ABSTRACT. We consider the estimation of dynamic discrete choice models in a semiparametric setting, in which the per-period utility functions are known up to a finite number of parameters, but the distribution of utility shocks is left unspecified. This semiparametric setup differs from most of the existing identification and estimation literature for dynamic discrete choice models. To show identification we derive and exploit a new Bellman-like recursive representation for the unknown quantile function of the utility shocks. Our estimators are straightforward to compute, and resemble classic closed-form estimators from the literature on semiparametric regression and average derivative estimation. Monte Carlo simulations demonstrate that our estimator performs well in small samples. To highlight features of this estimator, we estimate a structural model of dynamic labor supply for New York City taxicab drivers.

Keywords: Semiparametric estimation, Dynamic discrete choice model, Average derivative estimation, Taxicab industry, Labor supply

JEL: C14, D91, C41, L91
1. Introduction

The dynamic discrete choice (DDC) framework, pioneered by Wolpin (1984), Pakes (1986), Rust (1987, 1994), has gradually become the workhorse model for modelling dynamic decision processes in structural econometrics. Such models, which can be considered an extension of McFadden’s 1978; 1980 classic random utility model to a dynamic decision setting, have been used to model a variety of economic phenomenon ranging from labor and health economics to industrial organization, public finance, and political economy. More recently, the DDC framework has also been the starting point for the empirical dynamic games literature in industrial organization.

In this paper, we consider identification and estimation of a class of semiparametric dynamic binary choice models in which the utility indices are parametrically specified (as a linear index of observed variables) but the shock distribution is left unspecified. Since the utility shocks are typically interpreted as idiosyncratic and unpredictable shocks to preferences, which cause agents’ choices to vary over time even under largely unchanging economic environments, it is reasonable to leave their distribution unspecified. We study conditions under which the model structure (consisting of the finite-dimensional parameters in the utility indices, and the infinite-dimensional nonparametric shock distribution) is identified. Our identification argument is constructive, and we propose an estimator based upon it.

The semiparametric DDC framework that we focus on in this paper is novel relative to most of the existing literature on the identification and estimation of DDC models, which considers the case where the utility shocks are fully (parametrically) specified. This reflects an important result in Magnac and Thesmar (2002), who argue that in these models, the single-period utility indices for the choices are (nonparametrically) identifiable only when the distribution of the utility shocks is completely specified. Based on Magnac–Thesmar’s “impossibility” result, many recent estimators for and applications of DDC models have considered a structure in which the single-period utility indices are left unspecified, but the utility shock distribution is fully specified (and usually logistic, leading to the convenient multinomial logit choice probabilities).

For identification, we derive a new recursive representation for the unknown quantile function of the utility shocks. Accordingly, we obtain a single-index representation for the conditional choice probabilities in the model, which permits us to estimate the model using classic estimators.
from the existing semiparametric binary choice model literature. Specifically, we use Powell, Stock and Stoker’s (1989, PSS) kernel-based average-derivative estimator; we show that, under additional mild conditions, our estimator has the same asymptotic properties as PSS’s original estimator (which was applied to static discrete-choice models). Moreover, this estimator is computationally quite simple because it can be expressed in closed-form. Monte Carlo simulations demonstrate that our estimator performs well in small samples. To highlight features of this estimator, we estimate a structural model of dynamic labor supply for New York City taxicab drivers. We use our estimated model to develop insight into the labor-leisure tradeoff in light of the dynamic decision environment faced by drivers.

1.1. Literature. This paper builds upon several strands in the existing literature. The semiparametric binary choice literature (e.g. Manski (1975, 1985), Powell, Stock, and Stoker (1989), Ichimura and Lee (1991), Horowitz (1992), Klein and Spady (1993), and Lewbel (1998), among many others) is an important antecedent. There is an important substantive difference, however: because these papers focus on a static model, the shock distribution is treated as a nuisance element. As such, estimation of these shocks is not considered. In contrast, the shock distribution in a dynamic model must be estimated since it affects the beliefs that decision makers have regarding their future payoffs. Hence, the need to estimate both the utility parameters as well as the shock distribution represents an important point of divergence between our paper and the previous semiparametric discrete choice literature; nevertheless, as we will point out, the estimators we propose take a very similar form to the estimators in these papers.

To our knowledge, only a handful of papers consider identification of dynamic models in which the error distribution is left unspecified. Norets and Tang (2014) focus on the discrete state case, and derive (joint) bounds on the error distribution and per-period utilities which are consistent with an observed vector of choice probabilities. We consider the case with continuous state variables, and discuss nonparametric identification and estimation. With a discrete state space, there can never be point identification when the error distribution has continuous support. When the state space is continuous, however, point identification is possible under some support conditions and a location–scale normalization on the error distribution, as we show.
Aguirregabiria (2010) shows the joint nonparametric identification of utilities and the shock distribution in a class of finite-horizon dynamic binary choice models. His identification argument relies on the existence of a final period in the decision problem, and hence may not apply to infinite-horizon models as considered in this paper. Chen (2014) considers the identification of dynamic models, and, as we do here, obtains estimators for the model parameters which resemble familiar estimators in the semiparametric discrete choice literature. His approach exploits exclusion restrictions (that is, that a subset of the state variables affect only current utility, but not agents’ beliefs about future utilities). Blevins (2014) considers a very general class of dynamic models in which agents can make both discrete and continuous choices, and the shock distribution can depend on some of the state variables. Under exclusion restrictions, he shows the nonparametric identification of both the per-period utility functions as well as the error distribution. Unlike these papers, we do not use exclusion restrictions for identification, but rather exploit the optimality conditions to derive a new recursive representation of the quantile function for the unobserved shocks which allows us to identify and estimate both the model parameters as well as the shock distribution.

Another important related paper is Srisuma and Linton (2012), who pioneered the use of tools for solving type 2 integral equations for estimating dynamic discrete-choice models. We show that, besides the Bellman equation, other structural relations in the dynamic model also take the form of type 2 integral equations. In particular, when viewed as a function of the choice probability, the (unknown) quantile function for the utility shocks can also be recursively characterized via a Bellman-type equation, and hence methods for solving for the value function in the “usual” Bellman equation (either value function iteration or “forward simulation”) can also be applied in order to solve for this quantile function.

2. SINGLE AGENT DYNAMIC DISCRETE CHOICES MODEL

Following Rust (1987), we consider a single-agent infinite-horizon binary decision problem. At each time period $t$, the agent observes state variables $X_t \in \mathcal{X} \subseteq \mathbb{R}^k$, and chooses a binary decision $Y_t \in \{0, 1\}$ to maximize her expected utility. The per-period utility is given by

$$u_t(Y_t, X_t, \epsilon_t) = \begin{cases} W_1(X_t)^T \theta_1 + \epsilon_{1t}, & \text{if } Y_t = 1; \\ W_0(X_t)^T \theta_0 + \epsilon_{0t}, & \text{if } Y_t = 0. \end{cases} \quad (1)$$
In the above, $W_0(X_t) \in \mathbb{R}^{k_0}$ (resp. $W_1(X_t) \in \mathbb{R}^{k_1}$) denotes known transformations of the state variables $X_t$ which affect the per–period utility from choosing $Y_t = 0$ (resp. $Y_t = 1$), and $\epsilon_t \equiv (\epsilon_{0t}, \epsilon_{1t})^T \in \mathbb{R}^2$ are the agent’s action-specific payoff shocks. The structural parameters which are of interest are $\theta_d \in \mathbb{R}^{k_d}$, for $d \in \{0, 1\}$. In what follows, let $W(X) \equiv \{W_0(X), W_1(X)\}$ denote the full set of transformed state variables at $X$. For notational simplicity, we will use the shorthand $W_d$ for $W_d(X)$ ($d = 0, 1$) and suppress the explicit dependence upon the state variables $X$ when possible.

This specification of the per-period utility functions in Eq. (1), as single-indices of the transformed state variables $W(X)$ encompasses a majority of the existing applications of dynamic discrete-choice models, and thus imposes little loss in generality. The utility of action 0 is not normalized to be zero for reasons discussed in Norets and Tang (2014). Moreover, let $\beta \in (0, 1)$ be the discount factor (which is assumed to be known for purposes of identification and estimation)\(^1\) and $f_{X_{t+1}, \epsilon_{t+1}|X_t, \epsilon_t, Y_t}$ be the Markov transition probability density function that depends on the state variable as well as the decision.

The agent maximizes the expected discounted sum of the per-period payoffs:

$$
\max_{y_t, y_{t+1}, \ldots} \mathbb{E}\left\{ \sum_{s=t}^{\infty} \beta^{s-t} u_s(y_s, X_s, \epsilon_s) | X_t, \epsilon_t \right\}, \quad \text{s.t.} \quad f_{X_{s+1}, \epsilon_{s+1}|X_s, \epsilon_s, Y_s}.
$$

We assume stationarity of the problem, which implies that the problem is invariant to the period $t$. Because of this, we can omit the $t$ subscripts and use primes (‘) to denote next period values. Let $V(X, \epsilon)$ be the value function given $X$ and $\epsilon$. By Bellman’s equation, the value function can be written as

$$
V(X, \epsilon) = \max_{y \in \{0, 1\}} \left\{ \mathbb{E}[u(y, X, \epsilon)|X, \epsilon] + \beta \mathbb{E}[V(X', \epsilon')|X, \epsilon, Y = y] \right\}, \quad \text{and then the agent’s optimal decision is given by}
$$

$$
Y = \operatorname{argmax}_{y \in \{0, 1\}} \left\{ \mathbb{E}[u(y, X, \epsilon)|X, \epsilon] + \beta \mathbb{E}[V(X', \epsilon')|X, \epsilon, Y = y] \right\}.
$$

Unlike much of the existing literature, we do not assume the distribution of the utility shocks $(\epsilon_{0t}, \epsilon_{1t})$ to be known, but treat their distribution as a nuisance element for the estimation of $\theta$. In

\(^1\)The assumption that $\beta$ is known is commonplace in the applied DDC literature. See Magnac and Thesmar (2002) and Fang and Wang (2015), among others, for discussion on the identifiability of $\beta$. 
a static setting, such flexibility may not be necessary, as a flexible specification of \( u(X, Y) \) may be able to accommodate any observed pattern in the choice probabilities even when the distribution of utility shocks is parametric.\(^2\) However, in a dynamic setting, the distribution of utility shocks also plays the role of agents’ beliefs about the future evolution of state variables (i.e. they are a component in the transition probabilities \( f_{X',\epsilon'|X,\epsilon,Y} \)) and hence parametric assumptions on this distribution are not innocuous.

2.1. Characterization of the value function. In this subsection, we characterize the value function \( V(X, \epsilon) \) and the expected value function given \( X \), i.e., \( V^e(X) \equiv \mathbb{E}[V(X, \epsilon)|X] \). Both value functions are useful to characterize the equilibrium in our dynamic model. Let \( F_A \) and \( F_{A|B} \) denote the CDF and the conditional CDF for generic random variables \( A \) and \( B \), respectively.

**Assumption A** (Conditional Independence Assumption). *The transition density satisfies the following condition: \( F_{X',\epsilon'|X,\epsilon,Y} = F_{\epsilon'} \times F_{X'|X,Y} \). Moreover, \( F_{\epsilon'} = F_{\epsilon} \).*

Assumption A is strong, as it establishes that the shocks \( \epsilon \) are fully independent of the observed state variables \( X \). This rules out heteroskedasticity in the unobserved shocks, which is accommodated in other papers in the DDC literature (eg. Magnac and Thesmar (2002), Aguirregabiria (2010), among others). It is possible, as in Blevins (2014), to relax the independence assumption to one where the state variables can be divided into two groups \( X = (X_A, X_B) \) such that \( \epsilon \perp X_B|X_A \) (\( \epsilon \) is independent of \( X_B \) given \( X_A \)), which allows for some degree of heteroskedasticity in \( \epsilon \). The identification and estimation procedure described in this paper follow through, with the additional conditioning on \( X_A \) at every step.

Under assumption A, the value function can be written as

\[
V(X, \epsilon) = \max \left\{ W_1^T \theta_1 + \epsilon_1 + \beta \mathbb{E}[V(X', \epsilon')|X, Y = 1], \ W_0^T \theta_0 + \epsilon_0 + \beta \mathbb{E}[V(X', \epsilon')|X, Y = 0] \right\}.
\]

Let \( \eta = \epsilon_0 - \epsilon_1 \). Then the equilibrium decision maximizing the value function can be written as

\[
Y = 1 \{ \eta \leq \eta^*(X) \},
\]

\(^2\)McFadden and Train (2000) show such properties for the mixed logit specification of static multinomial choice models.
where the cutoff $\eta^*(X)$ is defined as

$$
\eta^*(X) \equiv W_1^T \theta_1 - W_0^T \theta_0 + \beta \{ \mathbb{E}[V(X', \epsilon') | X, Y = 1] - \mathbb{E}[V(X', \epsilon') | X, Y = 0] \}.
$$

(2)

Moreover, let $u^e(X)$ be the expected per–period utility conditional on $X$, i.e.,

$$
u^e(X) \equiv \mathbb{E}(\epsilon_0) + W_1^T \theta_1 \cdot F_{\eta}(\eta^*(X)) + W_0^T \theta_0 \cdot [1 - F_{\eta}(\eta^*(X))] - \mathbb{E}\{\eta \cdot 1 [\eta \leq \eta^*(X)]\},
$$

(3)

where $F_{\eta}$ is the CDF of $\eta$. Thus, the Bellman equation can be rewritten as

$$
V^e(X) = u^e(X) + \beta \cdot \mathbb{E}[V^e(X') | X].
$$

(4)

It is worth pointing out that eq. (4) is essentially a Fredholm Integral Equation of the second kind (FIE–2); See e.g. Zemyan (2012). Essentially, FIE–2 is a linear equation system in functional space, which is well–known to have a unique analytic solution under some sufficient and necessary conditions.

**Assumption B.** For all $s \geq 1$, we have $\mathbb{E}(\|W^{|s|}_d\| | X) < \infty$ a.s., where $|s|$ denotes the next $s$ period values.

Assumption B holds when $W_d(\cdot)$ are bounded functions.

Given these assumptions, the next lemma applies the Fredholm theorem to obtain a solution of the expected value function to the Bellman equation. (Similar results are utilized in Srisuma and Linton (2012).)

**Lemma 1.** Suppose assumptions A and B hold. Then, we have

$$
V^e(x) = u^e(x) + \beta \int_{\mathcal{X}} R^e(x', x; \beta) \cdot u^e(x') dx', \ \forall x \in \mathcal{X},
$$

(5)

where $R^e(x', x; \beta) = \sum_{s=1}^{\infty} \beta^{s-1} f_{X}[s, X](x' | x)$ is the resolvent kernel generated by the FIE eq. (4).

More succinctly, eq. (5) can be rewritten as

$$
V^e(X) = u^e(X) + \sum_{s=1}^{\infty} \beta^s \cdot \mathbb{E}[u^e(X^{|s|}) | X].
$$

(6)

In operator notation, eq. (6) denotes exactly the “forward integration” representation of the value function, which is familiar from many two-step procedures for estimating dynamic discrete
choice models (see e.g. Hotz and Miller, 1993; Bajari, Benkard, and Levin, 2007; Hong and Shum, 2010). In the special case when the state variables $X$ are finite and discrete-valued (taking $k < \infty$ values), the Bellman equation is a system of linear equations which can be solved for the value function (cf. Aguirregabiria and Mira, 2007; Pesendorfer and Schmidt-Dengler, 2008) and in that case, the resolvent kernel is just the inverse matrix $(I - \beta F_{X|X})^{-1}$ where $F_{X|X}$ denotes the $k \times k$ transition matrix for $X$.

2.2. Equilibrium Condition. To characterize the equilibrium, the key of our approach is to solve for the cutoff value $\eta^*$ that depends on the state variables $X$ (through the transformations $W_1(X)$ and $W_0(X)$). By using eq. (6), along with Lemma 1, eq. (2) becomes

$$\eta^*(X) = W_1^T \theta_1 - W_0^T \theta_0 + \sum_{s=1}^{\infty} \beta^s \left\{ E[u^e(X[s])|X, Y = 1] - E[u^e(X[s])|X, Y = 0] \right\}. \quad (7)$$

Moreover, let $\phi_d(X) \equiv (-1)^{d+1}W_d + \sum_{s=1}^{\infty} \beta^s \left\{ E[W_d[s]1_{Y[s]=d}|X, Y = 1] - E[W_d[s]1_{Y[s]=d}|X, Y = 0] \right\}$. Then, it follows from (3),

$$\eta^*(X) = \phi^T(X) \cdot \theta$$

$$- \sum_{s=1}^{\infty} \beta^s \left\{ E[\eta^*[s]1(\eta^*[s] \leq \eta^*(X[s]))|X, Y = 1] - E[\eta^*[s]1(\eta^*[s] \leq \eta^*(X[s]))|X, Y = 0] \right\}, \quad (8)$$

where $\phi(X) = (\phi_0^T(X), \phi_1^T(X))^T$ and $\theta = (\theta_0^T, \theta_1^T)^T$.

Eq. (8) characterizes the equilibrium decision rule in the single–agent infinite-horizon binary decision problem. Alternatively, we can rewrite it using a resolvent kernel:

$$\eta^*(x) = \phi^T(x) \cdot \theta - \int_{\mathcal{X}} E[\eta^*[x'] \cdot 1(\eta^*[x'] \leq \eta^*(x')) \cdot g(x', x; \beta)dx', \forall x \in \mathcal{X},$$

where $g(x', x; \beta) = \sum_{s=1}^{\infty} \beta^{s-1}[f_{X|x}[x,y](x'|x, 1) - f_{X|x}[x,y](x'|x, 0)]$. Given the structural parameters $\theta_0, \theta_1, F_\eta$ and $f_{X|x,y}$, in principal one can solve the threshold $\eta^*(\cdot)$.

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3However, if one were to use this equation to solve for $\eta^*(\cdot)$ via simulation or computation, note that $g(x', x; \beta)$ also contains $\eta^*(\cdot)$ implicitly through the transition density $f_{X|x,y}(\cdot|\cdot)$.
3. Identification

Next, we develop an identification strategy that does not involve solving the Markov equilibrium in the dynamic decision problem. To clarify ideas, we first provide identification of structural parameter \(\theta \in \Theta \subseteq \mathbb{R}^{k_\theta}\) (where \(k_\theta = k_0 + k_1\)) in a fully parametric model, i.e., assuming \(F_\eta\) is known. Then, we establish semiparametric identification of our model by a two–step approach: we first identify \(F_\eta\) up to the finite dimensional parameter \(\theta\). In the second step, we represent the agent’s choice by a single–index representation. Therefore, the identification of \(\theta\) follows the literature.

A key feature in our semiparametric identification is that we require (at least) one argument in the state variables \(X_t\) to have continuous variation, which is also the case in the semiparametric identification of the single–index binary response model in the static setting. See e.g. Manski (1975). Moreover, we show that the quantile function of \(F_\eta\) is identified on the support of the agent’s choice probabilities under a location–scale normalization. This result also corresponds to the findings in static binary response models.

3.1. Intermediate step: Parametric Identification. As a building block towards the more general semiparametric results below, we first consider parametric identification of the model, assuming that the researcher know \(F_\eta\), the distribution function for the utility shocks. (This is only an intermediate step; in the sections to follow we show how identification obtains even when \(F_\eta\) is not known.) Parametric identification in this setting (with continuous state variables) has also been established previously in Srisuma and Linton (2012). Our identification of \(\theta\) is constructive, which has an single index structure and suggests an OLS-like (i.e. closed form) estimator. To begin with, we introduce the following assumption.

**Assumption C.** Let \(\eta\) be continuously distributed with the full support \(\mathbb{R}\).

Assumption C is a weak condition widely used in semiparametric binary response models (see e.g. Horowitz, 2009). Under assumption C, \(F_\eta\) is strictly increasing on its support \(\mathbb{R}\). Let \(Q\) be the quantile function of \(F_\eta\), i.e., \(Q = F_\eta^{-1}\).

Let \(p(x) = \mathbb{P}(Y = 1|X = x)\), which obtains directly from the data. Under assumption C, \(0 < p(x) < 1\) for all \(x \in \mathcal{X}\) and \(\eta^*(x) = Q(p(x))\). Moreover, using the substitution \(\tau \rightarrow Q(\tau)\),
we have
\[ E[\eta \cdot 1(\eta \leq Q(p))] = \int \tau \cdot 1(\tau \leq Q(p))dF_\eta(\tau) = \int_0^p Q(\tau)d\tau. \]

From the above discussion, it is straightforward that we obtain the following lemma.

**Lemma 2.** Suppose assumptions A to C hold. Then we have

\[ Q(p(X)) + \sum_{s=1}^{\infty} \beta^s \left\{ E\left[ \int_0^{p(X^{[s]})} Q(\tau)d\tau | X, Y = 1 \right] - E\left[ \int_0^{p(X^{[s]})} Q(\tau)d\tau | X, Y = 0 \right] \right\} = \phi^T(X) \cdot \theta. \]  

(9)

Eq. (9) is the key restriction for our identification and estimation analysis, where the number of restrictions equals to the size of the support \( \mathcal{S}_X \).

When \( Q_\eta \) is given, then everything in (9) is known except for \( \theta \). If, in addition, the matrix \( E[\phi(X)\phi^T(X)] \) is invertible, then \( \theta \) can be estimated using nonlinear least-squares on eq. (9). This approach is related to Pesendorfer and Schmidt-Dengler (2008). The full rank of \( E[\phi(X)\phi^T(X)] \) requires that if the transformed state variables \( W_0(X) \) and \( W_1(X) \) contain a common component \( W_c(X) \), then \( E[W_c(X')|X, Y = 0] \neq E[W_c(X')|X, Y = 1] \). Such a necessary condition rules out the case that variables without any dynamic transition (e.g. the constant) are included in both transformed state variables.\(^4\)

3.2. **Semiparametric Identification.** Without making any distributional assumptions on \( \eta \), we now discuss the identification of \( \theta \) as well as \( Q_\eta \). Intuitively, the number of restrictions imposed by (9) depends on the richness of the support \( \mathcal{S}_X \). For identification of \( Q_\eta \) (up to \( \theta \)), we only exploit variation in the choice probabilities \( p(X) \). For notational simplicity, we assume the choice probability \( p(X) \) is continuously distributed on a closed interval.\(^5\)

**Assumption D.** (i) Let \( p(X) \) be continuously distributed; (ii) let the support of \( p(X) \) be a closed interval, i.e., \([p, \overline{p}] \subseteq [0, 1]\).

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\(^4\)In this case, it is also required that the discount rate \( \beta \neq 0 \), otherwise \( \phi_0(X) = W_0(X) \) and \( \phi_1(X) = W_1(X) \), which clearly invalidates the rank condition due to the common term \( W_c(X) \).

\(^5\)This interval–support restriction can be relaxed at expositional expense. For instance, suppose \( \mathcal{S}_{p(X)} \) is a non-degenerate compact subset of \([0, 1]\). All of our identification arguments below still hold by replacing the integral region \([p, \overline{p}] \) with \( \mathcal{S}_{p(X)} \).
This assumption requires the state variables $X$ to contain some continuous components. Letting $X^D$ (resp. $X^C$) denote the discrete (resp. continuous) components of $X$, a more primitive statement of Assumption D would be that, for fixed values of the discrete components (say) $X^D = x^d$, the support of $p(X^C, x^d)$ is a closed interval in $[0, 1]$. As is well–known, the continuity of covariates is crucial for the semiparametric identification in the static binary response model; this is still the case in our dynamic binary decision model.

In contrast, when $p(X)$ only has discrete variation (which typically arises when the state variables $X$ themselves have only discrete variation), Norets and Tang (2014) show that the distribution of $\eta$ is partially identified.

For each $p \in [p, \bar{p}]$, let $z(p) = \mathbb{E}[\phi(X)|p(X) = p]$. We now take the conditional expectation given $p(X) = p$ on both sides of eq. (9). By the law of iterated expectation, we have

$$Q(p) + \sum_{s=1}^{\infty} \beta^s \left\{ \mathbb{E} \left[ \int_0^{p(X^i)} Q(\tau) d\tau | p(X) = p, Y = 1 \right] - \mathbb{E} \left[ \int_0^{p(X^i)} Q(\tau) d\tau | p(X) = p, Y = 0 \right] \right\} = z(p)^\top \cdot \theta.$$ 

The above discussion is summarized by the following lemma.

**Lemma 3.** Suppose assumptions A to D hold. Then we have

$$Q(p) + \beta \int_{\underline{p}}^{\bar{p}} \int_{\underline{p}}^{p'} Q(\tau) d\tau \cdot \pi(p', p; \beta) dp' = z(p)^\top \cdot \theta, \quad \forall \ p \in [p, \bar{p}],$$

where $\pi(p', p; \beta) = \sum_{s=1}^{\infty} \beta^{s-1} [f_{p(X^i)|p(X), Y}(p'|p, 1) - f_{p(X^i)|p(X), Y}(p'|p, 0)].$

By definition, $\pi(p', p; \beta)$ is the difference of the discounted aggregate densities of the future choice probabilities, conditional on the current choice probability and (exogenously given) action, which can be obtained directly from the data.
Note that eq. (10) is also an FIE–2. To see this, let $\Pi(p', p; \beta) \equiv \sum_{s=1}^{\infty} \beta^{s-1} |F_{p(X)}|_p X_y(p'|p, 1) - F_{p(X)|p(x), y(p'|p, 0)}$. Then, the second term of eq. (10) can be rewritten as

$$\int_{p}^{p'} Q(\tau) \cdot \pi(p', p; \beta) dp' = \int_{p}^{1} Q(\tau) \cdot \int_{p}^{p'} 1(\tau \leq p') \cdot \pi(p', p; \beta) dp' d\tau$$

$$= - \int_{0}^{1} Q(\tau) \cdot \left[ \int_{p}^{p'} 1(\tau < \tau) \pi(p', p; \beta) dp' \right] d\tau$$

$$= - \int_{p}^{p'} Q(\tau) \cdot \Pi(\tau, p; \beta) d\tau,$$

where the second step comes from the fact $\int_{p}^{p'} \pi(p', p; \beta) dp' = 0$ and the last step is because $\Pi(p', p, \beta) = 0$ for all $p' \notin [p, p]$. Hence, we obtain the following FIE–2:

$$Q(p) - \beta \int_{p}^{p'} Q(\tau) \cdot \Pi(\tau, p; \beta) d\tau = z(p) \cdot \theta, \quad \forall p \in [p, p]. \quad (11)$$

By solving this equation, we can identify $Q(\cdot)$ on $[p, p]$ up to the finite dimensional parameter $\theta$.

**Assumption E.** Let $\beta^2 \cdot \int_{p}^{p} \int_{p}^{p} \Pi^2(p', p; \beta) dp' dp < 1$.

Assumption E ensures that the mapping in Eq. (11) is a contraction, so that the solution is unique. Note that this assumption is not a model restriction, but an identification condition, involving both structural primitives as well as variations of observed state variables. Though high-level, it is testable in principle as it depends only on data.

**Lemma 4.** Suppose assumptions A to E hold. Then, $Q_\eta$ is point identified on $[p, p]$ up to the finite dimensional parameter $\theta$:

$$Q(p) = \left\{ z(p) - \beta \int_{p}^{p} R(p', p; \beta) \cdot z(p') dp' \right\} \cdot \theta, \quad \forall p \in [p, p] \quad (12)$$

where $R(p', p; \beta) = \sum_{s=1}^{\infty} (-\beta)^{s-1} K_s(p', p; \beta)$, in which $K_s(p', p; \beta) = \int_{0}^{1} K_{s-1}(p', \tilde{p}; \beta - \Pi(\tilde{p}, p; \beta) dp' and K_1(p', p; \beta) = \Pi(p', p; \beta)$.

The solution (12) is proportional to $\theta$, which is due to the linearity of the FIE system. Therefore, (12) can also be represented by a sequence of “basis” solutions. To see this, let $z_{\ell}(p)$ be the $\ell$–th
argument of $z(p)$. For $\ell = 1, \cdots, k_0$, let $b_\ell^*(\cdot)$ be the (unique) solution to the following equation

$$b_\ell(p) + \beta \int_p^{p'} \int_p^{p'} b_\ell(\tau) d\tau \cdot \pi(p', p; \beta) dp' = z_\ell(p).$$

(13)

By a similar argument to Lemma 4, we have

$$b_\ell^*(p) = z_\ell(p) - \beta \int_p^{p'} R(p', p; \beta) \cdot z_\ell(p') dp', \quad \forall \ p \in [\underline{p}, \bar{p}]$$

as the unique solution to (13). Let $B(\cdot) = (b_1^*(\cdot), \cdots, b_{k_0}^*(\cdot))^T$ be the sequence of solutions supported on $[\underline{p}, \bar{p}]$. Thus, the solution in eq. (12) can be written as

$$Q(p) = B(p)^\top \cdot \theta, \quad \forall \ p \in [\underline{p}, \bar{p}].$$

(14)

By Lemmas 1 to 4, we obtain a single–index representation of the semiparametric dynamic decision model, which is stated in the following theorem.

**Theorem 1.** Suppose assumptions A to E hold. Then, the agent’s dynamic decision can be represented by a static single–index model:

$$\mathbb{P}(Y = 1|X) = F_\eta(m(X)^\top \cdot \theta)$$

where

$$m(X) = \phi(X) - \sum_{s=1}^{\infty} \beta_s \left\{ \mathbb{E} \left[ \int_p^{p_{X^{[s]}}} B(\tau) d\tau | X, Y = 1 \right] - \mathbb{E} \left[ \int_p^{p_{X^{[s]}}} B(\tau) d\tau | X, Y = 0 \right] \right\},$$

or alternatively, $m(X) = B(p(X))$.

Note that $\mathbb{P}(Y = 1|X) = F_\eta(Q(p(X)))$. Then, Theorem 1 obtains by combining eqs. (9) and (14). Given the identification of $B(\cdot)$ on the support $[\underline{p}, \bar{p}]$, $m(\cdot)$ is then constructively identified on $\mathcal{S}_X$. Therefore, the identification of $\theta$ simply follows the single-index model literature, see e.g. Manski (1975, 1985).

It is worth noting that any constant term in $W_d$ remains a constant in the transformed linear–index $m(X)$. In other words, suppose, w.l.o.g., $W_{11} = 1$. Then the corresponding argument in $m(X)$ also equals 1. To see this, note that the first argument in $\phi(X)$ is given by

$$1 + \sum_{s=1}^{\infty} \beta_s \left\{ \mathbb{E}[p(X^{[s]})|X, Y = 1] - \mathbb{E}[p(X^{[s]})|X, Y = 0] \right\}.$$
which thereafter implies
\[
    z_1(p) = 1 + \sum_{s=1}^{\infty} \beta^s \left\{ \mathbb{E}[p(X^{[s]})|p(X) = p, Y = 1] - \mathbb{E}[p(X^{[s]})|p(X) = p, Y = 0] \right\} .
\]

Using (13), one can verify that the solution is: \( b_1^*(\cdot) = 1 \). Then, we plug this solution into the first element of \( m(X) \), which gives us \( m_1(X) = 1 \). By a similar argument, a constant in \( W_0 \) implies the corresponding term in \( m(X) \) equals \(-1\).

By a similar argument as in the static binary response model literature, the index parameter \( \theta \) is identified up to location and scale in the semiparametric setting. For notational simplicity, hereafter we assume the state vector \( X \) does not include a constant term in the semiparametric setting.\(^6\) Moreover, we will introduce a scale normalization on \( \theta \) which is also standard in the literature.

**Assumption F.** *We denote the first argument of \( m(X) \) by \( m_1(X) \) and the rest by \( m_{-1}(X) \). Moreover, let \( m_1(X) \) be continuously distributed on an interval given \( m_{-1}(X) \) which is a vector of either discrete and/or continuous random variables. Let \( f_{m_1(X)|m_{-1}(X)} \) be the conditional pdf. Moreover, the matrix \( \mathbb{E}[m(X)m(X)^\top] \) is invertible.*

In Assumption F, the first half condition requires at least one argument in \( X_1 \) to be continuously distributed conditional on others; this rules out cases where, e.g. all the state variables are functions of a single variable \( X_1 \) (as in (Rust, 1987), where mileage and mileage-squared enter as state variables). The second half of Assumption F is a testable rank condition. Assumption F is a strong assumption, but almost indispensable in the semiparametric single index model literature; See Horowitz (2009).

**Assumption G.** *Let \( \|\theta\| = 1 \).*

Assumption G is a scale normalization, which has also been used in PSS. Note that we implicitly normalize our location term by 0, since neither \( W_0 \) nor \( W_1 \) contains a constant term.

**Theorem 2.** *Suppose assumptions A to G hold. Then, the structural parameter \( \theta \) is point identified.*

\(^6\)In our semiparametric setting, any constant term in the utility function will be absorbed by the error term since the distribution of the latter is left unspecified.
4. SEMIPARAMETRIC ESTIMATION

In this section, we describe and motivate the semiparametric estimation of our structural model. For expository simplicity, we assume all variables in X are continuously distributed. A mixture of continuous and discrete regressors can be accommodated at the expense of notation. Let \( \{(Y_t, X_t^i) : t = 1, \ldots, T\} \) be our sample of the Markov decision process. Our estimation procedure parallels the identification strategy, which takes multiple steps. Throughout, we use \( K \) and \( h \) to denote a Parzen–Rosenblatt kernel and a bandwidth, respectively.

First, we nonparametrically estimate the choice probabilities \( p(\cdot) \) and the generated regressor \( \hat{\phi}(\cdot) \). In particular, let

\[
\hat{\phi}(X_s) = \frac{\sum_{t=1}^{T} Y_t \times K_p \left( \frac{X_t - X_s}{h_p} \right)}{\sum_{t=1}^{T} K_p \left( \frac{X_t - X_s}{h_p} \right)} , \quad \forall s = 1, \ldots, T.
\]

As is standard, we choose an optimal bandwidth, i.e., \( h_p = 1.06 \times \hat{\sigma}(X) \times T^{-\frac{1}{5+\pi}} \), where \( \hat{\sigma}(X) \) is the sample standard deviation of \( X_t \) and \( t \geq 2 \) is the order of the kernel function \( K_p \). For example, if we choose \( K_p \) to be the pdf of the standard normal distribution, then \( t = 2 \). In addition, the support \([p, \bar{p}]\) of \( p(X) \) can be estimated by \([\min_{1 \leq s \leq T} \hat{p}(X_s), \max_{1 \leq s \leq T} \hat{p}(X_s)]\).

Moreover, recall that the transformed state variables \( W_d(X) \) \((d = 0, 1)\) are known. Then, for \( s = 1, \ldots, S_T \), where \( S_T = T - \ell_T \) for some integer \( \ell_T \) satisfying \( \ell_T \to +\infty \) and \( S_T \to +\infty \) as \( T \to +\infty \), let \( \delta_{dt} = \sum_{s=1}^{\ell_T} \beta^s \cdot W_d(X_{t+s})Y_{t+s}^d(1 - Y_{t+s})^{1-d} \). For \( s = 1, \ldots, T \), let further

\[
\hat{\phi}_d(X_s) = (-1)^{d+1} W_d(X_s) + \frac{\sum_{t=1}^{S_T} \delta_{dt} \cdot K_p \left( \frac{X_t - X_s}{h_p} \right) \mathbb{1}(Y_t = 1) \sum_{t=1}^{S_T} \delta_{dt} \cdot K_p \left( \frac{X_t - X_s}{h_p} \right) \mathbb{1}(Y_t = 0)}{\sum_{t=1}^{S_T} K_p \left( \frac{X_t - X_s}{h_p} \right) \mathbb{1}(Y_t = 1) - \sum_{t=1}^{S_T} K_p \left( \frac{X_t - X_s}{h_p} \right) \mathbb{1}(Y_t = 0)}.
\]

Similarly, we can choose \( h_{\phi} \) in an optimal way. In above expression, the summation includes only the first \( S_T \) observations. This is because \( \delta_{dt} \) is not well defined for all \( t > S_T \). In practice, we choose \( \ell_T \) in a way such that \( \delta_{dt} - \sum_{s=1}^{+\infty} \beta^s W_d(X_{t+s})Y_{t+s}^d(1 - Y_{t+s})^{1-d} \) is negligible relative to the sampling error, which is feasible because the former converges to zero at an exponential rate.
In the second stage, we estimate \( z(\cdot) \) and \( \mathcal{B}(\cdot) \) on the support \([\underline{p}, \overline{p}]\). First, let

\[
\hat{z}(p) = \frac{\sum_{t=1}^{T} \hat{\phi}(X_t) \cdot K_z \left( \frac{\hat{p}(X_t) - p}{h_z} \right)}{\sum_{t=1}^{T} K_z \left( \frac{\hat{p}(X_t) - p}{h_z} \right)}, \quad \forall \ p \in [\min_{1 \leq s \leq T} \hat{p}(X_s), \max_{1 \leq s \leq T} \hat{p}(X_s)].
\]

According to Guerre, Perrigne, and Vuong (2000, Theorem 2), we choose an oversmoothing bandwidth \( h_z \), since \( p(X) \) is nonparametrically estimated. Specifically, \( h_z = 1.06 \times \hat{\sigma}(p(X)) \times T^{-\frac{1}{2n+3}} \).

To estimate \( \hat{b}_t^*(\cdot) \) on the support \([\underline{p}, \overline{p}]\), we note that eq. (13) can be rewritten as

\[
b_t(p) + \sum_{s=1}^{\infty} \beta^s \cdot \mathbb{E} \left[ \int_{p}^{p(X[i])} b_t(\tau) d\tau | p(X) = p, Y = 1 \right]

- \sum_{s=1}^{\infty} \beta^s \cdot \mathbb{E} \left[ \int_{p}^{p(X[i])} b_t(\tau) d\tau | p(X) = p, Y = 0 \right] = z_t(p).
\]

This suggests an estimator \( \hat{b}_t^{*}(\cdot) \) that solves

\[
\hat{b}_t^{*}(p) + \frac{\sum_{t=1}^{S_T} \xi_t(\hat{b}_t) \times K_\xi \left( \frac{\hat{p}(X_t) - p}{h_x} \right) \times Y_t}{\sum_{t=1}^{S_T} K_\xi \left( \frac{\hat{p}(X_t) - p}{h_x} \right) \times Y_t} - \frac{\sum_{t=1}^{S_T} \xi_t(\hat{b}_t) \times K_\xi \left( \frac{\hat{p}(X_t) - p}{h_x} \right) \times (1 - Y_t)}{\sum_{t=1}^{S_T} K_\xi \left( \frac{\hat{p}(X_t) - p}{h_x} \right) \times (1 - Y_t)} = \hat{z}_t(p),
\]

where \( \xi_t(b_t) = \sum_{s=1}^{\infty} \beta^s \int_{p}^{p(X[i])} b_t(\tau) d\tau \) for which the integration can be computed by numerical integration. Similarly, \( h_z = 1.06 \times \hat{\sigma}(p(X)) \times T^{-\frac{1}{2n+3}} \) is chosen sub-optimally. A numerical solution of \( \hat{b}_t^{*} \) can obtain using the iteration method: Let \( \hat{b}_t^{[0]} = \hat{z}_t(p) \). Then we set

\[
\hat{b}_t^{[1]}(p) = \hat{z}_t(p) - \left\{ \frac{\sum_{t=1}^{S_T} \xi_t(\hat{b}_t^{[0]}) \times K_\xi \left( \frac{\hat{p}(X_t) - p}{h_x} \right) \times Y_t}{\sum_{t=1}^{S_T} K_\xi \left( \frac{\hat{p}(X_t) - p}{h_x} \right) \times Y_t} - \frac{\sum_{t=1}^{S_T} \xi_t(\hat{b}_t^{[0]}) \times K_\xi \left( \frac{\hat{p}(X_t) - p}{h_x} \right) \times (1 - Y_t)}{\sum_{t=1}^{S_T} K_\xi \left( \frac{\hat{p}(X_t) - p}{h_x} \right) \times (1 - Y_t)} \right\}.
\]

Repeat such an iteration until it converges. Then we obtain \( \hat{b}_t^{*} = \hat{b}_t^{[\infty]}(\cdot) \) on \([\underline{p}, \overline{p}]\).

Next, we obtain the single–index variables \( m(X_s) \) by: for \( \ell = 1, \cdots, k_\theta \)

\[
\hat{m}_\ell(X_s) = \hat{\phi}_\ell(X_s) - \left\{ \frac{\sum_{t=1}^{S_T} \xi_t(\hat{b}_t^*) \times K_m \left( \frac{X_t - Y_t}{h_m} \right) \times Y_t}{\sum_{t=1}^{S_T} K_m \left( \frac{X_t - Y_t}{h_m} \right) \times Y_t} - \frac{\sum_{t=1}^{S_T} \xi_t(\hat{b}_t^*) \times K_m \left( \frac{X_t - Y_t}{h_m} \right) \times (1 - Y_t)}{\sum_{t=1}^{S_T} K_m \left( \frac{X_t - Y_t}{h_m} \right) \times (1 - Y_t)} \right\}.
\]

In particular, \( h_m = 1.06 \times \hat{\sigma}(X) \times T^{-\frac{1}{2n+3}} \) is chosen optimally.
Finally, we apply PSS to estimate \( \theta \) (up to scale). Specifically, we define

\[
\hat{\theta} = -\frac{2}{T(T-1)} \sum_{t=1}^{T} \sum_{s \neq t} \left[ \frac{1}{h_{\theta}^{k+1}} \times \nabla \theta \left( \frac{\hat{m}(X_t) - \hat{m}(X_s)}{h_{\theta}} \right) \right] \cdot Y_s. \tag{15}
\]

Following the standard kernel regression literature, we can show \( \hat{\theta} \) is consistent given that \( \sup_{x \in X} |\hat{m}(x) - m(x)| = o_p(h_{\theta}), h_{\theta} \to 0 \) and \( Th_{\theta}^{k+1} \to \infty \) as \( T \to \infty \).

Similar to PSS, it is of particular interest to establish \( \sqrt{T} \)-consistency of \( \hat{\theta} \). The argument follows closely to that in PSS. In particular, we need to choose a high order kernel \( K_{\theta} \) and an under–smoothed bandwidth \( h_{\theta} \). However, it is more delicate in our setting because of the generated regressor \( \hat{m}(X) \) contained in the kernel function of our estimator (15). Due to the first–stage estimation error, we must make the following additional assumptions on the convergence of \( \hat{m}(X) \) to \( m(X) \):

**Assumption H.** \( h_{\theta} = T^{-\frac{1}{7}} \) where \( k_{\theta} + 2 < \gamma < k_{\theta} + 3 + 1(k_{\theta} \text{ is even}) \).

**Assumption I.** The support of the kernel function \( K_{\theta} \) is a convex subset of \( \mathbb{R}^{k_{\theta}} \) with nonempty interior, with the origin as an interior point. \( K_{\theta} \) is a bounded differentiable function that obeys: \( \int K_{\theta}(u)du = 1 \), \( K_{\theta}(u) = 0 \) for all \( u \) belongs to the boundary of its support, \( K_{\theta}(u) = K_{\theta}(-u) \) and

\[
\int u_{\ell_1} \cdots u_{\ell_{\ell'}} K_{\theta}(u) du = 0, \quad \text{for } \ell_1 + \cdots + \ell_{\ell'} < \frac{k_{\theta} + 3 + 1(k_{\theta} \text{ is even})}{2}, \text{ and}
\]

\[
\int u_{\ell_1} \cdots u_{\ell_{\ell'}} K_{\theta}(u) du \neq 0, \quad \text{for } \ell_1 + \cdots + \ell_{\ell'} = \frac{k_{\theta} + 3 + 1(k_{\theta} \text{ is even})}{2}.
\]

where \( u_{\ell} \) is the \( \ell \)-th argument of \( u \).

**Assumption J.** (i) \( \mathbb{E}||\hat{m}(X) - m(X)||^2 = o(T^{-\frac{1}{2}} h_{\theta}^3) \);

(ii) \( \mathbb{E}||\hat{m}(X)|X - m(X)|| = o(T^{-\frac{1}{2}} h_{\theta}^3) \);

(iii) \( \hat{m}(X_t) - \hat{m}_{t,-s} = o_p(T^{-\frac{1}{2}} h_{\theta}^3) \), where \( \hat{m}_{t,-s} \) is the nonparametric estimator \( \hat{m}(X_t) \), except for leaving the \( s \)-th observation out of the sample in its construction.

Assumptions H and I are introduced by PSS for the choice of bandwidth and kernel, respectively, to control the bias term in the estimation of \( \theta \).\(^8\) The restriction on the bandwidth Assumption H

\(^7\)One could also use alternative methods e.g. Klein and Spady (1993) and Ichimura (1993) to estimate \( \theta \).

\(^8\)Note that we implicitly assume that Assumptions 1 – 3 in PSS hold, which impose smoothness conditions on \( f_m(X) \) and \( \mathbb{P}(Y_t = 1|m(X_t) = m) \) as well as other regularity conditions.
implies that \( h_\theta \) is not an optimal bandwidth sequence (rather it is undersmoothed) such that the bias of estimating \( \theta \) goes to zero faster than \( \sqrt{T} \).

Moreover, Assumption J encompasses high–level conditions that could be further established under primitive conditions. In particular, Assumption J(i) requires \( \hat{m}(\cdot) \) to converge to \( m(\cdot) \) faster than \( T^{-\frac{1}{4}} \). By Assumption J(ii), the bias term in the estimation of \( m \) uniformly converges to zero faster than \( T^{-\frac{1}{2}} \). Hence, we need to use a higher order kernel in the estimation of \( m(\cdot) \). Assumption J(iii) is not essential, which could be dropped if we exclude both \( t \)-th and \( s \)-th observations in the argument \( \hat{m}(X_t) - \hat{m}(X_s) \) of the kernel function in (15). Assumption J is standard in the literature for the regular convergence of finite–dimensional parameters in semiparametric models (e.g. Ai and Chen, 2003), except for the polynomial terms of \( h_\theta \) in the \( o(\cdot) \) or \( o_p(\cdot) \) which arises due to the average derivate estimator in the second stage.

Given these assumptions, we can show the following result (the proof is in the appendix):

**Theorem 3.** Suppose assumptions H to J hold. Then, for some scalar \( \lambda > 0 \) specified below, \( \sqrt{T}(\hat{\theta} - \lambda \cdot \theta) \)
has a limiting multivariate normal distribution defined in Powell, Stock, and Stoker (1989, Theorem 3.1):

\[
\sqrt{T}(\hat{\theta} - \lambda \cdot \theta) \xrightarrow{d} N(0, \Sigma)
\]

where \( \Sigma \equiv 4 \mathbb{E}(\zeta \cdot \zeta^\top) - 4\lambda^2 \times \theta \cdot \theta^\top, \zeta = f_m(m(X)) \cdot f_\eta(\eta^*(X)) \cdot \theta - \left[ Y - F_\eta(\eta^*(X)) \right] \cdot f_m(m(X)) \) and \( \lambda = \mathbb{E}[f_m(m(X)) \cdot f_\eta(m(X)^\top \cdot \theta)] \).

In the above theorem, recall \( \mathbb{P}(Y = 1|X) = F_\eta(\eta^*(X)) \) and \( \eta^*(X) = m(X)^\top \cdot \theta \) by Theorem 1. Our estimator \( \hat{\theta} \) (as defined in Eq. (15)) has not imposed the scale restriction in Assumption G; thus \( \lambda \in \mathbb{R} \) in the above theorem denotes the probability limit of \( ||\hat{\theta}|| \); i.e., \( ||\hat{\theta}|| = \lambda + O_p(T^{-1/2}) \).

Therefore, by rescaling our estimator \( \hat{\theta} \) as \( \hat{\theta}^* = \hat{\theta}/\lambda \), we obtain that

\[
\sqrt{T}(\hat{\theta}^* - \theta) \xrightarrow{d} N(0, \Sigma/\lambda^2).
\]

Given \( \hat{\theta}^* \), a nonparametric estimator of \( Q(\cdot) \) directly follows from Eq. (12). Namely, let

\[
\hat{Q}(p) = \hat{z}^\top(p) \times \hat{\theta}^*, \quad \forall \ p \in \left[ \min_{1 \leq s \leq T} \hat{p}(X_s), \ max_{1 \leq s \leq T} \hat{p}(X_s) \right].
\]
Because of the $\sqrt{T}$-consistency of $\hat{\theta}^*$, the estimator $\hat{Q}_\mu(p)$ is asymptotically equivalent to $\hat{z}^T(p) \times \theta$, which converges at a nonparametric rate. Given the asymptotic normality established in this section, bootstrap inference is valid and we will use it for constructing standard errors in our empirical application below.

4.1. Monte Carlo. The focus of our Monte Carlo is on the semiparametric estimation. In our experiments, let $u_t(0, X_t, \epsilon_t) = \theta_0 + \epsilon_0t$ and $u_t(1, X_t, \epsilon_t) = X_{1t}\theta_1 + X_{2t}\theta_2 + \epsilon_{1t}$, where $X_{1t}, X_{2t}$ are random variables and $\theta_0, \theta_1, \theta_2 \in \mathbb{R}$. Moreover, we set the conditional distribution of $X_{t+1}$ given $X_t$ and $Y_t$ as follows: for $k = 1, 2$

$$X_{k,t+1} = \begin{cases} X_{kt} + \nu_{kt}, & \text{if } Y_t = 0 \\ \nu_{kt} & \text{if } Y_t = 1 \end{cases},$$

where $\nu_{kt}$ conforms to $\ln \mathcal{N}(0, 1)$ and $\nu_{1t} \perp \nu_{2t}$. Moreover, let $\epsilon_{dt}$ be i.i.d. across $d = 0, 1$ and $t$, and conform to an extreme value distribution with the density function $f(e) = \exp(-e) \exp[-\exp(-e)]$.

We set $\beta = 0.9$ and the parameter value as follows: $\theta_0 = -5, \theta_1 = -1$ and $\theta_2 = -2$.

Because we cannot estimate the constant $\theta_0$ in the semiparametric framework, then we treat $\theta_0$ as a nuisance parameter. Let $\theta = (\theta_1, \theta_2)^T$. As a matter of fact, $\theta$ is only identified up to scale in the semiparametric setting. To compare the performance of the semiparametric estimators, we assume the scale of $\theta$ is known, i.e., $||\theta|| = \sqrt{5}$, rather than imposing a different normalization, as assumption G. We present in Table 1 the bias and standard deviation of the semiparametric estimator.

5. Empirical Application: Dynamic Labor Supply for NYC Taxi Drivers

A growing literature has sprung up which seeks to estimate labor supply elasticities in markets where labor supply is continuously adjustable; several of these papers have studied the market for taxi rides, because taxi drivers choose their own hours. We next consider an application which aims to characterize the dynamic labor supply of taxi drivers. We first pose and estimate a model of taxi driver’s labor supply as a dynamic discrete choice over quitting for the day. Our model highlights the tradeoffs between working longer to earn extra income versus...
TABLE 1. Monte Carlo Results

<table>
<thead>
<tr>
<th>Sample Obs</th>
<th>Parameter</th>
<th>True Value</th>
<th>Estimate</th>
<th>Std. Dev.</th>
<th>Bias</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>$\theta_1$</td>
<td>-1</td>
<td>-1.0182</td>
<td>0.3636</td>
<td>0.0182</td>
</tr>
<tr>
<td></td>
<td>$\theta_2$</td>
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<td>-1.9457</td>
<td>0.2158</td>
<td>-0.0543</td>
</tr>
<tr>
<td>2000</td>
<td>$\theta_1$</td>
<td>-1</td>
<td>-1.0163</td>
<td>0.2913</td>
<td>0.0163</td>
</tr>
<tr>
<td></td>
<td>$\theta_2$</td>
<td>-2</td>
<td>-1.9618</td>
<td>0.1854</td>
<td>-0.0382</td>
</tr>
<tr>
<td>4000</td>
<td>$\theta_1$</td>
<td>-1</td>
<td>-0.9985</td>
<td>0.2344</td>
<td>-0.0015</td>
</tr>
<tr>
<td></td>
<td>$\theta_2$</td>
<td>-2</td>
<td>-1.9836</td>
<td>0.1176</td>
<td>-0.0164</td>
</tr>
</tbody>
</table>

This table presents Monte Carlo results for different sample sizes. For each sample size, reported estimates, standard deviations and bias are computed as the mean across 150 simulation draws. Estimation takes on average 6, 12, and 25 seconds respectively for each replication on a 4Ghz i7 computer.

inquiring increasing costs of effort. Estimating the structural model allows us to both analyze the labor-leisure tradeoff in a richer way compared to previous studies, and also to showcase some features of our semiparametric estimation procedure.

This work builds on a literature on labor supply in the taxi industry. Camerer, Babcock, Loewenstein, and Thaler (1997) found evidence of strong negative wage elasticities; they argued that negative elasticities reflected the presence of income-targeting on the part of drivers: for example, a labor supply policy of the form “I will work today until I earn $200.” Farber (2005, 2008, 2014) consider static models of labor supply. The latter two papers integrate reference-dependent utility, which is the notion that agents’ utility is not only a function of income but also reference-points or targets, where the marginal utility of income increases more quickly before the target is met than after it is met. Originally, Farber (2008) finds mixed evidence for the existence of reference-dependence, but Farber (2014) uses more comprehensive data and finds strong evidence that labor supply behavior is driven by the standard neoclassical prediction of upward sloping supply curves, as opposed to income-targeting and its associated negative elasticities. Crawford and Meng (2011) specify and estimate a dynamic model of labor supply incorporating reference-dependence in both income and hours-worked during a shift.

We estimate a dynamic structural model in which drivers solve a dynamic optimization problem to determine their hours worked, as a function of cumulative earned income and
cumulative time spent working. Our model is based on the taxi labor supply model of Frechette, Lizzeri, and Salz (2015) [FLS], in which taxi drivers solve a dynamic competitive game by choosing the optimal starting times and length of time to stay on a shift.\(^\text{10}\) As with FLS, our taxi drivers will decide how long to work by weighing the utility of earning revenue against the disutility of working longer. FLS utilizes the MPEC method to solve a dynamic entry game in an equilibrium framework, allowing the market to equilibrate via the waiting times experienced by passengers and taxis. While we do not consider these general equilibrium forces, we take advantage of our computationally light, semi-parametric estimation method to test for a variety of taxi driver stopping behaviors posited by previous authors. Thus our approach is partial equilibrium, as agents in our model take the waiting times and arrival of customers as given rather than endogenously determined in a dynamic equilibrium.

Both models can also be viewed as a stopping rule framework akin to the classic paper by Rust (1987). Rust models the decision to replace bus engines, which weighs routine maintenance costs against full engine replacement, the latter option preventing higher probabilities of catastrophic engine failure. In this setting, after each trip given, a taxi driver must weigh the opportunity for additional fares against a rising cost of fatigue in each day.\(^\text{11}\) Similarly to Rust, we use our estimation method to recover parameters defining driver’s cost functions.

5.1. Behavioral model.

5.1.1. Period revenues and costs. Taxi drivers are assumed to have costs of effort that are increasing in hours-worked each day. Each period is a ride. After each ride given, drivers face a discrete decision to continue searching for passengers or quit for the day. In this sense, their labor supply decision boils down to a comparison between the expected profit of searching for an additional unit of time versus the disutility of driving for that much more time. The period payoff function for driver \(i\) depends on the decision to quit \((y_{it} = 1)\) or keep working \((y_{it} = 0)\), and takes the following form:

\(^{10}\)The more taxis that are working, the less revenue is earned as a result of lower probabilities of finding a passenger.

\(^{11}\)In other words, drivers experience increasingly large marginal utility of leisure as the remaining hours of leisure fall.
Table 2. Taxi Trip and Fare Summary Statistics

<table>
<thead>
<tr>
<th>Trips/Shifts</th>
<th>Variable</th>
<th>Obs.</th>
<th>10%ile</th>
<th>Mean</th>
<th>90%ile</th>
<th>S.D.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trip Statistics</td>
<td>Trip Revenue ($)</td>
<td>10,000</td>
<td>5.88</td>
<td>12.47</td>
<td>21.80</td>
<td>8.86</td>
</tr>
<tr>
<td></td>
<td>Trip Time (min.)</td>
<td>10,000</td>
<td>4.00</td>
<td>11.72</td>
<td>22.0</td>
<td>8.17</td>
</tr>
<tr>
<td>Shift Statistics</td>
<td>Shift Revenue ($)</td>
<td>381</td>
<td>210.49</td>
<td>327.34</td>
<td>446.62</td>
<td>105.49</td>
</tr>
<tr>
<td></td>
<td>Shift Time (min.)</td>
<td>381</td>
<td>368.00</td>
<td>559.39</td>
<td>730.17</td>
<td>204.90</td>
</tr>
</tbody>
</table>

Taxi trip and fare data come from New York Taxi and Limousine Commission (TLC) and refer to February 2012 data. The first set of statistics relates to individual taxi trips. The second set of statistics relate to cumulative earnings and time spent in individual driver shifts.

This dynamic labor supply model is an optimal stopping model, in which the taxi driver’s dynamic problem ends once he decides to end his current shift. The terminal utility from ending the shift is given in the upper prong of the utility specification above. In this terminal utility, the term \( \theta_u \cdot s_{it} \) captures the utility from earnings enjoyed by the taxi driver after ending his shift, and \( \theta_c,01 \cdot h_{it} + \theta_c,02 \cdot h_{it}^2 \) likewise captures the post-shift utility depending on the cumulative hours worked. When a driver continues to drive \( (y_{it} = 0) \), our specification assumes that he receives (dis-)utility from doing so, which depends on the cumulative hours worked so far in this shift. This lower prong of the utility function measures the cost of the effort exerted by the working driver, which may change as the cumulative hours \( h_{it} \) increases.

5.2. Data. In 2009, The Taxi and Limousine Commission of New York City (TLC) initiated the Taxi Passenger Enhancement Project, which mandated the use of upgraded metering and information technology in all New York medallion cabs. The technology includes the automated data collection of taxi trip and fare information. We use TLC trip data on all New York City medallion cab rides given in February, 2012. The sample analyzed here consists of 10,000 observations, or about 0.1% of the data. Each row in the data is information related to a single cab ride. Data include driver and medallion identifiers, the exact time and date of pickup and drop-offs, trip distance, and trip time for 10,000 individual taxi rides. Table 2 provides summary statistics.
5.3. Results. The estimation results are presented in Table 3. For estimation, we scaled the cumulative time variable to be in units of five-minutes. We find that the terminal utility upon ending a shift grows with earnings, which is weighed against a negative effect of cumulative hours worked, the latter accumulating in each period of continued work. It is important to note that the relatively small coefficient on hours-worked is to be expected, since this utility accrues in every period that a driver continues working, while the utility benefit of earned income is only received once, when the driver stops working for the day.

Given these parameter estimates, in Figure 1 we graph the implied quantile function for the difference in utility shocks \( \eta \equiv \epsilon_1 - \epsilon_0 \).\(^{12}\) The density of \( \hat{p} \) is plotted as well, which highlights a range over which choice probabilities are actually observed. Outside of this range, we are unable to identify the corresponding quantile function, and in the figure the blue dotted lines represent possible values of the quantile function outside the identified range. Using the density of \( \hat{p} \) as a guide, we can recover the quantile function for the range of percentiles approximated by \([0.05, 0.25]\). A thin vertical dotted line depicts this range. The shocks take (even very large) positive values, with magnitudes in the hundreds; this may imply that there is a large fixed positive component to the terminal utility from quitting.\(^{13}\)

This feature that, as shown in Figure 1, our approach only yields an incomplete estimate of the error distribution, may be problematic for evaluating some counterfactual policies. For certain counterfactuals, knowledge of the entire distribution of the utility shocks is required, as this distribution feeds agents’ beliefs about the future. In ongoing work, we are exploring ways for extrapolating this distribution beyond the range identified by our approach.

While the empirical specification in Table 3 is simple, the behavioral implications of the dynamic model, which we illustrate in Figure 2, are quite rich. Theorem 1 shows that \( m(X)'\theta \) corresponds to the difference in the choice-specific value functions for quitting and continuing.

\(^{12}\)This contrasts with much of the existing semiparametric estimation literature for discrete-choice models, in which the error distribution is treated as a completely nuisance component, and it is not straightforward to recover estimates of it even given estimates of the model parameters. Since we derive analytical expressions for the error distribution as part of our identification argument, we are able to estimate it once we have estimated the model parameters.

\(^{13}\)Note that in estimating the quantile function, we have not fixed the scale and location for the utility shock difference \( \eta \); we have this flexibility because we imposed a scale normalization on the parameter vector \( \beta \). In contrast, parametric estimation approaches for DDC models typically do not impose normalization on the parameters, but implicitly the researcher must set the scale and location for the utility shocks (a common assumption is zero mean and unit variance).
TABLE 3. Parameter Estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Estimate</th>
<th>Std. Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_u )</td>
<td>Earnings (upon quitting)</td>
<td>0.9907</td>
<td>0.0118</td>
</tr>
<tr>
<td>( \theta_c,01 )</td>
<td>Cumul. hours (while working)</td>
<td>-0.1359</td>
<td>0.0759</td>
</tr>
<tr>
<td>( \theta_c,02 )</td>
<td>Cumul. hours squared (while working)</td>
<td>-0.0004</td>
<td>0.0002</td>
</tr>
</tbody>
</table>

Note: Standard errors are computed by first sampling, with replacement, from each driver-shift (on average there are roughly 24 observations per driver-shift) to generate 200 resamples of approximately identical size to our original sample. We re-estimate the model for each resample and report the standard deviation of estimates.

FIGURE 1. Estimated Quantile Function

\[ m(X)'\theta = V_1(X) - V_0(X), \] at each value of the state variables \( X = (\text{hours worked}, \text{income}) \). Hence, in Figure 2, we plot the estimated \( m(X)'\theta \) function at different levels of hours-worked. Clearly, the \( m(X)'\theta \) curves are increasing in both income and hours-worked (except at hours-worked equal to ten). That is, for most values of the state variables \( X \), the continuation benefits from quitting, \( V_1(X) \), are increasing faster than the continuation benefits from continuing to drive, \( V_0(X) \), as income rises and hours-worked increases. This implies that, holding hours-worked fixed, drivers are more likely to quit as their income increases; similarly, holding income fixed, drivers are more likely to quit as their hours-worked increases.
We report the estimates of the \( m(X)'\theta \) function, as in Theorem 1, for \( X = (\text{hours worked}, \text{income}) \). For fixed values of hours-worked, we graph \( m(X)'\theta \) as a function of income. Stars (*) mark average income earned by drivers for a given hours-worked, as observed in the raw data.

In Figure 2, we also plot, using stars, the actual average income in the raw data earned by drivers who quit at different values of hours-worked.\(^{14}\) As we described above, much of the existing empirical literature on taxicab driver behavior has focused on testing whether drivers’ wage elasticities are positive or negative, where positive elasticities are viewed as a corroboration of the classic model of labor-leisure choice, and negative elasticities are taken as evidence of a behavioral “income targeting” model. In both types of models, the wage rate is taken to be exogenous by the drivers and unchanging throughout the course of the day. Drivers then decide how many hours to work for each given wage rate.

\(^{14}\)This illustrates the ranges of income for which the \( m(X)'\theta \) functions would be estimated most precisely. Currently, we do not include standard error bands in Figure 2 as that would complicate the picture substantially.
In our model, however, income evolves stochastically; drivers’ “wage rates” are random and vary across the shift. Accordingly, a driver reconsiders the decision to end the shift after each fare. Hence, implied wage elasticities are not straightforward to compute in our modelling framework. That caveat aside, the results in Figure 2 do have implications for wage elasticities if we associate higher cumulative income, conditional on hours worked, with a higher wage rate (as is done in the existing literature). Under this interpretation, the graphs suggest that by holding hours fixed, the quitting probability is increasing in wage rate, which is consistent with negative wage elasticities. To illustrate this, consider a thought experiment where we have two drivers who have only driven four hours; the first driver already has income of $300, while the second only has income of $180. The bottom line in the graph is higher at $300 than at $180, graphs suggesting that the benefits from quitting are relatively larger for the first rather than the second driver, and hence that the first should quit more readily than the second.

A similar pattern manifests in the reduced-form evidence from the raw data (as illustrated by the stars in the graph). We see that, for those drivers who quit at four hours (the bottom line), the average hourly wage was around $40 (≈ $160/4); at seven hours, it was around $36 (≈ $255/7), and at ten hours it was around $34 (≈ $345/10). Thus, we conclude that hours-worked is, in a sense, decreasing in wage rate – the same conclusion we draw from the structural estimates. The caveat here is that our model shows that interpreting this evidence as a negative labor supply elasticity may be inappropriate: drivers here do not experience persistent changes to their wage rate, but rather a series of random revenue shocks. From the prior literature’s perspective on intra-day labor supply elasticities, however, our results show that once dynamics are modeled it is possible to obtain “nonstandard” (i.e., negative) wage elasticities from a model in which drivers’ utility functions do not have any explicitly “nonstandard” features, such as reference dependence or loss aversion.

6. Conclusions

In this paper we consider the estimation of dynamic binary discrete choice models in a semiparametric setting, in which the per-period utility functions are parameterized as single-index functions, but the distribution of the utility shocks is left unspecified and treated as nuisance components of the model. This setup differs from most of the existing work on
estimation and identification of dynamic discrete choice models. For identification, we derive a new recursive representation for the unknown quantile function of the utility shocks; our argument requires no additional exclusion restrictions beyond the conditional independence conditions assumed in the typical parametric dynamic-discrete choice literature (e.g. Rust (1987, 1994)). Accordingly, we obtain a single-index representation for the conditional choice probabilities in the model, which permits us to estimate the model using classic estimators from the existing semiparametric binary choice literature.

In particular, we use Powell, Stock and Stoker’s (1989) kernel-based estimator to estimate the dynamic discrete choice model. We show that the estimator has the same asymptotic properties as PSS’s original estimator (for static discrete-choice models), under mild conditions. Significantly, the computational procedure is quite simple, because the estimator for the parameters can be expressed in closed-form. Monte Carlo simulations show that the estimator works well even in moderately-sized samples. We provide an empirical application to estimate the dynamic labor supply problem for taxicab drivers in New York City.

More broadly, the analysis in this paper has opened possibilities for the use of classic closed-form estimators from the semiparametric literature, which were proposed for estimation of static models, to dynamic models. We will continue exploring these possibilities in future work.

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APPENDIX A. PROOFS

A.1. Proof of Lemma 1.

Proof. First, note that the resolvent kernel \( R^* \) is well-defined. This is because \( \beta^{s-1} f_{X|X}(x'|x) \rightarrow 0 \) as \( s \rightarrow +\infty \). Under assumption B, the solution \( V^\varepsilon(x) \) is also well defined.

Because it is straightforward to verify that the solution in the lemma solves eq. (4), Hence, it suffices to show the uniqueness of the solution. Eq. (4) can be rewritten as

\[
V^\varepsilon(x) = u^\varepsilon(x) + \beta \int_{\mathcal{X}} V^\varepsilon(x') \cdot f_{X|X}(x'|x) dx', \quad \forall \ x \in \mathcal{X},
\]

which is an FIE-2. Then, we apply the method of Successive Approximation (see e.g. Zemyan, 2012). Specifically, let \( V^*(\cdot) \) be an alternative solution to (4). Then, we have

\[
V^*(x) = u^\varepsilon(x) + \beta \int_{\mathcal{X}} V^*(x') \cdot f_{X|X}(x'|x) dx'.
\]

Let \( \nu(x) = V^\varepsilon(x) - V^*(x) \). Then \( \nu(x) \) satisfies the following equation:

\[
\nu(x) = \beta \int_{\mathcal{X}} \nu(x') \cdot f_{X|X}(x'|x) dx'.
\]

It suffices to show that \( \nu(\cdot) \) has the unique solution: \( \nu(x) = 0 \). To see this, we substitute the left-hand side as an expression of \( \nu \) into the integrand:

\[
\nu(x) = \beta^2 \int_{\mathcal{X}} \int_{\mathcal{X}} \nu(\tilde{x}) \cdot f_{X|X}(\tilde{x}|x') d\tilde{x} \cdot f_{X|X}(x'|x) dx' = \beta^2 \int_{\mathcal{X}} \nu(x') \cdot f_{X|X}(x'|x) dx'.
\]

Repeating this process, then we have: for all \( t \geq 1 \)

\[
\nu(x) = \beta^t \int_{\mathcal{X}} \nu(x') \cdot f_{X|X}(x'|x) dx'.
\]

For the stationary Markov equilibrium, \( f_{X|X}(x'|x) \) converges to \( f_{X}(x') \) as \( t \rightarrow \infty \). Hence, the right-hand side converges to zero as \( t \) goes to infinity. It follows that \( \nu(x) = 0 \) for all \( x \in \mathcal{X} \). \( \square \)


Proof. The result follows the Theorem of Successive Approximation (see e.g. Zemyan, 2012). \( \square \)

A.3. Proof of Theorem 3. The estimator is defined in (15). For the consistency of \( \hat{\theta} \), we need \( h_\theta \rightarrow 0 \), \( Th_\theta^{k+1} \rightarrow \infty \) and \( \mathbb{E} |\hat{m}(X) - m(X)| = o(h_\theta) \) as \( T \rightarrow \infty \). Note that the last condition ensures the estimation error in \( \hat{m} \) is negligible.

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Let \( \hat{\theta} \) be the infeasible estimator

\[
\hat{\theta} = -\frac{2}{T(T-1)} \sum_{t=1}^{T} \sum_{s \neq t} \left[ \frac{1}{h_{\theta}^{k_{0}+2}} \times \nabla K_{\theta} \left( \frac{m(X_t) - m(X_s)}{h_{\theta}} \right) \times Y_s \right].
\]

The asymptotic analysis for \( \hat{\theta} \) was done in Powell, Stock, and Stoker (1989). They show that the variance term in \( \hat{\theta} \) has the order \( T^{-1} \) if \( T h_{\theta}^{k_{0}+2} \to \infty \), while the bias term has the order \( h_{\theta}^{T} \). Therefore, if \( T^{1/2} h_{\theta}^p \to 0 \), then the bias term disappear faster than \( T^{-1/2} \). The leading term left is the variance term – the \( \hat{\theta} \) converges at the rate \( T^{-1/2} \). Our arguments piggybacks off of this argument, as we will show here that \( T^{1/2}(\hat{\theta} - \theta) \) is identical to \( T^{1/2}(\tilde{\theta} - \theta) \) by a negligible factor; that is, our estimator and the infeasible estimator have the same limiting distribution (corresponding to that derived in Powell, Stock, and Stoker (1989)).

By Taylor expansion, we have

\[
\hat{\theta} = \tilde{\theta} - \frac{2}{T(T-1)} \sum_{t=1}^{T} \sum_{s \neq t} \left[ \frac{1}{h_{\theta}^{k_{0}+2}} \times \nabla K_{\theta} \left( \frac{m(X_t) - m(X_s)}{h_{\theta}} \right) \times Y_s \times (\hat{m}(X_t) - m(X_t)) \right]
+ \frac{2}{T(T-1)} \sum_{t=1}^{T} \sum_{s \neq t} \left[ \frac{1}{h_{\theta}^{k_{0}+2}} \times \nabla^2 K_{\theta} \left( \frac{m(X_t) - m(X_s)}{h_{\theta}} \right) \times Y_s \times (\hat{m}(X_s) - m(X_s)) \right]
+ O_p(h_{\theta}^{-3} \cdot \mathbb{E}[\|\hat{m}(X) - m(X)\|^2]) \equiv \hat{A}_1 + \hat{A}_2 + \mathbb{B} \quad (17)
\]

We will show that \( \hat{A}_1 + \hat{A}_2 + \mathbb{B} \) are all \( o_p(T^{-1/2}) \) implying \( T^{1/2}(\hat{\theta} - \tilde{\theta}) \) is negligible. First, by Assumption J(i), we have

\[
h_{\theta}^{-3} \times \mathbb{E}[\|\hat{m}(X) - m(X)\|^2] = h_{\theta}^{-3} \times o_p(T^{-1/2} h_{\theta}^3) = o_p(T^{-1/2}). \quad (18)
\]

Then, \( \mathbb{B} = o_p(T^{-1/2}) \).

Next we show \( \hat{A}_1 \) and \( \hat{A}_2 = o_p(T^{-1/2}) \). For simplicity, we only provide an argument for \( \hat{A}_1 \) (that for \( \hat{A}_2 \) is analogous).

Define

\[
\tilde{\theta}_1 \equiv -\frac{2}{T(T-1)} \sum_{t=1}^{T} \sum_{s \neq t} \left[ \frac{1}{h_{\theta}^{k_{0}+2}} \times \nabla^2 K_{\theta} \left( \frac{m(X_t) - m(X_s)}{h_{\theta}} \right) \times Y_s \times [\mathbb{E}[\hat{m}(X_t)|X_t, X_s = m(X_s)]] \right]
\]

Clearly \( \mathbb{E}(\tilde{\theta}_1) = \mathbb{E}(\tilde{\theta}_1) \). Following Powell, Stock, and Stoker (1989), we now establish two properties:

\[
(a) : \tilde{\theta}_1 = o_p(T^{-1/2});
(b) : T \times \text{Var}(\hat{A}_1 - \tilde{\theta}_1) \to 0,
\]

which together imply \( \hat{A}_1 = o_p(T^{-1/2}) \).
For property (a), by Assumption J(iii),

\[ E[\hat{m}(X_t)|X_t, X_s] = E[\hat{m}_{t-s}|X_t, X_s] + o_p(T^{-1/2}h_0^2) = E[\hat{m}(X_t)|X_t] + o_p(T^{-1/2}h_0^2). \]

Then, we have

\[
\hat{A}_1 = -\frac{2}{T(T-1)} \sum_{t=1}^{T} \sum_{s \neq t} \phi_{T,s,t} \times \left[ \hat{m}(X_t) - E[\hat{m}(X_t)|X_t] \right] \leq C_1 + o_p(T^{-1/2}).
\]

Because

\[
E[C_1] \leq 2E\left[ \frac{1}{h_0^{k_0+2}} \nabla^2 K_\theta \left( \frac{m(X_t) - m(X_s)}{h_0} \right) \right] \leq 2C \times \frac{1}{h_0^2} E \| E[\hat{m}(X) - m(X)|X] \|
\]

for some positive \( C < \infty \). Hence, by Assumption J(ii), property (a) obtains.

For property (b), note that

\[
A_1 - \hat{A}_1 = -\frac{2}{T(T-1)} \sum_{t=1}^{T} \sum_{s \neq t} \phi_{T,s,t} \times \left[ \hat{m}(X_t) - E[\hat{m}(X_t)|X_t] \right] + o_p(T^{-1/2}) \equiv C_2 + o_p(T^{-1/2})
\]

where \( \phi_{T,s,t} = \frac{1}{h_0^{k_0+2}} \nabla^2 K_\theta \left( \frac{m(X_t) - m(X_s)}{h_0} \right) \).

Clearly,

\[
\text{Var}(C_2) = \frac{4}{T^2(T-1)^2} \sum_{t=1}^{T} \sum_{s \neq t} \text{Var}(\phi_{T,s,t} \times \left[ \hat{m}(X_t) - E[\hat{m}(X_t)|X_t] \right])
\]

\[
+ \frac{4}{T^2(T-1)^2} \sum_{t=1}^{T} \sum_{s \neq t} \sum_{t', s' \neq t,t,s} \text{Cov}(\phi_{T,s,t} \left[ \hat{m}(X_t) - E[\hat{m}(X_t)|X_t] \right], \phi_{T,s',t} \left[ \hat{m}(X_t) - E[\hat{m}(X_t)|X_t] \right])
\]

\[
+ \frac{4}{T^2(T-1)^2} \sum_{t=1}^{T} \sum_{s \neq t} \sum_{t', s' \neq t,t,s} \sum_{t''} \text{Cov}(\phi_{T,s,t} \left[ \hat{m}(X_t) - E[\hat{m}(X_t)|X_t] \right], \phi_{T,s',t'} \left[ \hat{m}(X_{t'}) - E[\hat{m}(X_{t'})|X_{t'}] \right])
\]

\[
= O(T^{-2}h_0^{-k_0-4}) \times E\{\hat{m}(X) - E[\hat{m}(X)|X]\}^2 + \frac{4}{T} \text{Cov}(\phi_{T,2,1} \left[ \hat{m}(X_1) - E[\hat{m}(X_1)|X_1] \right], \phi_{T,3,1} \left[ \hat{m}(X_1) - E[\hat{m}(X_1)|X_1] \right])
\]

\[
+ 4 \text{Cov}(\phi_{T,2,1} \left[ \hat{m}(X_1) - E[\hat{m}(X_1)|X_1] \right], \phi_{T,4,3} \left[ \hat{m}(X_3) - E[\hat{m}(X_3)|X_3] \right]).
\]
Note that

\[
\text{Cov}\left(\phi_{T,2,1}[\hat{m}(X_1) - E[\hat{m}(X_1)|X_1]], \phi_{T,4,3}[\hat{m}(X_3) - E[\hat{m}(X_3)|X_3]]\right)
= \mathbb{E}\left\{\phi_{T,2,1}\phi_{T,4,3}[\hat{m}(X_1) - E[\hat{m}(X_1)|X_1]] \times [\hat{m}(X_3) - E[\hat{m}(X_3)|X_3]]\right\}
- \mathbb{E}\left\{\phi_{T,2,1}[\hat{m}(X_1) - E[\hat{m}(X_1)|X_1]] \times \mathbb{E}\left\{\phi_{T,4,3}[\hat{m}(X_3) - E[\hat{m}(X_3)|X_3]]\right\}\right\}.
\]

By Assumption J(iii),

\[
\mathbb{E}\left\{\phi_{T,2,1}[\hat{m}(X_1) - E[\hat{m}(X_1)|X_1]]\right\}
= \mathbb{E}\left\{\phi_{T,2,1}[\hat{m}_{1,-2} - E[\hat{m}_{1,-2}|X_1]]\right\} + O_p(h^{-2}) \times o_p(T^{-1/2}h^2) = o_p(T^{-1/2}).
\]

Furthermore, by the law of iterated expectation (conditioning on the sigma algebra: \(\mathcal{F}_2, \mathcal{F}_4, \mathcal{F}_5, \ldots, n\)),

\[
\mathbb{E}\left\{\phi_{T,2,1}\phi_{T,4,3}[\hat{m}(X_1) - E[\hat{m}(X_1)|X_1]] \times [\hat{m}(X_3) - E[\hat{m}(X_3)|X_3]]\right\}
= O_p(h^{-4}) \times o_p(T^{-1/2}h^2) \times o_p(T^{-1/2}h^2)
= o_p(T^{-1}),
\]

where the term \(o_p(T^{-1/2}h^2)\) is due to the differences \(\hat{m}(X_1) - \hat{m}_{1, -2}\) and \(\hat{m}(X_3) - \hat{m}_{3, -1}\). Therefore, the last term in \(\text{Var}(\hat{\theta})\) is \(o_p(T^{-1})\).

Moreover, because

\[
\frac{1}{T}\text{Cov}\left(\phi_{T,2,1}[\hat{m}(X_1) - E[\hat{m}(X_1)|X_1]], \phi_{T,3,1}[\hat{m}(X_1) - E[\hat{m}(X_1)|X_1]]\right)
= \frac{1}{T}\mathbb{E}\left\{\phi_{T,2,1}\phi_{T,3,1}[\hat{m}(X_1) - E[\hat{m}(X_1)|X_1]]^2\right\}
= O(T^{-1}h^{-4}) \times \mathbb{E}\{\hat{m}(X) - E[\hat{m}(X)|X]\}^2.
\]

Then a sufficient condition for property (b) is

\[
\mathbb{E}\{\hat{m}(X) - E[\hat{m}(X)|X]\}^2 = o(h^4).
\]

Note that this condition is implied by Assumption J(i).

Hence, we have shown that our estimator \(\hat{\theta}\) and the infeasible estimator \(\tilde{\theta}\) differ by an amount which is \(o_p(T^{-1/2})\). Hence, the asymptotic properties for \(\hat{\theta}\) are the same as those for the infeasible estimator \(\tilde{\theta}\), which were previously established in Powell, Stock, and Stoker (1989).