Outline

1. The Two Type Case

2. Empirical calibration: Cable TV Markets

3. The Continuum of Types Case
The Basic Model

- A firm produces a single good at marginal cost $c$.

- Consumers receive utility $\theta V(q) - T(q)$ if they purchase a quantity $q$ and utility 0 otherwise.

- Two cases:
  1. $\theta \in \{\theta_1, \theta_2\}$, $\theta_1 < \theta_2$; “lo” and “hi” types
  2. $\theta \in [\underline{\theta}, \overline{\theta}]$
The Two Type Case

- Monopolist offers two bundles (we assume the monopolist serves both types; \( \lambda \) is sufficiently large):
  - \((q_1, T_1)\), directed at type \( \theta_1 \) consumers (in proportion \( \lambda \)), and
  - \((q_2, T_2)\), directed at type \( \theta_2 \) consumers (in proportion \( 1 - \lambda \)).

- The monopolist’s profit is
  \[
  \Pi^m = \lambda(T_1 - cq_1) + (1 - \lambda)(T_2 - cq_2)
  \]

- Monopolist faces two types of constraints. The **individual rationality constraint** for type \( \theta \) \((IR(\theta))\) requires that consumers of type \( \theta \) are willing to buy.
  \[
  IR(\theta_1) : \theta_1 V(q_1) - T_1 \geq 0; \quad IR(\theta_2) : \theta_2 V(q_2) - T_2 \geq 0.
  \]

  Since \( \theta_2 \) consumers can always buy the \( \theta_1 \) bundle, only \( IR(\theta_1) \) is relevant
The incentive compatibility constraint for type $\theta$ (IC(\(\theta\))) requires that consumers of type $\theta$ prefer the bundle designed for them rather than that designed for type $\theta'$

$$IC(\theta_2) : \theta_2 V(q_2) - T_2 \geq \theta_2 V(q_1) - T_1$$  \hspace{1cm} (2)

$$IC(\theta_1) : \theta_1 V(q_1) - T_1 \geq \theta_1 V(q_2) - T_2$$ \hspace{1cm} (3)

The relevant IC constraint is that of the high-valuation consumers, IC($\theta_2$). Assume this for now.

In fact, we will proceed ignoring IC($\theta_1$) and then show that the solution of the subconstrained problem satisfies it.

We thus solve the problem: max $\Pi^m$ s.t. IC($\theta_2$) and IR($\theta_1$)
Since $\text{IR}(\theta_1)$ will hold with equality at the optimum, we can rewrite (2) as

$$T_2 \leq \theta_2 V(q_2) - [\theta_2 V(q_1) - T_1] = \theta_2 V(q_2) - (\theta_2 - \theta_1)V(q_1)$$

$T_1$ can be chosen to appropriate the type-$\theta_1$ surplus entirely, but $T_2$ must leave some net surplus to the type $\theta_2$ consumers, because they can always buy the bundle $(q_1, T_1)$ and have net surplus

$$\theta_2 V(q_1) - T_1 = (\theta_2 - \theta_1)V(q_1)$$
Substituting this into the objective function, the monopolist solves the following unconstrained problem

$$\max_{q_1, q_2} \lambda (\theta_1 V(q_1) - c q_1) + (1 - \lambda) [\theta_2 V(q_2) - c q_2 - (\theta_2 - \theta_1) V(q_1)]$$

First Order Conditions are:

$$\theta_1 V'(q_1) = \frac{c}{1 - \frac{1-\lambda}{\lambda} \frac{\theta_2 - \theta_1}{\theta_1}} \quad (4)$$

$$\theta_2 V'(q_2) = c \quad (5)$$

It follows from (5) that the quantity purchased by the high value consumers is socially optimal (marginal utility equal marginal cost) (absence of distortion at the top),

and from (4) that the quantity consumed by the low-demand consumers is socially suboptimal: $\theta_1 V'(q_1) > c$ and their consumption is distorted downwards.
It remains to check that the low demand consumers do not want to choose the high demand consumers’ bundle. Because they have zero surplus, we require that $0 \geq \theta_1 V(q_2) - T_2$. But this condition is equivalent to

$$0 \geq - (\theta_2 - \theta_1)[V(q_2) - V(q_1)],$$

which is satisfied
Monopolist attempt to extract the high demand consumers’ large surplus faces threat of personal arbitrage:

- High demand consumer can consume the low-demand consumers’ bundle if his own bundle does not generate enough surplus.

To relax this personal arbitrage constraint, the monopolist offers a relatively low consumption to the low demand consumers.

- OK because high demand consumers suffer more from a reduction in consumption than low demand ones (single crossing property).

- Since low demand consumers are not tempted to exercise personal arbitrage, no distortion at the top (recall welfare gains can be captured by the monopolist through an increase in $T_2$).
2.1. Continuous Types but Discrete Qualities

The theory described in the previous section applies also to the case of continuous types but to discrete qualities. To see this, suppose instead that consumer types are continuously distributed on with probability density function $f(t)$ but that the monopolist has decided to offer just two qualities regardless. He or she may do so for a number of reasons. There may be fixed costs associated with the design, production, or marketing of products of different qualities. Or there may be incremental (especially marketing) costs of offering numerous goods. If these are large, the monopolist will offer only those products that can cover his or her fixed costs, limiting the number of products in the market (Spence 1980; Dixit and Stiglitz 1977).

Suppose the firm offered arbitrary qualities $q$ and $q_1$. Who would buy these goods? All consumers for whom $u(q, t) \geq u(q_1, t)$ would buy the first good. Because of the structure of the problem—notably the single-crossing condition—only the first of these constraints would bind. Let denote the consumer type that is just indifferent between purchasing the two goods and denote the analogous consumer type just indifferent between purchasing good $q_1$ and the outside (or no) good. Then the share of the distribution of consumer types that purchase each good, $\mu_i$, is given by the integral under the distribution $f(t)$.

Figure 1. Quality degradation with two types adapted from Maskin and Riley (1984)

indifference curves for $U(q) - C(q)$
### Table 3

Preliminary Evidence of Quality Degradation

<table>
<thead>
<tr>
<th>Total Prices/Channels</th>
<th>Three-Good Markets</th>
<th>Two-Good Markets</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Difference</td>
</tr>
<tr>
<td>$p_3$/channel</td>
<td>1.06 (.03)</td>
<td>-.04 (.01)</td>
</tr>
<tr>
<td>$p_2$/channel</td>
<td>1.10 (.03)</td>
<td>-.13 (.02)</td>
</tr>
<tr>
<td>$p_1$/channel</td>
<td>1.23 (.04)</td>
<td>. . .</td>
</tr>
<tr>
<td>$N$</td>
<td>72</td>
<td>239</td>
</tr>
</tbody>
</table>

**Note.** Reported are the average price per channel for each offered cable service. Channels include all top 40 satellite channels and, for the lowest quality service, all major broadcast networks. Ratios are formed with total price and total channels. Values in the Difference columns are the difference in price per channel in that row and the row that follows. The cable system in one two-good market included no satellite or broadcast networks in its lowest quality service. Standard errors are in parentheses.
### Table 5
**Recovered Parameter Values and Implied Qualities**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Three-Good Markets</th>
<th>Two-Good Markets</th>
<th>One-Good Markets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Net type distribution:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tilde{f}_3$</td>
<td>.47</td>
<td>.61</td>
<td>.70</td>
</tr>
<tr>
<td>$\tilde{f}_2$</td>
<td>.12</td>
<td>.04</td>
<td>. .</td>
</tr>
<tr>
<td>$\tilde{f}_1$</td>
<td>.04</td>
<td>. .</td>
<td>. .</td>
</tr>
<tr>
<td>$\tilde{f}_0$</td>
<td>.37</td>
<td>.35</td>
<td>.30</td>
</tr>
<tr>
<td>$\tilde{t}_3$</td>
<td>5.15</td>
<td>4.77</td>
<td>4.35</td>
</tr>
<tr>
<td>$\tilde{t}_2$</td>
<td>4.99</td>
<td>4.65</td>
<td>. .</td>
</tr>
<tr>
<td>$\tilde{t}_1$</td>
<td>4.90</td>
<td>. .</td>
<td>. .</td>
</tr>
<tr>
<td>Qualities:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$q_3$</td>
<td>5.15</td>
<td>4.77</td>
<td>4.35</td>
</tr>
<tr>
<td>$q_2$</td>
<td>4.43</td>
<td>2.57</td>
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</tr>
<tr>
<td>$q_1$</td>
<td>3.42</td>
<td>. .</td>
<td>. .</td>
</tr>
<tr>
<td>% Degradation:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(\tilde{t}_3 - q_3)/\tilde{t}_3$</td>
<td>.00</td>
<td>.00</td>
<td>.00</td>
</tr>
<tr>
<td>$(\tilde{t}_2 - q_2)/\tilde{t}_2$</td>
<td>.11</td>
<td>.45</td>
<td>. .</td>
</tr>
<tr>
<td>$(\tilde{t}_1 - q_1)/\tilde{t}_1$</td>
<td>.30</td>
<td>. .</td>
<td>. .</td>
</tr>
<tr>
<td>Price/quality ratio</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$q_3/p_3$</td>
<td>.20</td>
<td>.21</td>
<td>.23</td>
</tr>
<tr>
<td>$q_2/p_2$</td>
<td>.21</td>
<td>.21</td>
<td>. .</td>
</tr>
<tr>
<td>$q_1/p_1$</td>
<td>.21</td>
<td>. .</td>
<td>. .</td>
</tr>
<tr>
<td>$N$</td>
<td>72</td>
<td>240</td>
<td>730</td>
</tr>
</tbody>
</table>

**Note.** Parameters of net type distribution are obtained using the procedure in Section 4.2. Quality measures are calculated using equation (12). Percentage of degradation evaluated at cut types is defined as the marginal type just inclined to purchase that quality.
Insights from previous example generalize to case where there are an infinite number of types.

Now let $\theta$ be distributed with density $f(\theta)$ (and CDF $F$) on an interval $[\theta, \bar{\theta}]$.

Monopolist offers a nonlinear tariff $T(q)$. A consumer with type $\theta$ purchases $q(\theta)$ and pays $T(q(\theta))$.

Monopolist’s profit is

$$\Pi^m = \int_{\theta}^{\bar{\theta}} [T(q(\theta)) - cq(\theta)] f(\theta) d\theta$$

The monopolist maximizes his profit subject to two types of constraints.
The Continuum of Types Case

IR constraints

- For all $\theta$,

$$\theta V(q(\theta)) - T(q(\theta)) \geq 0$$

- As before, it suffices that IR ($\theta$) holds:

$$\theta V(q(\theta)) - T(q(\theta)) \geq 0$$

(6)

- If (6) holds, any type $\theta$ can realize a nonegative surplus consuming $\theta$'s bundle:

$$\theta V(q(\theta)) - T(q(\theta)) \geq (\theta - \theta)V(q(\theta)) \geq 0$$
IC constraints

- $\theta$ should not consume the bundle designed for $\tilde{\theta}$ ($\tilde{\theta} \neq \theta$)
- IC($\theta$): for all $\theta, \tilde{\theta}$:

\[ U(\theta) \equiv U(\theta, \theta) = \theta V(q(\theta)) - T(q(\theta)) \geq \theta V(q(\tilde{\theta})) - T(q(\tilde{\theta})) \equiv U(\theta, \tilde{\theta}) \tag{7} \]

These constraints are not very tractable in this form. However, we can show that it suffices to require that the ICs are satisfied “locally”; i.e., a necessary and sufficient condition for

\[ \theta = \arg\max_{\tilde{\theta}} U(\theta, \tilde{\theta}) = \theta V(q(\tilde{\theta})) - T(q(\tilde{\theta})) \]

is given by the FOC (evaluated at the true type $\theta$):

\[ \theta V'(q(\theta)) = T'(q(\theta)) \tag{8} \]

This says that a small increase in the quantity consumed by they type $\theta$ consumer generates a marginal surplus $\theta V'(q(\theta))$ equal to the marginal payment $T'(q(\theta))$. Thus, the consumer does not want to modify the quantity at the margin.
Equation (8) can be used to characterize the payment function once the quantity function \( q(\theta) \) is known. Assume for now that \( q(\theta) \) is strictly monotonic in \( \theta \) (we’ll show this later). Letting \( \alpha(\cdot) = q^{-1}(\cdot) \), i.e., \( \alpha(q(\theta)) = \theta \), then (8) is

\[
T'(q) = \alpha(q)V'(q)
\]

Now from the IC

\[
U(\theta) = \max_{\tilde{\theta}} \theta V(q(\tilde{\theta})) - T(q(\tilde{\theta}))
\]

Using the envelope theorem,

\[
\frac{\partial U(\theta)}{\partial \theta} = V(q(\theta))
\]

Thus we can write (use \( U(\theta) = 0 \))

\[
U(\theta) = \int_\theta^\theta \frac{\partial U(t)}{\partial t} dt + U(\theta) = \int_\theta^\theta V(q(t)) dt
\]
• Note that consumer’s utility grows with $\theta$ at a rate that increases with $q(\theta)$.

• This is important for the derivation of the optimal quantity function, as it implies that higher quantities “differentiate” different types more, in that the utility differentials are higher.

• Since leaving a surplus to the consumer is costly to the monopolist (recall $T(q(\theta)) = \theta V(q(\theta)) - U(\theta)$), the monopolist will tend to reduce $U$ and to do so, will induce (most) consumers to consume a suboptimal quantity.

• Intuition from the previous equations is that optimality will imply a bigger distortion for low-$\theta$ consumers.
Since $T(q(\theta)) = \theta V(q(\theta)) - U(\theta) = \theta V(q(\theta)) - \int_{\theta}^{\bar{\theta}} V(q(t))dt$, we can write

$$\Pi_m = \int_{\theta}^{\bar{\theta}} \left( \theta V(q(\theta)) - \int_{\theta}^{\bar{\theta}} V(q(t))dt - c(q(\theta)) \right) f(\theta) d\theta \quad (9)$$
Recall: Integration by Parts

If $u$ and $v$ are continuous functions on $[a, b]$ that are differentiable on $(a, b)$, and if $u$ and $v$ are integrable on $[a, b]$, then

$$\int_a^b u(x)v'(x)dx + \int_a^b u'(x)v(x)dx = u(b)v(b) - u(a)v(a)$$

Let $g = uv$. Then $g' = uv' + vu'$. By the fundamental theorem of calculus,

$$\int_a^b g'(x)dx = g(b) - g(a).$$

Then

$$\int_a^b g'(x)dx = u(b)v(b) - u(a)v(a),$$

and the result follows.
• Now, integrating by parts, with \( f(\theta) = F'(\theta) \) and \( \int_{\theta}^{\Theta} V(q(t))dt = G(\theta) \),

\[
\int_{\theta}^{\Theta} \left[ \int_{\theta}^{\Theta} V(q(t))dt \right] f(\theta)d\theta
\]

is equal to

\[
\int_{\theta}^{\Theta} V(q(\theta))d\theta - 0 - \int_{\theta}^{\Theta} V(q(\theta))F(\theta)d\theta = \int_{\theta}^{\Theta} V(q(\theta))[1 - F(\theta)]d\theta
\]

• Then going back to (9)
We can write

$$\Pi^m = \int_{\underline{\theta}}^{\overline{\theta}} \left( \theta V(q(\theta)) - \int_{\underline{\theta}}^{\theta} V(q(t)) dt - cq(\theta) \right) f(\theta) d\theta$$

as

$$\Pi^m = \int_{\underline{\theta}}^{\overline{\theta}} ([\theta V(q(\theta)) - cq(\theta)] f(\theta) - V(q(\theta))[1 - F(\theta)]) d\theta$$

Now max $\Pi^m$ w.r.t. $q(\cdot)$ requires that the term under the integral be maximized w.r.t. $q(\theta)$ for all $\theta$, yielding:

$$[\theta V'(q(\theta)) - c]f(\theta) - V'(q(\theta))[1 - F(\theta)] = 0$$

or equivalently

$$\theta V'(q(\theta)) = c + \frac{[1 - F(\theta)]}{f(\theta)} V'(q(\theta)) \quad (10)$$
\[ \theta V'(q(\theta)) = c + \frac{[1 - F(\theta)]}{f(\theta)} V'(q(\theta)) > c \]

Thus marginal willingness to pay for the good \( \theta V'(q(\theta)) \) exceeds the marginal cost \( c \) for all but the highest value consumer \( \theta = \bar{\theta} \).
Moreover, for this specification of preferences, we can get a simple expression for the price-cost margin.

Let $T'(q) \equiv p(q)$ denote the price of an extra unit when the consumer already consumes $q$ units.

From consumer optimization

\[ T'(q(\theta)) = \theta V'(q(\theta)) \Rightarrow V'(q(\theta)) = \frac{T'(q(\theta))}{\theta} = \frac{p(q(\theta))}{\theta} \]

Substituting in (10), which I write again here

\[ \theta V'(q(\theta)) = c + \frac{[1 - F(\theta)]}{f(\theta)} V'(q(\theta)) \]

we have:

\[ \frac{p(q(\theta)) - c}{p(q(\theta))} = \frac{[1 - F(\theta)]}{\theta f(\theta)} \] (11)
• We will assume that the “hazard rate” of the distribution of types, \( \frac{f(\theta)}{[1-F(\theta)]} \), is increasing in \( \theta \) (common assumption, satisfied by a variety of distributions).

• We can rewrite (10) as:

\[
\left( \theta - \frac{[1-F(\theta)]}{f(\theta)} \right) V'(q(\theta)) = c
\]

• Or letting \( \Gamma(\theta) \equiv \left( \theta - \frac{[1-F(\theta)]}{f(\theta)} \right) \), simply as:

\[
\Gamma(\theta)V'(q(\theta)) = c
\]

• where \( \Gamma'(\theta) > 0 \) by our increasing-hazard-rate assumption
\[ \Gamma(\theta)V'(q(\theta)) = c \]

- Then totally differentiating (with variables \( q(\theta) \) and \( \theta \)), we obtain:

\[
\frac{dq(\theta)}{d\theta} = -\frac{\Gamma'(\theta)}{\Gamma(\theta)} \frac{V'(q(\theta))}{V''(q(\theta))} > 0,
\]

using \( V \) concave and \( \Gamma'(\theta) > 0 \)

- Thus \( q'(\theta) > 0 \), \( q(\theta) \) increases with \( \theta \)
Now consider the price-cost margin. Recall

$$\frac{p(q(\theta)) - c}{p(q(\theta))} = \frac{[1 - F(\theta)]}{\theta f(\theta)} = \frac{1}{\theta \Gamma(\theta)}$$

The derivative of the RHS with respect to $\theta$ is:

$$-1 \frac{1}{[.]^2} [\Gamma(\theta) + \theta \Gamma'(\theta)] < 0$$

Thus $\frac{p-c}{c}$ decreases with consumer type, and therefore with output.
Finally, recall $T'(q) = p(q)$. Hence

$$T''(q) = \frac{dp}{dq} = \frac{dp/d\theta}{dq/d\theta} = \left(\begin{array}{c} - \\ + \end{array}\right) < 0$$

Thus $T(q)$ is concave. As a result:

- Average price per unit $T(q)/q$ decreases with $q$ (Maskin and Riley’s quantity discount result).

- Because a concave function is the lower envelope of its tangents, the optimal nonlinear payment schedule can also be implemented by offering a menu of two part tariffs (where the monopolist lets the consumer choose among the continuum of two-part tariffs).
We argued that it was enough to require that the ICs are satisfied “locally”; i.e., that a necessary and sufficient condition for

\[ \theta = \arg\max_{\tilde{\theta}} U(\theta, \tilde{\theta}) = V(q(\tilde{\theta}), \theta) - T(q(\tilde{\theta})) \]

is given by the FOC (evaluated at the true type \(\theta\)):

\[ \frac{\partial V(q(\theta), \theta)}{\partial q} = T'(q(\theta)) \]

This is because of the single crossing property, \( \frac{\partial^2 V(q(\theta), \theta)}{\partial q \partial \theta} > 0 \)
Decompose the aggregate demand function into independent demands for marginal units of consumption. Fix a quantity $q$ and consider the demand for the $q^{th}$ unit of consumption. By definition, the unit has price $p$. The proportion of consumers willing to buy the unit is

$$D_q(p) \equiv 1 - F(\theta_q^*(p))$$

where $\theta_q^*(p)$ denotes the type of consumer who is indifferent between buying and not buying the $q^{th}$ unit at price $p$:

$$\theta_q^*(p)V'(q) = p$$  \hspace{1cm} (12)

The demand for the $q$-th unit is independent of the demand for the $\tilde{q}$th unit for $\tilde{q} \neq q$ (due to no income effects). We can thus apply the inverse elasticity rule.
The optimal price for the $q$th unit is given by

$$\frac{p - c}{p} = -\frac{dp}{dD_q} \frac{D_q}{p}$$

However

$$\frac{dD_q}{dp} = -f(\theta^*_q(p)) \frac{d\theta^*_q(p)}{dp}$$

and from (12)

$$\frac{d\theta^*_q(p)}{\theta^*_q(p)} = \frac{dp}{p}$$

We thus obtain

$$\frac{p - c}{p} = \frac{1 - F(\theta^*_q(p))}{\theta^*_q(p)f(\theta^*_q(p))}$$

which is equation (11), thus unifying the theories of second degree and third degree price discrimination.