a unique joint distribution of \((U, V)\), given by\(^3\)

\[
F_{U,V}(u, v, \beta_\epsilon, \beta_\eta, \theta) = \min(F_U(u, \beta_\epsilon, \theta), F_V(v, \beta_\eta, \theta)).
\]

(5)

Also, the joint distribution of \((U, V)\) is related to that of \((X, Y)\) through the following integral equation:

\[
F^*_X(x, y, \beta_\epsilon, \beta_\eta, \theta) = F_Y(y) \int \left[ \int F_{X|Y,\eta}(x|y,a) da \right] dF_\eta(a).
\]

(6)

Equation (5) uniquely determines \(F_{U,V}(x,\epsilon|g(Y,\eta)|u|v)\) but that is not sufficient to uniquely determine \(F_{X|Y,\eta}(x|e|y,a)\), causing \(F^*_X(x, y, \beta_\epsilon, \beta_\eta, \theta)\) to be under determined.

The indeterminacy problem arises from two features of the transferable utility matching model that we have set up. First, because \(\epsilon\) and \(\eta\) are unobservable, it is not possible to directly identify the parameters by matching \(F_{U,V}\) to the model-implied joint distribution of \((U, V)\). Second, because of the single-index assumption, there are many values of characteristics which map to a given value of the index, which necessarily imply a one-to-many mapping from \(F_{U,V}\) to \(F_{X,Y}\).

3 Introducing Random Matching

In all the above examples, the identification problem is that, because of the index assumption, the set of spouses that a given agent is matched to in the optimal assortative matching – the “matching set”, as it is called in the literature\(^4\) – is not a singleton. This leads to multiple values for the equilibrium joint distribution of the observed characteristics \((X, Y)\); this multiplicity breaks down the “moment matching” identification strategy described in Section 1.

We propose one resolution of this problem using the notion of “random matching”. Roughly speaking, what we mean by random matching is that, for agents who have matching sets which are not singletons, they are randomly matched to each element of their matching sets. Formally, “random matching” is specified in the following assumption:\(^5\)

\(^3\)This corresponds to the Fréchet upper-bound copula (see, eg. Joe (1997)).

\(^4\)See e.g. Shimer and Smith (2000), Atakan (2006).

\(^5\)Note that in the assumption and the rest of this section, we ignore the parameters \(\beta_\epsilon, \beta_\eta, \theta\) for notational simplicity.
Assumption 2 (Random Matching). The joint distribution of $(X, \epsilon, Y, \eta)$ for a married couple satisfies

$$F^*_X(x|y,a) = F^*_{X|g(y,\eta)}(x|g(y,a)).$$

That is, a woman with characteristics $(y,a)$ and a woman with characteristics $(y',a')$ have equal opportunity among men, as long as $g(y,a) = g(y',a').$

Comments. (a) Our random matching only allow the matches to occur randomly among potential matches with the same surplus. Thus, the randomness does not affect the optimality of the matching outcome.

(b) Assumption 2 may be micro-founded as limit versions of the search and matching models of (i) Shimer and Smith (2000), as we take the agents’ patience to infinity; or (ii) Atakan (2006), as the agents’ search cost approaches zero. Indeed, the random arrival of potential partners in those search models provides a natural randomization mechanism, and the ex ante symmetry of the agents guarantees the “equal opportunity” part of the assumption.

Next we show that Assumption 2 is sufficient for Assumption 1 and thus sufficient for enacting the identification strategy in Section 1. We firsts resolve the two simple examples above, before proceeding to the general argument.

Bernoulli Example, continued. In this example, random matching implies that the $(1,0)$ and $(0,1)$ women match “randomly” with the $(1,0)$ and $(0,1)$ men. Formally, under Assumption 2, the matched joint distribution of $(X, \epsilon, Y, \eta)$ satisfies

(i) $\Pr(U = V) = 1$ (assortative matching); and

(ii) $\Pr(X = 1|Y = y, \eta = a) = \Pr(X = 1|Y + \eta = y + a)$ (random matching)

The two conditions uniquely determine the joint distribution of $X$ and $Y$ in the matched population by the following derivation. First observe that

$$\Pr(X = 1|Y = 1, \eta = 1) = \Pr(X = 1|V = 2) = \Pr(X = 1|U = 2) = \Pr(X = 1|X = 1, \epsilon = 1) = 1$$

where the first equality holds by condition (ii) and the second equality holds by condition (i). Similarly,

$$\Pr(X = 1|Y = 1, \eta = 0) = \Pr(X = 1|V = 1) = \Pr(X = 1|U = 1)$$
\[ \Pr(X = 1|X = 1, \epsilon = 0 \text{ or } X = 0, \epsilon = 1) = 0.5 \quad (8) \]

\[ \Pr(X = 1|Y = 0, \eta = 1) = \Pr(X = 1|V = 1) = \Pr(X = 1|U = 1) = \Pr(X = 1|X = 1, \epsilon = 0 \text{ or } X = 0, \epsilon = 1) = 0.5 \quad (9) \]

\[ \Pr(X = 1|Y = 0, \eta = 0) = \Pr(X = 1|V = 0) = \Pr(X = 1|U = 0) = \Pr(X = 1|X = 0, \epsilon = 0) = 0 \quad (10) \]

Therefore,

\[ \Pr(X = 1|Y = 1) = 0.5 \Pr(X = 1|Y = 1, \eta = 1) + 0.5 \Pr(X = 1|Y = 1, \eta = 0) = 0.75 \]

\[ \Pr(X = 1|Y = 0) = 0.5 \Pr(X = 0|Y = 1, \eta = 1) + 0.5 \Pr(X = 1|Y = 1, \eta = 0) = 0.25 \quad (11) \]

These fully and uniquely characterize the joint distribution of \( X \) and \( Y \) under (i) and (ii).

\[ \text{Bivariate normal example, continued.} \]

For the bivariate normal example, under Assumption 2, the matched joint distribution of \((X, \epsilon, Y, \eta)\) satisfies

(i) \( \Pr(U = V) = 1; \) and

(ii) \( f_{X|Y,\eta}(x|y, a) = f_{X|Y+\eta}(x|y + a). \)

Now we show that in the optimal match defined by (i) and (ii), the unique distribution of observed characteristics \( X, Y \) is bivariate normal with variance covariance matrix \([1, 0.5; 0.5, 1]\).

Observe that

\[ f_{X|Y,\eta}(x|y, a) = f_{X|Y+\eta}(x|y + a) = f_{X|X+\epsilon}(x|y + a) \]

\[ = \frac{f_{X,X+\epsilon}(x, y + a)}{f_{X+\epsilon}(y + a)} \]

\[ = \frac{(2\pi)^{-1}\exp(-2^{-1}(2x^2 - 2x(y + a) + (y + a)^2))}{(4\pi)^{-1/2}\exp(-1(y + a)^2)} \]

\[ = \pi^{-1/2}\exp(-x^2 + xy + xa - (y + a)^2/4) \quad (12) \]

Thus,

\[ f_{X|Y}(x|y) = \int_{-\infty}^{\infty} f_{X|Y,\eta}(x|y, a)f_{\eta}(a)da \]

\[ = \int_{-\infty}^{\infty} \pi^{-1/2}\exp(-x^2 + xy + xa - (y + a)^2/4)(2\pi)^{-1/2}\exp(-a^2/2)da \]

\[ = \frac{1}{\sqrt{2\pi}}\exp(-x^2 + xy - y^2/4) \int_{-\infty}^{\infty} \exp(xe - ya/2 - 3a^2/4)da \quad (13) \]
Therefore,

\[
f_{X,Y}(x, y) = f_{X|Y}(x|y)f_Y(y)
= \frac{1}{2\pi^{3/2}} \exp(-x^2 + xy - 3y^2/4) \int_{-\infty}^{\infty} \exp(xa - ya/2 - 3a^2/4) da
= \frac{1}{\sqrt{3\pi}} \exp(-(x,y)\Sigma^{-1}(x,y)'/2),
\]

(14)

where \(\Sigma = [1, 0.5; 0.5, 1]\). This shows that \((X, Y) \sim N(0, \Sigma)\). ■

**General Case, continued.** In the general case, under Assumption 2, the optimal matching can be defined by the following two conditions:

(i) \(\Pr(F_U^*(U) = F_V(V)) = 1\); and

(ii) \(F_{X|Y,\eta}^*(x|y, a) = F_{X|g(V,\eta)}^*(x|g(y, a))\).

Suppose that both \(U\) and \(V\) have strictly increasing distributions. Due to the strict monotonicity, the inverse function \(F_U^{-1}\) is well-defined. Then we have

\[
F_{X|Y,\eta}^*(y, a) = F_{X|V}^*(x|g(y, a)) = F_{X|U}^*(x|F_U^{-1}(F_V(g(y, a))))
\]

(15)

Thus,

\[
F_{X,Y}^*(x, y) = F_{X|Y}^*(x|y)F_Y(y) = F_Y(y) \int F_{X|Y,\eta}^*(x|y, a)dF_\eta(a)
= F_Y(y) \int F_{X|g(V,\eta)}(x|g(y, a))dF_\eta(a)
= F_Y(y) \int F_{X|f(X,\epsilon)}(x|F_U^{-1}(F_V(g(y, a))))dF_\eta(a).
\]

(16)

That is, \(F_{X,Y}^*\) can be analytically (and uniquely) determined by \(F_{X,\epsilon}\) and \(F_{Y,\eta}\).

**Estimation strategies.** The analytical representation (16) suggests a possible computational strategy to obtain the identified value(s) of \(\beta_\epsilon, \beta_\eta\) and \(\theta\). Given values for these parameters, the distribution of the indices \(F_V\) and \(F_U\) can be derived from the joint distributions \(F_{X,\epsilon,\beta_\epsilon}\) and \(F_{Y,\eta,\beta_\eta}\) using convolution techniques. Subsequently, \(F_{X,Y}^*(x, y, \beta_\epsilon, \beta_\eta, \theta)\) be obtained by numerical computation of the integral in equation (16). By setting this \(F_{X,Y}^*(x, y, \beta_\epsilon, \beta_\eta, \theta)\) equal to the observed joint distribution \(F_{X,Y}\) (as in Eq. (3)), one can solve for the value(s) of the parameters that are consistent with the data.

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\(^6\)This is without loss of generality because if \(F_U\) is not strictly increasing, we can simply redefine \(U\) to be \(F_U\), and work with the redefined \(U\) instead.
However, the procedure sketched above may impose formidable computational hurdles. In that case, we suggest a simpler simulation-based method here. For each $\beta_\epsilon, \beta_\eta, \theta,$

**Step 1.** Independently draw $N$ i.i.d. observations of $(X, \epsilon)$ and $(Y, \eta)$ from $F_{X,\epsilon}(\cdot, \beta_\epsilon)$ and $F_{Y,\eta}(\cdot, \beta_\eta)$ respectively for a large number $N$. Denote the draws by $\{X_i, \epsilon_i\}_{i=1}^N$ and $\{Y_i, \eta_i\}_{i=1}^N$.

**Step 2.** Compute $\{U_i = f(X_i, \epsilon_i, \theta)\}_{i=1}^N$ and $\{V_i = g(Y_i, \eta_i, \theta)\}_{i=1}^N$.

**Step 3.** Sort both $\{(X_i, U_i)\}$ and $\{(Y_i, V_i)\}$ in decreasing order of $U_i$ and $V_i$ respectively, and when doing so, make sure that the sorting mechanism randomly sorts tied values. Let the sorted data be denoted $\{\tilde{X}_i, \tilde{\epsilon}_i, \tilde{Y}_i, \tilde{\eta}_i\}_{i=1}^N$.

**Step 4.** Then, compute the simulated $F_{X,Y}^*(x,y; \beta_\epsilon, \beta_\eta, \theta)$ as follows:

$$F_{X,Y}^*(x,y; \beta_\epsilon, \beta_\eta, \theta) = N^{-1} \sum_{i=1}^N 1(\tilde{X}_i \leq x, \tilde{Y}_i \leq y).$$

(17)

### 4 Concluding Remarks

We focus on assortative matching in this paper for simplicity. But our analysis can be extended to the general transferable utility framework as studied in Choo and Siow (2006) and Galichon and Salanie (2011). In particular, we expect that the fast equilibrium-solving algorithm developed in Galichon and Salanie can be used in combination with our identification strategy to yield useful estimation procedures in practice.

In this note, we assume precise knowledge of $F_{X,Y}$, although in practical situations, typically only a finite sample estimator of $F_{X,Y}$ is available. One can take into account the sample variability based on the asymptotic properties of the distribution estimator. This is beyond the scope of this note and is left for future research.

### References


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7See also Fox (2010), Uetake and Watanabe (2012), Menzel (2013), and Graham (2013), among others, for work utilizing structural econometric two-sided matching models.


