Implicit assumptions

In the traditional analysis of demand functions, there are a number of implicit assumptions that if not made render the analysis vacuous. Among these are:

U.1 The utility function $u: \mathbb{R}^n_+ \rightarrow \mathbb{R}$ is continuous, and twice continuously differentiable on $\mathbb{R}^n_+$.

U.2 At each point $x \gg 0$, we have $u'(x) \gg 0$, which is a strong monotonicity condition.

U.3 The utility satisfies the following strong quasiconcavity condition. At each $x \gg 0$, the Hessian is negative definite on tangent planes to indifference curves. That is, for all $v \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} u_{ij}(x)v_i v_j < 0 \quad v \neq 0 \text{ and } u'(x) \cdot v = 0.$$

(Here and throughout these notes we adopt the notational convention that subscripts can be used to denote partial differentiation, so that $u_i$ denotes $D_i u = \frac{\partial u}{\partial x_i}$, and $u_{ij}$ denotes $D_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j}$.)

This is equivalent to

$$\begin{vmatrix} u_{11} & \ldots & u_{1p} & u_1 \\ \vdots & \vdots & \vdots \\ u_{p1} & \ldots & u_{pp} & u_p \\ u_1 & \ldots & u_p & 0 \end{vmatrix} > 0 \quad p = 2, \ldots, n.$$

In particular,

$$\begin{vmatrix} u_{11} & \ldots & u_{1n} & u_1 \\ \vdots & \vdots & \vdots \\ u_{n1} & \ldots & u_{nn} & u_n \\ u_1 & \ldots & u_n & 0 \end{vmatrix} \neq 0.$$
U.4 If $x \gg 0$, its indifference curve never approaches a point on the boundary of $R_n^+$. Attention is restricted to strictly positive price vectors $p$ and strictly positive income $m$. Under these conditions, the utility maximizing consumption is always unique, satisfies the budget with equality, and is strictly positive. Furthermore, utility maximization and expenditure minimization are equivalent. That is, if $x^*$ maximizes $u(x)$ subject to $m - p \cdot x \geq 0$ and if $\hat{x}$ minimizes $p \cdot x$ subject to $u(x) \geq u^*$, where $u^* = u(x^*)$, then $x^* = \hat{x}$ and $p \cdot x^* = m$. Throughout these notes, $p \in R^n_+$ is a vector of strictly positive prices, and $m > 0$ is a strictly positive.

Utility maximization

The constrained maximization problem is

$$\text{maximize } u(x) \text{ subject to } m - p \cdot x = 0.$$ 

We know there is a unique interior maximizer $x^*$. The gradient of the constraint is $-p \neq 0$, so the Lagrange Multiplier Theorem applies. Thus there is a Lagrange multiplier $\lambda^*$ so that the first-order conditions

$$u_i(x^*) - \lambda^* p_i = 0 \quad i = 1, \ldots, n$$

are satisfied. Since $p \gg 0$ and each $u_i > 0$ by assumption U.2, we have

$$\lambda^* > 0.$$ 

The second-order conditions are that the Hessian matrix $[u_{ij}]$ be negative semidefinite under constraint, more specifically

$$\sum_i \sum_j u_{ij}(x^*) v_i v_j \leq 0 \quad \text{for all } v \text{ such that } \sum_i (-p_i) v_i = 0.$$ 

Since the first-order condition implies $u_i(x^*) = \lambda^* p_i$, we see that assumption U.3 guarantees that the strong second-order conditions are satisfied.

Consider the function $g: R^{n+1} \times R^{n+1} \rightarrow R^{n+1}$ defined by the first-order conditions,

$$g^i(x, \lambda; p, m) = u_i(x) - \lambda p_i$$

for $i = 1, \ldots, n$, and the constraint,

$$g^{n+1}(x, \lambda; p, m) = m - p \cdot x.$$ 

The Jacobian of this function with respect to $(x, \lambda)$ is

$$\begin{vmatrix} u_{11} & \cdots & u_{1n} & -p_1 \\ \vdots & \vdots & \vdots & \vdots \\ u_{n1} & \cdots & u_{nn} & -p_n \\ -p_1 & \cdots & -p_n & 0 \end{vmatrix} = \begin{vmatrix} u_{11} & \cdots & u_{1n} & -\frac{1}{\lambda} u_1 \\ \vdots & \vdots & \vdots & \vdots \\ u_{n1} & \cdots & u_{nn} & -\frac{1}{\lambda} u_n \\ -\frac{1}{\lambda} u_1 & \cdots & -\frac{1}{\lambda} u_n & 0 \end{vmatrix} = \frac{1}{\lambda^2} \begin{vmatrix} u_{11} & \cdots & u_{1n} & u_1 \\ \vdots & \vdots & \vdots & \vdots \\ u_{n1} & \cdots & u_{nn} & u_n \\ u_1 & \cdots & u_n & 0 \end{vmatrix}.$$
where the first equality follows from the first-order conditions, and the second by multiplying
the last row and last column by $-\lambda$. It follows from $\lambda^* > 0$ and U.3 this determinant
is nonzero at $(x^*, \lambda^*)$, so by the Implicit Function Theorem, since $u$ is $C^2$, then $x^*$ and $\lambda^*$ are $C^1$
functions of $(p, m)$, at least locally. Thus the first-order conditions imply that

$$u_i(x^*(p, m)) - \lambda^*(p, m)p_i = 0 \quad i = 1, \ldots, n.$$  \hspace{1cm} (FOC)

and

$$m - \sum_i p_ix_i^*(p, m) = 0$$

for all $(p, m)$.

The left-hand side of each of these first-order conditions can be viewed as a constant function
of $(p, m)$, namely the zero function. So for each commodity $i$ and each price $p_j$, differentiate
the left-hand side of the first order condition for $x_i$ with respect to $p_j$ to get

$$\sum_k u_{ik} \frac{\partial x^*_k}{\partial p_j} - \frac{\partial \lambda^*}{\partial p_j} p_i - \lambda^* \delta_{ij} = 0 \quad i = 1, \ldots, n, j = 1, \ldots, n.$$  \hspace{1cm} (1'')

and the constraint to get

$$-\sum_k p_k \frac{\partial x^*_k}{\partial p_j} - x^*_j = 0 \quad j = 1, \ldots, n.$$  \hspace{1cm} (2'')

Differentiate each left-hand side with respect to $m$ to get

$$\sum_k u_{ik} \frac{\partial x^*_k}{\partial m} - \frac{\partial \lambda^*}{\partial m} p_i = 0 \quad i = 1, \ldots, n.$$  \hspace{1cm} (3'')

and

$$1 - \sum_k p_k \frac{\partial x^*_k}{\partial m} = 0.$$  \hspace{1cm} (4'')

For aesthetic reasons that will become clear in a moment, I want to use the first-order
conditions $u_i = \lambda^* p_i$ and do a little regrouping:

$$\sum_k u_{ik} \frac{\partial x^*_k}{\partial p_j} + u_i \frac{-1}{\lambda^*} \frac{\partial \lambda^*}{\partial p_j} p_i = \lambda^* \delta_{ij} \quad i = 1, \ldots, n, j = 1, \ldots, n.$$  \hspace{1cm} (1')

$$\sum_k u_k \frac{\partial x^*_k}{\partial p_j} = -\lambda^* x^*_j \quad j = 1, \ldots, n.$$  \hspace{1cm} (2')

$$\sum_k u_{ik} \frac{\partial x^*_k}{\partial m} + u_i \frac{-1}{\lambda^*} \frac{\partial \lambda^*}{\partial m} = 0 \quad i = 1, \ldots, n.$$  \hspace{1cm} (3')

$$\sum_k u_k \frac{\partial x^*_k}{\partial m} = \lambda^*.$$  \hspace{1cm} (4')
We can view these in terms of $n$-vectors and rewrite them as

$$
\left[ u_{1i}, \ldots, u_{in}, u_i \right] \cdot \left[ \frac{\partial x_1^*}{\partial p_j}, \ldots, \frac{\partial x_n^*}{\partial p_j} \right] = \lambda^* \delta_{ij} \quad i = 1, \ldots, n, \quad j = 1, \ldots, n. \tag{1}
$$

$$
\left[ u_1, \ldots, u_n, 0 \right] \cdot \left[ \frac{\partial x_1^*}{\partial p_j}, \ldots, \frac{\partial x_n^*}{\partial p_j} \right] = -\lambda^* x_j^* \quad j = 1, \ldots, n. \tag{2}
$$

$$
\left[ u_{1i}, \ldots, u_{in}, u_i \right] \cdot \left[ \frac{\partial x_1^*}{\partial m}, \ldots, \frac{\partial x_n^*}{\partial m} \right] = 0 \quad i = 1, \ldots, n. \tag{3}
$$

$$
\left[ u_1, \ldots, u_n, 0 \right] \cdot \left[ \frac{\partial x_1^*}{\partial m}, \ldots, \frac{\partial x_n^*}{\partial m} \right] = \lambda^* \quad i = 1, \ldots, n. \tag{4}
$$

Arranging all this in matrix terms gives

$$
\begin{bmatrix}
  u_{11} & \cdots & u_{1n} & u_1 \\
  \vdots & \ddots & \vdots & \vdots \\
  u_{n1} & \cdots & u_{nn} & u_n \\
  u_1 & \cdots & u_n & 0
\end{bmatrix}
\begin{bmatrix}
  \frac{\partial x_1^*}{\partial p_1} & \cdots & \frac{\partial x_n^*}{\partial p_1} & \frac{\partial x_1^*}{\partial m} \\
  \vdots & \ddots & \vdots & \vdots \\
  \frac{\partial x_n^*}{\partial p_1} & \cdots & \frac{\partial x_n^*}{\partial p_n} & \frac{\partial x_n^*}{\partial m} \\
  \frac{-1}{\lambda^*} \frac{\partial \lambda^*}{\partial p_1} & \cdots & \frac{-1}{\lambda^*} \frac{\partial \lambda^*}{\partial p_n} & \frac{-1}{\lambda^*} \frac{\partial \lambda^*}{\partial m}
\end{bmatrix}
= \begin{bmatrix}
  \lambda^* & 0 & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \lambda^* & 0 \\
  -\lambda^* x_1^* & \cdots & -\lambda^* x_n^* & \lambda^*
\end{bmatrix}. \tag{5}
$$

The entries in the $(n+1) \times (n+1)$ right-hand side matrix correspond to the equations (1)–(4) according to this scheme:

$$
\begin{bmatrix}
  (1)_{n\times n} & (3)_{n\times 1} \\
  (2)_{1\times n} & (4)_{1\times 1}
\end{bmatrix}
$$

Solving this gives

$$
\begin{bmatrix}
  \frac{\partial x_1^*}{\partial p_1} & \cdots & \frac{\partial x_n^*}{\partial p_1} & \frac{\partial x_1^*}{\partial m} \\
  \vdots & \ddots & \vdots & \vdots \\
  \frac{\partial x_n^*}{\partial p_1} & \cdots & \frac{\partial x_n^*}{\partial p_n} & \frac{\partial x_n^*}{\partial m} \\
  \frac{-1}{\lambda^*} \frac{\partial \lambda^*}{\partial p_1} & \cdots & \frac{-1}{\lambda^*} \frac{\partial \lambda^*}{\partial p_n} & \frac{-1}{\lambda^*} \frac{\partial \lambda^*}{\partial m}
\end{bmatrix}
^{-1}
\begin{bmatrix}
  u_{11} & \cdots & u_{1n} & u_1 \\
  \vdots & \ddots & \vdots & \vdots \\
  u_{n1} & \cdots & u_{nn} & u_n \\
  u_1 & \cdots & u_n & 0
\end{bmatrix}
= \begin{bmatrix}
  \lambda^* & 0 & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \lambda^* & 0 \\
  -\lambda^* x_1^* & \cdots & -\lambda^* x_n^* & \lambda^*
\end{bmatrix}. \tag{6}
$$
Set
\[ A = \begin{bmatrix}
  u_{11} & \ldots & u_{1n} & u_1 \\
  \vdots & \ddots & \vdots & \vdots \\
  u_{n1} & \ldots & u_{nm} & u_n \\
  u_1 & \ldots & u_n & 0
\end{bmatrix}^{-1}, \]
which exists by assumption U.3. Then we have
\[ \frac{\partial x^*_i}{\partial m} = \lambda^{*} a_{i,n+1} \]
and
\[ \frac{\partial x^*_i}{\partial p_j} = \lambda^{*} a_{ij} - \lambda^{*} x^*_j a_{i,n+1} = \lambda^{*} a_{ij} - x^*_j \frac{\partial x^*_i}{\partial m}. \]
In particular,
\[ \frac{\partial x^*_i}{\partial p_i} = \lambda^{*} a_{ii} - x^*_i \frac{\partial x^*_i}{\partial m}. \] (7)

The natural question is, what is the economic interpretation of \( \lambda^{*} a_{ij} \)? The answer lies in the expenditure minimization problem.

Utility maximization and expenditure minimization

Fix \((p, m)\) and let \( v = u(x^*(p, m)) = v(p, m) \). Consider the problem
\[ \min_{x} p \cdot x \quad \text{subject to} \quad u(x) \geq v. \]
When is this problem equivalent to the utility maximization problem?

To answer that let me introduce a new definition. We say that the utility function \( u: X \to R \) is \textbf{locally nonsatiated at} \( x \) if for every \( \varepsilon > 0 \), there is some \( z \in X \) satisfying \( \|x - z\| < \varepsilon \) and \( u(z) > u(x) \). (Note that this is a joint assumption on \( X \) and \( u \).)

**Lemma 1 (Budget exhaustion)** Let \( u: X \to R \) and suppose \( \bar{x} \in X \) maximizes \( u(x) \) over the budget set \( \beta(p, m) = \{ x \in X : p \cdot x \leq m \} \). If \( u \) is locally nonsatiated at \( \bar{x} \), then \( \bar{x} \) exhausts the budget, that is,
\[ p \cdot \bar{x} = m. \]

**Proof:** If \( p \cdot \bar{x} < m \), then there is an \( \varepsilon > 0 \) such that \( y \in X \) and \( \|y - \bar{x}\| < \varepsilon \) implies \( p \cdot y < m \), and thus \( y \in \beta(p, m) \). Thus \( u(\bar{x}) \geq u(y) \) for all such \( y \), so \( u \) is not locally nonsatiated at \( \bar{x} \). The lemma now follows by contraposition.

**Proposition 1** Let \( u: R^n_+ \to R \) be locally nonsatiated everywhere.

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1. If $\bar{x}$ maximizes $u(x)$ subject to $p \cdot x \leq m$, then $\bar{x}$ minimizes $p \cdot x$ subject to $u(x) \geq u(\bar{x})$.

2. If $u$ is also continuous, and $\bar{x}$ minimizes $p \cdot x$ subject to $u(x) \geq u(\bar{x})$, and $p \cdot \bar{x} > 0$, then $\bar{x}$ maximizes $u(x)$ subject to $p \cdot x \leq p \cdot \bar{x}$.

Proof: (1) Assume $\bar{x}$ maximizes $u(x)$ subject to $p \cdot x \leq m$. Then clearly $u(y) > u(\bar{x})$ implies $p \cdot y > p \cdot \bar{x}$. Now suppose by way of contradiction that $u(y) = u(\bar{x})$, but $p \cdot y < p \cdot \bar{x}$. Since $u$ is locally nonsatiated at $y$ there is some $z \in X$ close to $y$ with $p \cdot z < p \cdot \bar{x}$ and $u(z) > u(y) = u(x)$, which contradicts the maximality of $\bar{x}$ over he budget set.

(2) Assume $\bar{x}$ minimizes $p \cdot x$ subject to $u(x) \geq u(\bar{x})$, and $p \cdot \bar{x} > 0$. Then clearly $p \cdot y < p \cdot \bar{x}$ implies $u(y) < u(\bar{x})$. Now consider the case $p \cdot y = p \cdot \bar{x} > 0$. The for $0 < \lambda < 1$ we have $p \cdot \lambda y < p \cdot \bar{x}$, so $u(\lambda y) < u(\bar{x})$. Since $u$ is continuous $u(y) \leq u(\bar{x})$. Thus $p \cdot y \leq p \cdot \bar{x}$ implies $u(y) \leq u(\bar{x})$.

The assumption that $p \cdot \bar{x} > 0$ in part (2) above is needed, if we wish to allow nonnegative price vectors that are not strictly positive. For instance, let $u(x, y) = x + \sqrt{y}$, and $p = (0, 1)$. Then $(1, 0)$ minimizes $p \cdot (x, y)$ over $R^2_+$ subject to $u(x, y) \geq 1$ as $p \cdot (1, 0) = 0$, but it does not maximize $u$ subject to $p \cdot (x, y) \leq 0$, since $u(x, 0) = x$ and $p \cdot (x, 0) = 0$ for all $x$.

**Expenditure minimization**

Let $\hat{x}(p, v)$ minimize $p \cdot x$ subject to $u(x) - v = 0$, so that $\hat{x}$ minimizes the cost of achieving utility level $v$. The Lagrangean for this is

$$p \cdot x - \mu(u(x) - v)$$

and by the Lagrange Multiplier Theorem first-order conditions are (multiplying by $-1$)

$$p_i - \mu u_i(\hat{x}) = 0 \quad i = 1, \ldots, n,$$

and the second-order conditions for a minimum are

$$-\sum_i \sum_j u_{ij}(\hat{x})v_i v_j \geq 0 \quad \text{for all } v \text{ such that } \sum_i u_i(\hat{x})v_i = 0.$$

Again assumption U.3 guarantees the second-order conditions are satisfied, and that the Jacobian of the system is nonsingular.

Differentiate each of the first-order conditions with respect to $p_j$ to get

$$\delta_{ij} - \mu \sum_k u_{ik} \frac{\partial \hat{z}_k}{\partial p_j} - \frac{\partial \mu}{\partial p_j} u_i = 0 \quad i = 1, \ldots, n, \quad j = 1, \ldots, n$$

or, dividing by $\mu$,

$$\sum_k u_{ik} \frac{\partial \hat{z}_k}{\partial p_j} + \frac{\partial \mu}{\partial p_j} = \delta_{ij} \quad i = 1, \ldots, n, \quad j = 1, \ldots, n.$$

Differentiating with respect to $v$ to get

$$-\mu \sum_k u_{ik} \frac{\partial \hat{z}_k}{\partial v} - \frac{\partial \mu}{\partial v} u_i = 0 \quad i = 1, \ldots, n.$$
or, dividing by $-\dot{\mu}$,
\[
\sum_k u_{ik} \frac{\partial \dot{x}_k}{\partial v} + \frac{\partial \dot{\mu}}{\partial v} u_i = 0 \quad i = 1, \ldots, n.
\]
Now take the constraint $u(\dot{x}) - \nu = 0$, and differentiate with respect to $p_j$ to get
\[
\sum_k u_k \frac{\partial \dot{x}_k}{\partial p_j} = 0 \quad j = 1, \ldots, n,
\]
so $\sum_k \frac{\partial \dot{x}_k}{\partial p_j} u_i = 0$, and differentiate with respect to $\nu$ to get
\[
\sum_i u_i \frac{\partial \dot{x}_i}{\partial \nu} - 1 = 0.
\]
Arranging in matrix terms gives
\[
\begin{bmatrix}
  u_{11} & \cdots & u_{1n} & 1/\mu u_1 \\
  \vdots & \ddots & \vdots & \vdots \\
  u_{n1} & \cdots & u_{nn} & 1/\mu u_n \\
  1/\mu u_1 & \cdots & 1/\mu u_n & 0
\end{bmatrix}
\begin{bmatrix}
  \frac{\partial \dot{x}_1}{\partial p_1} & \cdots & \frac{\partial \dot{x}_1}{\partial p_n} & \frac{\partial \dot{x}_1}{\partial \nu} \\
  \vdots & \ddots & \vdots & \vdots \\
  \frac{\partial \dot{x}_n}{\partial p_1} & \cdots & \frac{\partial \dot{x}_n}{\partial p_n} & \frac{\partial \dot{x}_n}{\partial \nu} \\
  1/\mu \frac{\partial \mu}{\partial p_1} & \cdots & 1/\mu \frac{\partial \mu}{\partial p_n} & 1/\mu \frac{\partial \mu}{\partial \nu}
\end{bmatrix}
= \begin{bmatrix}
  1/\mu & 0 & 0 \\
  \vdots & \ddots & \vdots \\
  0 & 1/\mu & 0 \\
  0 & \cdots & 0 & 1/\mu
\end{bmatrix}
\]
Once again, let’s rearrange things to get
\[
\begin{bmatrix}
  \frac{\partial x^*_1}{\partial p_1} & \cdots & \frac{\partial x^*_1}{\partial p_n} & \frac{\partial x^*_1}{\partial \nu} \\
  \vdots & \ddots & \vdots & \vdots \\
  \frac{\partial x^*_n}{\partial p_1} & \cdots & \frac{\partial x^*_n}{\partial p_n} & \frac{\partial x^*_n}{\partial \nu} \\
  1/\mu^* \frac{\partial \mu^*}{\partial p_1} & \cdots & 1/\mu^* \frac{\partial \mu^*}{\partial p_n} & 1/\mu^* \frac{\partial \mu^*}{\partial \nu}
\end{bmatrix}
= \frac{1}{\mu}
\begin{bmatrix}
  u_{11} & \cdots & u_{1n} & u_1 \\
  \vdots & \ddots & \vdots & \vdots \\
  u_{n1} & \cdots & u_{nn} & u_n \\
  u_1 & \cdots & u_n & 0
\end{bmatrix}^{-1}
\]
We know from our results on matrices negative definite under constraint that the matrix
\[
\begin{bmatrix}
  \frac{\partial \dot{x}_1}{\partial p_1} & \cdots & \frac{\partial \dot{x}_1}{\partial p_n} \\
  \vdots & \ddots & \vdots \\
  \frac{\partial \dot{x}_n}{\partial p_1} & \cdots & \frac{\partial \dot{x}_n}{\partial p_n}
\end{bmatrix}
\]
is negative semidefinite of rank $n - 1$. Consequently, for each $i$,
\[
\frac{\partial \dot{x}_i}{\partial p_i} \leq 0.
\]
Moreover we also know that the null space of this matrix is the one-dimensional linear space spanned by $u'(\dot{x})$. 

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Combining the two

Look at equivalent expenditure minimization and utility maximization problems. That is, set \( v = u(x^*(p, m)) \). Then

\[
x^*(p, m) = \hat{x}(p, v) \quad \text{and} \quad \lambda^* = \frac{1}{\hat{\mu}}.
\]

Thus we have just shown in (7) that

\[
\frac{\partial x_i^*}{\partial p_j} = \frac{\partial \hat{x}_i}{\partial p_j} - \frac{x_j^*}{m} \frac{\partial x_i^*}{\partial m},
\]

a formula known as the **Slutsky decomposition**.

Now define the **expenditure function**

\[
e(p, v) = p \cdot \hat{x}(p, v),
\]

and observe that

\[
\frac{\partial e}{\partial p_j} = \sum_{i=1}^n p_i \frac{\partial \hat{x}_i}{\partial p_j} + \hat{x}_j.
\]

Now cleverly notice that

\[
\begin{bmatrix}
\frac{\partial \hat{x}_1}{\partial p_1} & \cdots & \frac{\partial \hat{x}_1}{\partial p_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial \hat{x}_n}{\partial p_1} & \cdots & \frac{\partial \hat{x}_n}{\partial p_n}
\end{bmatrix}
\begin{bmatrix}
p_1 \\
\vdots \\
p_n
\end{bmatrix} = 0
\]

since \( p = \hat{\mu}u'(\hat{x}) \) is in the null space. Thus,

\[
\frac{\partial e(p, v)}{\partial p_j} = \hat{x}_j(p, v).
\]

Also, defining the **indirect utility function**

\[
v(p, m) = u(x^*(p, m)),
\]

we have

\[
\frac{\partial v}{\partial p_j} = \sum_{i=1}^n u_i \frac{\partial x_i^*}{\partial p_j} = \sum_{i=1}^n \lambda^* p_i \frac{\partial x_i^*}{\partial p_j},
\]

and

\[
\frac{\partial v}{\partial m} = \sum_{i=1}^n u_i \frac{\partial x_i^*}{\partial m} = \sum_{i=1}^n \lambda^* p_i \frac{\partial x_i^*}{\partial m} = \lambda^*.
\]
Therefore

\[
\frac{\partial v}{\partial p_j} = \frac{\partial v}{\partial m} \sum_{i=1}^{n} \lambda^*_i p_i \frac{\partial x_i^*}{\partial p_j}
\]

\[
= \sum_{i=1}^{n} \lambda^*_i p_i \left( \frac{\partial \hat{x}_i}{\partial p_j} - x_j^* \frac{\partial x_i^*}{\partial m} \right)
\]

\[
= \sum_{i=1}^{n} \lambda^*_i p_i \frac{\partial \hat{x}_i}{\partial p_j} - x_j^* \sum_{i=1}^{n} \lambda^*_i p_i \frac{\partial x_i^*}{\partial m}
\]

\[
= 0 - x_j^* \cdot (1).
\]

Which gives us Roy’s Law:

\[
x_j^* = -\frac{\partial v}{\partial p_j}. \quad \frac{\partial v}{\partial m}
\]