Identification and Estimation of Dynamic Games when Players’ Belief Are Not in Equilibrium

A Short Review of Aguirregabiria and Magesan (2010)

January 25, 2012
Dynamics of the game

- Two players, \{i, j\}
- \(T\) periods
- \(X_t\) is the vector of state variables
- \(Y_{it}\) is player \(i\)'s choice at time \(t\)
- \(\epsilon_{it}(Y_{it})\) is \(i\)'s private information; it is distributed by \(G\)
- \(\pi_i(Y_{it}, Y_{jt}, X_t)\) is a real value function of \(i\)'s own action, opponent's action, and state variables; it is the deterministic part of \(i\)'s payoff
- payoff function

\[
\Pi_{it}(Y_{it}, Y_{jt}, X_t) = \pi_i(Y_{it}, Y_{jt}, X_t) + \epsilon_i(Y_{it})
\]
State variables and transition functions

- \( X_t = (W_t, S_{1t}, S_{2t}) \)
- \( W_t \) are some exogenous, player independent market characteristics
- \( S_{it} \) is player \( i \)'s specific characteristics
- \( f^W(W_{t+1}|W_t) \) is the transition function of \( W \)
- \( f^S(S_{t+1}|Y_{it}, S_{it}) \) is the transition function of \( S \); it does not depend on \( j \)'s action or state
Strategies, choice probabilities, and beliefs

- $\sigma_{it}(X_t, \epsilon_{it})$ is $i$’s strategy at time $t$
- $b_{jt}^t(X_t, \epsilon_{jt})$ is player $i$’s belief at period $t_o$ about the strategy of player $j$ at time $t$
- $P_{it}(X_t) = Pr(\sigma_{it}(X_t, \epsilon_{it}) = 1|X_t)$ is $i$’s choice probability
- $B_{jt}(X_t) = Pr(b_{jt}(X_t, \epsilon_{it}) = 1|X_t)$ is $i$’s belief of player $j$’s behavior at time $t$
Model assumptions

MOD1: Players’ strategies depend on state variables
MOD2: Players maximize expected payoffs
MOD3: A player’s belief of his own action is consistent with his expectation of his actual actions.
’equil’: Players’ beliefs about other players’ actions are unbiased expectations of the actual actions of other players. That is,

\[ B_{jt}(X_t) = P_{jt}(X_t) \]

MOD4: If \( T < \infty \), \( B_{jt}^{(to)} = B_{jt} \); if \( T < \infty \), \( B_{jt}^{(to)} = B_j \)
i’s belief of j’s behavior is a T\times T matrix

<table>
<thead>
<tr>
<th>Period when beliefs are formed ($t_0$)</th>
<th>Period of the opponents’ behavior ($t$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t = 1$</td>
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<tr>
<td>$t_0 = 1$</td>
<td>$B_{j1}^{(1)}$</td>
</tr>
<tr>
<td>$t_0 = 2$</td>
<td>$B_{j2}^{(2)}$</td>
</tr>
<tr>
<td>$t_0 = 3$</td>
<td>$B_{j3}^{(3)}$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$t_0 = T - 1$</td>
<td>$B_{j,T-1}^{(T-1)}$</td>
</tr>
<tr>
<td>$t_0 = T$</td>
<td>$B_{jT}^{(T)}$</td>
</tr>
</tbody>
</table>
Best response (1)

- Given a belief $B_j$, player $i$ best responds by maximizing her expected utility payoff
- The key optimization criterion is the Bellman equation

$$V_{it}^B(X_t, \epsilon_{it}) = \max_{Y_{it}} \left( Y_{it}\pi^B_i(X_t - \epsilon_{it}) + \beta \int V^B_{i}(X_{t+1}, \epsilon_{it+1}) f^B dG \right)$$

where:

$$\pi^B_i(X_t) = B_{jt}(X_t)\pi_i(1, X_t) + (1 - B_{jt}(X_t))\pi_i(0, X_t)$$

$$f^B_{i}(X_{t+1}|Y_{it}, X_t) = f_i(X_{it+1}|Y_{it}, X_{it}) *$$

$$[B_{jt}(X_t)f_j(X_{jt+1}|1, X_{jt}) +$$

$$(1 - B_{jt}(X_t))f_j(X_{jt+1}|0, X_{jt})]$$
Best response (2)

- The best response function can be represented by the threshold function

\[
\{Y_{it} = 1\} \iff \{\epsilon_{it}(0) - \epsilon_{it}(1) \leq v^B_{it}(1, X_t) - v^B_{it}(0, X_t)\}
\]

where:

\[
v^B_{it} = \pi^B_{it}(Y_{it}, X_t) + \beta \int_{X', \epsilon'} V_i(X', \epsilon') f^B_i(X' | Y_{it}, X_t) dG(\epsilon')
\]

- Denote \( \Lambda \) as the best response function using the explicit distribution function \( (G) \) of \( \epsilon \), i.e.

\[
Pr(Y_{it} = 1 | X_t) = \Lambda(v^B_{it}(1, X_t) - v^B_{it}(0, X_t))
\]
Data

- There are $M$ markets. The econometrician observes

$$\{Y_{imt}, Y_{jmt}, X_{mt}\}_{t=1}^T$$

for every market $m$.

- We are going to suppress $m$ for our discussion of how to estimate the model.
Identification assumptions

ID1: $X_{mt} = X_t$, $B_{jmt}(X) = B_{jt}(X)$

ID2: Normalization of the payoff function $\pi(\cdot)$

ID3: There are two values of player $i$’s opponent’s state, $S_j^L$ and $S_j^H$, at which player $i$’s beliefs are in equilibrium; that is,

$$B_{jt}(W_t, S_i, S_j^L) = P_{jt}(W_t, S_i, S_j^L)$$

$$B_{jt}(W_t, S_i, S_j^H) = P_{jt}(W_t, S_i, S_j^H)$$
Estimation with the assumption 'equil'

Suppose $T = \infty$,

1. Observe the data $(Y_{it}, Y_{jt}, X_t)$; do not observe $\epsilon_{it}$
2. Assume that $G$ (hence $\Lambda$) and $\beta$ are known
3. Estimate $(\widehat{f}_t^S, \widehat{f}_t^S, \widehat{P}_{it}, \widehat{P}_{jt})$ non-parametrically
4. Inverts $\Lambda$ to obtain $\tilde{v}_{it}$
5. Solve the Bellman equation to obtain $\tilde{V}$ and $\tilde{\pi}$

$$V_{it}^B(X_t, \epsilon_{it}) = \max_{Y_{it}} \{v_{it}^B(Y_{it}, X_t) + \epsilon_{it}(Y_{it})\}$$

$$v_{it}^B(Y_{it}, X_t) = \pi_{it}^B(Y_{it}, X_t) + \beta \int V_{it+1}^B(X_{t+1}, \epsilon_{it+1}) f_{it}^B dG$$

note:

$$B_{it}(X_t) = \Lambda(v_{it}^B(1, X_t) - v_{it}^B(0, X_t))$$

$$B = P$$
Estimation using backward induction

Same as the last slide, but suppose $T < \infty$

- Define player $i$’s continuation payoff at time $t - 1$

\[ d_{it-1} = \beta \sum_{X'} \tilde{V}_{it}^B (X') f_{t-1}(X'|Y_i, Y_j, X) \]

- Let $\tilde{d}_{iT} = 0$
- Solve for $\tilde{\pi}_{iT-1}$ and $\tilde{V}_{iT-1}^B$
- Calculate $\tilde{d}_{iT-1}$
- Repeat
Identification assumptions without the assumption ’equil’

• Instead of the assumption ’equil’, we assume MOD4 and ID3, which states that there are two values of opponent’s state variable, \( S_j^L \) and \( S_j^H \), at which player \( i \)’s beliefs are in equilibrium.

• Proposition 2 states that these are sufficient conditions to non-parametrically estimate player \( i \)’s belief function and payoff function.
Estimation without the assumption ’equil’

1. Let $\tilde{d}_{iT} = 0$
2. Calculate $\hat{B}_{jt}$ by formula (30) in the paper
3. Calculate $\hat{V}_{iT}^B$ by formula (31)
4. Calculate $\tilde{d}_{iT-1}$ of the previous period
5. Repeat
Testing unbiased beliefs (1)

Under the assumptions MOD1, MOD2, MOD3, MOD4, ID1, and ID2, we can test the null of unbiased belief, i.e. player $i$’s belief of $j$’s behavior is consistent with the $j$’s actual behavior at time $t$, $B_{jt}(X_t) = P_{jt}(X_t)$.

- Define

$$q_{it}(X) = \Lambda^{-1}(P_{it}(X))$$

- Pick $X^a, X^b, X^c, X^d$ s.t. each value has the same value in the component of $(S_i, W)$, but different values of $S_j$. 
Testing unbiased beliefs (2)

- Define

\[ \delta = \left\{ q_{it}(X_a) - q_{it}(X_b) \right\} \]

\[ \frac{q_{it}(X_c) - q_{it}(X_d)}{P_{jt}(X_a) - P_{jt}(X_b)} \]

\[ - \left\{ \frac{P_{jt}(X_c) - P_{jt}(X_d)}{P_{jt}(X_a) - P_{jt}(X_b)} \right\} \]

- Further define

\[ D = \sum_{h=1}^{H} \left( \frac{\bar{\delta}_h}{se(\bar{\delta})} \right)^2 \]

where \( \bar{\delta} \) is the sample mean, then \( D \) is asymptotically distributed as Chi-square with \( H \) degrees of freedom. \( H \) is the number of all possible combinations of four different values of \( S_j \) with \( S_j^a \neq S_j^b \) and \( S_j^d \neq S_j^d \).
Empirically testing the null of unbiased belief

<table>
<thead>
<tr>
<th>Stores of MD</th>
<th>Stores of BK</th>
<th>B_{MD}</th>
<th>P_{MD}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>-0.17</td>
<td>(0.04)</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>-0.10</td>
<td>(0.06)</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>-0.08</td>
<td>(0.07)</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>-0.06</td>
<td>(0.10)</td>
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<th>Stores of BK</th>
<th>Stores of MD 0</th>
<th>Stores of MD 1</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>-0.03 (0.05)</td>
<td>0.02 (0.04)</td>
</tr>
<tr>
<td>2</td>
<td>0.03 (0.10)</td>
<td>0.04 (0.12)</td>
</tr>
</tbody>
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