Both Theorems 1 and 2 below have been described to me as Leibniz’ Rule.

1 The vector case

The following is a reasonably useful condition for differentiating a Riemann integral. The proof may be found in Dieudonné [6, Theorem 8.11.2, p. 177]. One thing you have to realize is that for Dieudonné a partial derivative can be taken with respect to a vector variable. That is, if \( f: \mathbb{R}^n \times \mathbb{R}^m \) where a typical element of \( \mathbb{R}^n \times \mathbb{R}^m \) is denoted \((x, z)\) with \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^m \). The partial derivative \( D_x f \) is a Fréchet derivative with respect to \( x \) holding \( z \) fixed.

1 Theorem Let \( A \subset \mathbb{R}^n \) be open, let \( I = [a, b] \subset \mathbb{R} \) be a compact interval, and let \( f \) be a (jointly) continuous mapping of \( A \times I \) into \( \mathbb{R} \). Then

\[
g(x) = \int_a^b f(x, t) \, dt
\]

is continuous in \( A \).

If in addition, the partial derivative \( D_x f \) exists and is (jointly) continuous on \( A \times I \), then \( g \) is continuously differentiable on \( A \) and

\[
g'(x) = \int_a^b D_x f(x, t) \, dt.
\]

The next, even more useful, result is listed as an exercise (fortunately with hint) by Dieudonné [6, Problem 8.11.1, p. 177].

2 Leibniz’s Rule Under the hypotheses of Theorem 1, let \( \alpha \) and \( \beta \) be two continuously differentiable mappings of \( A \) into \( I \). Let

\[
g(x) = \int_{\alpha(x)}^{\beta(x)} f(x, t) \, dt.
\]

Then \( g \) is continuously differentiable on \( A \) and

\[
g'(x) = \int_{\alpha(x)}^{\beta(x)} D_x f(x, t) \, dt + f(x, \beta(x))\beta'(x) - f(x, \alpha(x))\alpha'(x).
\]
2 The measure space case

This section is intended for use with expected utility, where instead if integrating with respect to a real parameter $t$ as in Theorem 1, we integrate over an abstract probability space. So let $(\Omega,\mathcal{F},\mu)$ be a measure space, let $A \subset \mathbb{R}^n$ be open. We are interested in the properties of a function $g : A \to \mathbb{R}$ defined by

$$g(x) = \int_{\Omega} f(x, \omega) \, d\mu(\omega).$$

We are particularly interested in when $g$ is continuous or continuously differentiable. It seems clear that in order for $g$ to be defined, the function $f$ must be measurable in $\omega$, and in order for $g$ to stand a chance of being continuous, the function $f$ needs to be continuous in $x$.

3 Definition Let $(\Omega,\mathcal{F},\mu)$ be a measure space, let $A$ be a topological space. We say that a function $f : A \times \Omega \to \mathbb{R}$ is a Carathéodory function if for each $x \in A$ the mapping $\omega \mapsto f(x, \omega)$ is $\mathcal{F}$-measurable, and for each $\omega \in \Omega$ the mapping $x \mapsto f(x, \omega)$ is continuous. (Sometimes we say that $f$ is continuous in $x$ and measurable in $\omega$.)

In order for the function $g$ defined by (1) to be finite-valued we need that for each $x$, the function $\omega \mapsto f(x, \omega)$ needs to be integrable. But this is not enough for our needs we need the following stronger property.

4 Definition The function $f : A \times \Omega \to \mathbb{R}$ is locally uniformly integrably bounded if for every $x$ there is a nonnegative measurable function $h_x : \Omega \to \mathbb{R}$ such that $h_x$ is integrable, that is, $\int_{\Omega} h_x(\omega) \, dP(\omega) < \infty$, and there exists a neighborhood $U_x$ of $x$ such that for all $y \in U_x$,

$$|f(y, \omega)| \leq h_x(\omega).$$

Note that since $x \in U_x$, if $f$ is locally uniformly integrably bounded, then we also have that $\omega \mapsto |f(x, \omega)|$ is integrable.

Note that if $\mu$ is a finite measure, and if $f$ is bounded, then it is also locally uniformly integrably bounded. The next result may be found, for instance, in [2, Theorem 24.5, p. 193], Billingsley [4, Theorem 16.8, pp.181–182], or Cramér [5, ¶ II, p. 67–68].

5 Proposition Let $(\Omega,\mathcal{F},\mu)$ be a measure space, let $A \subset \mathbb{R}^n$ be open, and let the function $f : A \times \Omega \to \mathbb{R}$ be a Carathéodory function. Assume further that $f$ is locally uniformly integrably bounded. Then the function $g : A \to \mathbb{R}$ defined by

$$g(x) = \int_{\Omega} f(x, \omega) \, d\mu(\omega)$$

is continuous.

Suppose further that for each $i$ and each $\omega$, the partial derivative $D_i f(x, \omega)$ with respect to $x_i$ is a continuous function of $x$ and $D_i f$ is locally uniformly integrably bounded. Then $g$ is continuously differentiable and

$$D_i g(x) = \int_{\Omega} D_i f(x, \omega) \, d\mu(\omega).$$

Proof: First we deal with continuity. Since $f$ is locally uniformly integrably bounded, for each $x$ there is a nonnegative integrable function $h_x : \Omega \to \mathbb{R}$, and a neighborhood $U_x$ of $x$ such that for all $y \in U_x$, we have $|f(y, \omega)| \leq h_x(\omega)$. Then $|g(x)| \leq \int_{\Omega} h_x(\omega) \, d\mu(\omega) < \infty$. Now suppose
For each $x_n \to x$. Since $f$ is continuous in $x$, $f(x_n, \omega) \to f(x, \omega)$ for each $\omega$. Eventually $x_n$ belongs to $U_x$, so for large enough $n$, $|f(x_n, \omega)| \leq h_x(\omega)$. Then by the Dominated Convergence Theorem,\(^1\)

$$g(x_n) = \int_{\Omega} f(x_n, \omega) \, d\mu(\omega) \to \int_{\Omega} f(x, \omega) \, d\mu(\omega) = g(x).$$

That is, $g$ is continuous.

For continuous differentiability, start by observing that $D_i f(x, \omega)$ is measurable in $\omega$ and hence a Carathéodory function. To see this, recall that

$$D_i f(x, \omega) = \lim_{t \to 0} \frac{f(x + te^i, \omega) - f(x, \omega)}{t}.$$  

For each $t$, the difference quotient is a measurable function of $\omega$, so its limit is measurable as well.

Assume that $D_i f(x, \omega)$ is uniformly bounded by the integrable $h_x(\omega)$ on a neighborhood $U_x$ of $x$. Let $e_i$ denote the $i$th unit coordinate vector. By the Mean Value Theorem,\(^2\) for each $\omega$ and for each nonzero $t$ there is a point $\xi(t, \omega)$ belonging to the interior of the segment joining $x$ and $x + te^i$ with

$$f(x + te^i, \omega) - f(x, \omega) = tD_i f(\xi(t, \omega), \omega).$$

Since both functions on the left hand side are measurable, the right-hand side is also a measurable function of $\omega$.\(^3\) For $|t|$ small enough, since $\xi(t, \omega)$ lies between $x$ and $x + te^i$, we must have that $\xi(t, \omega) \in U_x$, so

$$|tD_i f(\xi(t, \omega), \omega)| \leq h_x(\omega).$$

Now

$$g(x + te^i) - g(x) = \int_{\Omega} f(x + te^i, \omega) - f(x, \omega) \, d\mu(\omega) = \int_{\Omega} tD_i f(\xi(t, \omega), \omega) \, d\mu(\omega).$$

As $t \to 0$, we have $\xi(t, \omega) \to x$, so $D_i(\xi(t, \omega), \omega) \to D_i f(x, \omega)$ for each $\omega$. Dividing by $t$ and applying the Dominated Convergence Theorem yields

$$D_i g(x) = \lim_{t \to 0} \frac{g(x + te^i) - g(x)}{t} = \int_{\Omega} D_i f(x, \omega) \, d\mu(\omega).$$

The proof of continuity of $D_i g$ is the same as the proof of continuity of $g$. \hfill \blacksquare

### 3 An application to expected utility

The previous section dealt directly with a function $f$ defined on the Cartesian product of a subset of $\mathbb{R}^n$ and a measurable space $\Omega$. In practice the dependence on $\Omega$ is often via a random vector, which allows for conditions that easier to understand. Here is a common application of these results. See, for instance, Hildreth [9], who refers the reader to Hildreth and Tesfatsion [10] for proofs.

\(^1\)See, for example, Royden [12, Theorem 16, p. 267] or Aliprantis and Border [1, Theorem 11.21, p. 415].

\(^2\)See, for instance, Apostol [3, Theorem 4.5, p. 185]. It is also sometimes known as Darboux’s Theorem.

\(^3\)In fact, by the Stochastic Taylor’s Theorem 8 below we can show that $\omega \mapsto \xi(t, \omega)$ can be taken to be measurable with respect to $\omega$. But that theorem requires a lot of high-powered machinery for its proof, and contrary to my initial instincts we don’t need it for our purposes.
6 Corollary Let $I$ be an interval of the real line with interior $I^\circ$, and let $u: I \to \mathbb{R}$ be strictly increasing, continuous, and concave on $I$, and twice continuously differentiable on $I^\circ$. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $x, y: \Omega \to \mathbb{R}$ be measurable functions (random variables). Let $A$ be an open interval of the real line, and assume that for all $\alpha \in A$ and almost all $\omega \in \Omega$ that

$$x(\omega) + \alpha y(\omega) \in I^\circ.$$  

In addition, assume that for each $\alpha \in A$ that

$$\int_\Omega |u(x(\omega) + \alpha y(\omega))| \, dP(\omega) < \infty,$$

(ii)

$$\int_\Omega |u'(x(\omega) + \alpha y(\omega)) y(\omega)| \, dP(\omega) < \infty,$$

(iii)

$$\int_\Omega |u''(x(\omega) + \alpha y(\omega)) y^2(\omega)| \, dP(\omega) < \infty.$$  

(iv)

Define the function

$$g(\alpha) = \int_\Omega u(x(\omega) + \alpha y(\omega)) \, dP(\omega).$$

Then $g$ is continuously differentiable, and

$$g'(\alpha) = \int_\Omega u'(x(\omega) + \alpha y(\omega)) y(\omega) \, dP(\omega).$$

(1)

If in addition $u''$ is (weakly) increasing, then $g$ is twice continuously differentiable and

$$g''(\alpha) = \int_\Omega u''(x(\omega) + \alpha y(\omega)) y^2(\omega) \, dP(\omega).$$

(2)

Proof: Since $u$ is concave, $u'$ is (weakly) decreasing, and $u'' \leq 0$. It also follows that $u' > 0$ on $I^\circ$. Define $f: A \times \Omega \to \mathbb{R}$ by

$$f(\alpha, \omega) = u(x(\omega) + \alpha y(\omega)).$$

Then $f$ is clearly a Carathéodory function. In order to apply Proposition 5, we need to show that $f$ and $D_1 f$ are locally uniformly integrably bounded. So let $\alpha \in A$ and choose $\delta > 0$ so that $A' = [\alpha - \delta, \alpha + \delta] \subset A$. Since $u$ is strictly increasing,

$$|f(\alpha, \omega)| \leq |f(\alpha - \delta, \omega)| + |f(\alpha + \delta, \omega)| = h_{\alpha}(\omega)$$

for all $\alpha \in A'$. By (ii), $h_{\alpha}$ is integrable. Thus $f$ is uniformly locally integrably bounded, so $g$ continuous.

Similarly, since $u'$ is decreasing

$$|D_1 f(\alpha, \omega)| \leq |D_1 f(\alpha - \delta, \omega)| + |D_1 f(\alpha + \delta, \omega)|$$

for all $\alpha \in A'$, so (iii) implies $D_1 f$ is uniformly locally integrably bounded and the same reasoning implies that $g'$ is continuous and satisfies (1). You can now see how the remainder of the theorem is proven.  

---

This condition is known as prudence in the expected utility literature, as it implies a desire to save more in the face of increased risk. For the purposes of twice differentiability of $g$, we could have assumed that $u''$ is weakly decreasing, but there is no convincing economic interpretation of that condition.

Since $u$ is strictly increasing, $u' > 0$ and it cannot attain a maximum on $I^\circ$. But for concave $u$, the condition $u' = 0$ implies a maximizer. Thus $u' > 0$.  

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4 An illustrative (counter)example

To get an idea of what these conditions mean, consider the following example, taken from Gelbaum and Olmsted [7, Example 9.15, p. 123].

7 Example The following example shows what can go wrong when the hypotheses of the previous theorems are violated.

Define \( f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \) via

\[
f(x, t) = \begin{cases} 
\frac{x^3}{t^2} e^{-x^2/t} & t > 0, \\
0 & t = 0.
\end{cases}
\]

First observe that for fixed \( t \) the function \( x \mapsto f(x, t) \) is continuous at each \( x \), and for each fixed \( x \) the function \( t \mapsto f(x, t) \) is continuous at each \( t \), including \( t = 0 \). (This is because the exponential term goes to zero much faster than the polynomial term goes to zero as \( t \to 0 \).) The function is not jointly continuous though. On the curve \( t = x^2 \) we have \( f(x, t) = e^{-1/x} \), which diverges to \( \infty \) as \( x \downarrow 0 \) and diverges to \( -\infty \) as \( x \uparrow 0 \). See Figure 1.

Define

\[
g(x) = \int_0^1 f(x, t) \, dt
\]

\[
= x^3 \int_0^1 \frac{1}{t^2} e^{-x^2/t} \, dt.
\]

Consulting a table of integrals if necessary, we find the indefinite integral \( \int \frac{1}{t^2} e^{-a/t} \, dt = e^{-a/t}/a \).

Thus, letting \( a = x^2 \) we have

\[
g(x) = xe^{-x^2}
\]

This holds for all \( x \in \mathbb{R} \). Consequently

\[
g'(x) = (1 - 2x^2)e^{-x^2}
\]

again for all \( x \).

Now let’s compute

\[
\int_0^1 D_1 f(x, t) \, dt.
\]

For \( t = 0 \), \( f(x, t) = 0 \) for all \( x \), so \( D_1 f(x, 0) = 0 \). For \( t > 0 \), we have

\[
D_1 f(x, t) = \frac{3x^2}{t^2} e^{-x^2/t} + \frac{x^3}{t^2} e^{-x^2/t} (-2x/t)
\]

\[
= e^{-x^2/t} \left( \frac{3x^2}{t^2} - \frac{2x^4}{t^3} \right).
\]

So

\[
D_1 f(x, t) = \begin{cases} 
\frac{e^{-x^2/t} \left( \frac{3x^2}{t^2} - \frac{2x^4}{t^3} \right)}{t^2} & t > 0 \\
0 & t = 0.
\end{cases}
\]
Note that for fixed $x$ the limit of $D_1f(x,t)$ as $t \downarrow 0$ is zero, so for each fixed $x$, $D_1f(x,t)$ is continuous in $t$. But again, along the curve $t = x^2$, we have $D_1f(x,t) = e^{-1}(3x^{-2} - 2x^{-2}) = -e^{-1}/x^2$ which diverges to $\infty$ as $x \to 0$. Thus $D_1f(x,t)$ is not continuous at $(0,0)$. See Figure 2.

The integral
\[ I(x) = \int_0^1 e^{-x^2/t} \left( \frac{3x^2}{t^2} - \frac{2x^4}{t^3} \right) dt \]

satisfies $I(0) = 0$ and for $x > 0$ it can be computed as
\[
\int_0^1 D_1f(x,t) \, dt = \int_0^1 e^{-x^2/t} \left( \frac{3x^2}{t^2} - \frac{2x^4}{t^3} \right) dt
= 3x^2 \int_0^1 \frac{1}{t^2} e^{-x^2/t} \, dt - 2x^4 \int_0^1 \frac{1}{t^3} e^{-x^2/t} \, dt
\]
so dividing by $x^2 \neq 0$,
\[
= 3e^{-x^2/t} \bigg|_{t=1}^{t=0} - 2e^{-x^2/t} \left( 1 + \frac{x^2}{t} \right) \bigg|_{t=0}^{t=1}
= (1 - 2x^2)e^{-x^2}
\]
which holds for all $x > 0$.

Thus at $x = 0$, we have
\[
g'(0) = (1 - 2 \cdot 0^2)e^{-0^2} = 1 \neq 0 = I(0) = \int_0^1 D_1f(0,t) \, dt.
\]

The remarks above show that $f$ and $D_1f(x,t)$ fail to be continuous at $(0,0)$ so this example does not violate Leibniz’ Rule. How does it compare to the hypotheses of Proposition 5?

In this example $t$ plays the role of $\omega$ in Proposition 5, so locally uniform integrability requires that for each $x$ there is an integrable function $h_x$ and a neighborhood $U_x$ such that $\sup_{y \in U_x} |D_1f(y,t)| \leq h_x(t)$. Let’s check this for $x = 0$. We need to find a $\delta > 0$ so that $|y| < \delta$ implies $|D_1f(y,t)| \leq h_0(t)$. Now for $t > 0$,
\[
D_1f(y,t) = e^{-y^2/t} \left( \frac{3y^2}{t^2} - \frac{2y^4}{t^3} \right).
\]
Looking at points of the form $y = \sqrt{t}$, we see that $h_0(t)$ must satisfy
\[
h_0(t) \geq D_1f(\sqrt{t},t) = e^{-1} \left( \frac{3}{t} - \frac{2}{t^2} \right) = e^{-1}/t,
\]
which is not integrable over any interval $(0,\varepsilon)$, so the hypotheses of Proposition 5 are also violated by this example. □

5 A Stochastic version of Taylor’s Theorem

I used to think the following sort of result was necessary in the proof of Proposition 5, but I was wrong. But I spent a lot of effort figuring out the machinery needed to prove it, so I’m sharing it with you.
**8 Stochastic Taylor’s Theorem** Let $h: [a, b] \to \mathbf{R}$ be continuous and possess a continuous $n$th-order derivative on $(a, b)$. Fix $c \in [a, b]$ and let $X$ be a random variable on the probability space $(S, \mathcal{B}, \mathbf{P})$ such that $c + X \in [a, b]$ almost surely. Then there is a (measurable) random variable $\xi$ satisfying $\xi(s) \in [0, X(s)]$ for all $s$ (where $[0, X(s)]$ is the line segment joining 0 and $X(s)$, regardless of the sign of $X(s)$), and

$$h(c + X(s)) = h(c) + \sum_{k=1}^{n-1} \frac{1}{k!} h^{(k)}(c) X^k(s) + \frac{1}{n!} h^{(n)}(c + \xi(s)) X^n(s).$$

**Proof:** (See [1, Theorem 18.18, p. 603].) Taylor’s Theorem without remainder (see, for instance, Landau [11, Theorem 177, p. 120] or Hardy [8, p. 286]) is a generalization of the Mean Value Theorem that asserts that there is such a $\xi(s)$ for each $s$, the trick is to show that there is a measurable version. To this end define the correspondence $\varphi: S \to \mathbf{R}$ by $\varphi(s) = [0, X(s)]$. It follows from [1, Theorem 18.5, p. 595] that $\varphi$ is measurable and it clearly has compact values. Set $g(s) = h(c + X(s)) - h(c) - \sum_{k=1}^{n-1} \frac{1}{k!} h^{(k)}(c) X^k(s)$, $f(s, x) = \frac{1}{n!} h^{(n)}(c + x) X^n(s)$. Then $g$ is measurable and $f$ is a Carathéodory function. (See section 4.10 of [1] for the definition of measurable correspondences.) By Filippov’s Implicit Function Theorem [1, Theorem 18.17, p. 603] there is a measurable function $\xi$ such that for all $s$, $\xi(s) \in \varphi(s)$ and $f(s, \xi(s)) = g(s)$, and we are done.

**References**


Figure 1. Plots of $\frac{x^3}{t^2}e^{-x^2/t}$. 

Surface of graph. 

Contours.
Surface of graph.

Figure 2. Plots of \( e^{-x^2/t} \left( \frac{3x^2}{t^2} - \frac{2x^4}{t^6} \right) \).