Introduction: One-to-one matchings

A Solution to Matching with Preferences over Colleagues
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The Model

- a finite set $W$ of workers,
- a finite set $F$, disjoint from $W$, of firms,
- a preference profile $P = (P(a))_{a \in W \cup F}$, where $P(a)$ is a strict preference relation over $F \cup \{\emptyset\}$ if $a \in W$, and over $W \cup \{\emptyset\}$ if $a \in F$.

**Notation:** $b'R(a)b$ if $b' = b$ or $b'P(a)b$. 
A *matching* $\mu$ is a mapping from $F \cup W$ into $F \cup W \cup \{\emptyset\}$ s.t.

1. $\mu(w) \in F \cup \{\emptyset\}$.
2. $\mu(f) \in W \cup \{\emptyset\}$.
3. $f = \mu(w)$ iff $w = \mu(f)$. 
Stability

• A matching $\mu$ is *individually rational* if $\mu(a)R(a)\emptyset \forall a$.

• A pair $(w, f)$ *blocks* $\mu$ if $w \neq \mu(f)$,

  \[ wP(f)\mu(f) \text{ and } fP(w)\mu(w). \]

• A matching $\mu$ is *stable* if it is individually rational and there is no pair that blocks $\mu$.

\[
\text{Set of stable matchings} = \text{the core}
\]
A *prematching* $\nu$ is a mapping from $F \cup W$ into $F \cup W \cup \{\emptyset\}$ s.t.

1. $\nu (w) \in F \cup \{\emptyset\}$.
2. $\nu (f) \in W \cup \{\emptyset\}$.

Let \( \mathcal{V} = \text{set of all prematchings}. \)

_A prematching is a fantasy_
Construct $T : \mathcal{V} \rightarrow \mathcal{V}$.

- $U(f, \nu) = \{ w : fR(w) \nu(w) \} \cup \{ \emptyset \}$
- $V(w, \nu) = \{ f : wR(f) \nu(f) \} \cup \{ \emptyset \}$

\[
(T\nu)(f) = \max_{P(f)} U(f, \nu) \\
(T\nu)(w) = \max_{P(w)} V(w, \nu)
\]
\( \nu = T\nu \)
Order prematchings by $\leq_F$:

$\nu \leq_F \nu'$ iff

- $\nu'(f)R(f)\nu(f)$ for all $f$
- $\nu(w)R(w)\nu'(w)$ for all $w$
Let $\nu \leq_F \nu'$

$$w \in U(f, \nu) \implies fR(w)\nu(w)R(w)\nu'(w)$$
$$\implies w \in U(f, \nu')$$

So $U(f, \nu) \subseteq U(f, \nu')$.

Similarly, $V(w, \nu') \subseteq V(w, \nu)$.

$T$ is monotone increasing.
$\mathcal{E} = \{\nu : \nu = T\nu\}$

- $\mathcal{E}$ is a nonempty lattice
- $T$-algorithm finds a matching in $\mathcal{E}$. 
Matching with Preferences over Colleagues
The Model. \( \langle C, S, P \rangle \)

- a set \( C \) of colleges
- a set \( S \) of students
- preferences \( P(c) \) over \( 2^S \), for each \( c \)
  preferences \( P(s) \) over \( (C \times 2^S) \cup \{ (\emptyset, \emptyset) \} \)
\[ S_s = \{ S' \subseteq S : S' \ni s \} \]

A matching \( \mu \) is a mapping on \( C \cup S \) s.t.

\begin{itemize}
  \item \( \mu(s) \in C \times S_s \cup \{(\emptyset, \emptyset)\} \)
  \item \( \mu(c) \in 2^S \)
  \item \( s \in \mu(c) \Rightarrow \mu(s) = (c, \mu(c)) \)
  \item \( \mu(s) = (c, S') \Rightarrow \mu(c) = S' \).
\end{itemize}
\((B, c) \in 2^S \times C \) blocks* \(\mu\) if
\[B \cap \mu(c) = \emptyset\]
\[\exists A \subseteq \mu(c) \text{ s.t. } \forall s' \in A \cup B, (c, A \cup B)P(s')\mu(s')\]
\[A \cup BP(c)\mu(c).\]

\(\mu\) is in the \textit{core} if it is IR and there is no block* of \(\mu\).
Example – empty core.

\( C = \{c_1, c_2\}, \ A = \{s_1, s_2, s_3\} \)

\[
P(c_1) : \quad s_1s_2, s_1s_3, s_1, s_2 \\
P(c_2) : \quad s_2s_3, s_3, s_2
\]

\[
P(s_1) : \quad (c_1, s_1s_2), (c_1, s_1s_3), (c_1, s_1) \\
P(s_2) : \quad (c_2, s_2s_3), (c_1, s_1s_2), (c_1, s_2), (c_2, s_2) \\
P(s_3) : \quad (c_1, s_1s_3), (c_2, s_2s_3), (c_2, s_3)
\]
Need very strong assumptions to guarantee nonemptyness.

**RESULTS:**

- Algorithm finds the core match. if the exist.
- Algorithm is efficient when we can ensure nonemptyness.
- “Partial” solutions.
Fixed-point approach.

- prematchings
- $T$
- fixed points of $T = \text{core}$
\begin{align*}
U(c, \nu) &= \{S' \subseteq S : \forall s \in S', (c, S') R(s) \nu(s)\} \\
V(s, \nu) &= \{(c, S') \in C \times 2^S : s \in S', \forall s' \in S'\setminus\{s\}(c, S') R(s') \nu(s') \\
&\quad \text{and } S' R(c) \nu(c)\} \cup \{\emptyset \times \emptyset\} \\
(T \nu)(a) &= \max_{P(a)} \ldots
\end{align*}
Theorem. The core is the set of fixed points of $T$. 
• $\nu \leq \nu'$ if everyone prefers $\nu'$
• $T$ is decreasing
• $T^2$ is increasing

$\mathcal{E}(T^2)$ is a nonempty complete lattice
Algorithm: find matchings in $\mathcal{E}(T^2)$.

Will find the core, if nonempty.
May miss some matchings in $\mathcal{E}(T^2) \setminus \mathcal{E}(T)$.
Partial solutions

μ is in the core with singles if, for any block* \((c, D)\) of \(μ\),

\[
\mu(c) = \emptyset \\
\forall s \in D \; \mu(s) = (\emptyset, \emptyset)
\]

Let \(μ\) be a matching.

**Theorem.** \(μ \in \mathcal{E}(T^2) \Rightarrow \mu \text{ is in the core with singles.}\)

Let \(μ\) be a matching w/no single agents.

**Corollary.** \(μ \text{ is a core matching iff } μ \in \mathcal{E}(T^2).\)
Partial solutions — 2

Let \( C_\nu \subseteq C \) be the set \( c \) s.t. \((c, \nu(c)) = \nu(s) \forall s \in \nu(c)\). Let
\[ S_\nu = \bigcup_{c \in C_\nu} \nu(c). \]

Let \( \nu \in \mathcal{E}(T^2) \).

**Proposition.** \( \nu \) on \( C_\nu \cup S_\nu \) is in the core of \( \langle C_\nu, S_\nu, P|_{C_\nu \cup S_\nu} \rangle \).

**Proposition.** Let \( \mu \) be in the core with singles, and let \( C' \) and \( S' \) denote the agents who are single in \( \mu \). If \( \mu' \) is in the core with singles of \( \langle C', S', P|_{C' \cup S'} \rangle \), then the matching \((\mu, \mu')\), which matches \( C' \) and \( S' \) according to \( \mu' \), and \( C \setminus C' \) and \( S \setminus S' \) according to \( \mu \), is in the core with singles of \( \langle C, S, P \rangle \).
Restrictions on Preferences

$P$ satisfies the \textit{weak top-coalition property}: \exists a partition $(A_1, A_2, ..., A_k)$ of agents s.t.

\forall a \in A_1, A_1 \text{ is } a \text{'s top choice}
\forall a \in A_i, A_i \text{ is } a \text{'s top choice, within } C \cup S - A_1 - ... - A_{i-1}

$P$ is \textit{respecting} if \exists P_S \text{ over } 2^S, \text{ and } P_C \text{ over } C, \text{ s.t.}

1. \forall s \in S, (c, S)P(s)(c, S') \iff S P_S S'.
2. \forall s \in S, (c, S)P(s)(c', S) \iff c P_C c'.
3. \forall c \in C, S P(c)S' \iff S P_S S'.

\textbf{Proposition.} respecting \Rightarrow \textit{weak top-coalition property}. 
Restrictions on Preferences

$\langle C, S, P \rangle$ satisfies the weak top coalition prop.

**Theorem.** *There is a unique core matching $\mu$*

$\mu$ *is the largest fixed point of* $T^2$

*$T^2$ algorithm finds $\mu$ in at most $\lfloor k/2 \rfloor$ steps.*
Restrictions on Preferences

Order prematchings in usual way.
Suppose \( T \) is monotone.

**Proposition.** \( \exists \) core matchings \( \underline{\mu} \) and \( \bar{\mu} \) s.t. \( \forall \nu \in \mathcal{E}(T^2) \),

\[
\bar{\mu}(c) R(c) \nu(c) R(c) \underline{\mu}(c)
\]

\[
\underline{\mu}(s) R(s) \nu(s) R(s) \bar{\mu}(s)
\]
Comparing algorithms

- $T^2$ algorithm: speed depends on number of iterations.
- exhaustive search: search all matchings
e.g. with 1200 students and 9 colleges, there are $1.233 \times 10^{1145}$ matchings.
Couples

Extension of our model to matching with couples.

- Algorithm.
- New result:
  Substitutability $\Rightarrow$ Core $=$ Pairwise Stab.