Optimal investing with perceived mispricing *

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Abstract

We derive optimal portfolio weights for an investor who has a strong belief on the distribution of the stock price at a future time. That distribution may be in disagreement with standard equilibrium pricing models, and the investor wants to take advantage of the perceived mispricing and attractive risk premium. We compute numerically optimal weights for models in which the investor believes that there is a range in which the price is likely to remain. As practiced by active managers, the optimal strategy is to take significant long/short positions as the price nears its lower/upper boundary. The risk, expected stock return and the optimal investments strategy are thus dependent on the stock price relative to the boundaries.

Keywords: Investor beliefs, mispricing, optimal portfolios, range reversion, risk premium.

JEL Classification:

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1 Introduction

Actively managed mutual funds are an essential part of the financial industry. From the standpoint of an active investor, there is a fundamental paradox with respect to the application of equilibrium asset pricing models. To model the relation between risk and expected returns, asset pricing models employ highly restrictive assumptions. The most stringent of these assumptions, employed by most standard models, are that investors have homogeneous beliefs and defined over investments opportunities that are either stationary or whose variation is determined by stationary stochastic processes. Given these assumptions, equilibrium asset pricing equations can be derived that provide a precise definition of risk and its relation to asset expected returns\(^1\). The paradox is that the reason for being an active investor is the belief that some combination of asset price volatility and nonstationarity, incomplete and asymmetric information, and heterogeneous information processing capabilities and beliefs, lead to inefficient pricing that can be exploited\(^2\). This implies that, from the standpoint of such active investors, asset pricing models must be incorrect. However, if standard asset pricing models are not applicable, how is the active investor to measure risk? This problem has been recognized in various contexts by numerous professional investors. For example, as reported by Penman (2007), famed investor Warren Buffett puts the matter this way: “The CAPM says that if the price of a stock drops more than the market, it has a high beta: It’s high risk. But if the price goes down because the market is mispricing the stock relative to other stocks, then the stock is not necessarily high risk: The chance of making an abnormal return has increased, and paying attention to fundamentals makes the investor more secure, not less secure.”\(^3\) In a similar vein, Morningstar (2004), a leading provider of investment analysis, rejects asset pricing based measures of risk, stating that, “In deciding the rate to discount future cash flows, we ignore stock-price volatility (which drives most estimates of beta) because we welcome volatility if it offers opportunities to buy a stock at a discount to

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1See Cochrane (2005) for detailed derivations.
2Friedman (1953) argues that security prices reflect fundamental values. Otherwise, if securities are mispriced, as a result of irrational investors’ behavior, rational investors will take advantage of the mispricing, and therefore push prices to their fundamentals. Other researchers, on the other hand, argue that security prices may persistently diverge from their fundamentals. Shiller (1984), De Long et al. (1990) argue that when the transaction costs exceed potential profits then the prices may not converge to their fundamentals. Campbell and Shiller (2001) show that deviations from the fundamentals have provided valuable forecasting information for future stock prices. When the Price/Earnings ratio has been above its historical mean, stock prices tended to fall. Likewise, when the Dividends/Price ratio has been above its mean, stock prices tended to rise.
3This is consistent with the Fama and French (1992) observation that Book-to-Market (B/M) ratios are positively correlated with subsequent stock returns, a relation that is known as the book-to-market effect. That is, value stocks - the stocks with lower B/M ratios yield higher returns compared to growth stocks - stocks with higher B/M ratios.
its fair value. Instead, we focus on the fundamental risks facing a company’s business. Ideally, we’d like our discount rates to reflect the risk of permanent capital loss to the investor. When assigning a cost of equity to a stock, our analysts score a company in the following areas:

- **Financial leverage** - The lower the debt, the better.
- **Cyclicality** - The less cyclical the firm, the better.
- **Size** - We penalize very small firms.
- **Free cash flow** - The higher as a percentage of sales and the more sustainable, the better.”

Both Buffet and Morningstar are getting at the same fundamental point. If markets are inefficient, then it makes little sense to rely on measures of risk derived from equilibrium asset pricing models. At a minimum, such models fail to take account of the fact that when the price of an asset is low, relative to the fundamentals as assessed by the investor, then the risk will be less. Given the focus on construction of equilibrium models, financial economists have not spent much time developing techniques for assessing risk in inefficient markets. This is no doubt due in part to the fact the problem faced by an active investor of measuring risk in an “inefficient” market cannot be unambiguously defined. It depends on the precise nature of the inefficiency that the investor believes exists. Early efforts to integrate perceived superior information into the context of the CAPM floundered on this problem. Despite the inherent ambiguities, modern derivative pricing techniques can provide active investors with useful tools for assessing risk in markets which they believe reflect significant mispricing. With that goal in mind, and in the spirit of the quotes from Warren Buffett and Morningstar, this paper examines a situation in which the active investor believes that he can place bounds on the variation of the price of certain specific stocks through the application of fundamental valuation analysis.

More specifically, we consider two related types of market inefficiency from the standpoint of the active investor. In the first instance, the investor believes that he can determine fundamental, more or less loose bounds on the value of a stock during the time interval, (0,T). If the market price penetrates either of those boundaries, it reverts back toward the boundary. The speed of reversion is varied in our analysis. We refer to this first case as range reversion or RR.

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4 Under the assumption of existence of market frictions and imperfections, Basak and Croitoru (2000) show that mispricing can be sustainable in general equilibrium.

5 As Campbell and Shiller (2001) state it, the mean reversion in the valuation ratios and its forecasting ability for future prices are not new concepts. Mean reversion in fundamentals has been frequently discussed as a forecasting tool for price movements over the last century.

6 Our model reflects the assumption that stock prices can diverge from their fundamentals for prolonged periods. Theorizing this has been a challenge for financial economists. Campbell and Shiller (2001) review the proposed solutions to this problem. Our model is consistent with the Campbell and Shiller (2001) observation that prices rather than fundamentals do most of the adjustment in bringing the ratios back
In the second related case, the active investor believes that as of time 0 he can place strict bounds on the distribution of a stock price at time T. As a result, rather than being lognormal, the time T distribution of the stock price is truncated at the boundaries. More generally, we allow the investor to specify any continuous distribution for the price at time T. This case is referred to as range distribution, RD. In both cases, it is intuitively clear that the active investors will perceive that both the risk and expected return of the stock will be a function of price.

This is in distinction to standard models such as the CAPM or the Black-Scholes-Merton option pricing model in which risk and expected return are not correlated with the price of a security. Furthermore, the distinction between the standard models and the inefficient market models that we analyze becomes more pronounced as the price approaches a boundary. Naturally, the relation of the price to the boundary also affects the optimal holdings of the active investor. We analyze those holdings in two contexts, in which the investor invests in a risky stock, an index and a risk-free asset. The first context is a mean-variance framework in which the portfolio holdings are set at time 0 and cannot be altered. The second is a dynamic optimization context in which the portfolio holdings can be adjusted continuously.

Our analysis is presented as follows. We begin with a formal mathematical discussion of both cases. To simplify the reading, the results are presented without proofs which are available from the authors. In situations where there are no closed form solutions to the problems, we provide Monte Carlo simulations. We report on numerical computations of optimal portfolios in the models, useful for estimating the economic magnitude of the effects we consider. In particular, we quantify the more aggressive behavior of the investor who has a strong belief on the range of the stock price. We also graphically present the dependence of betas, and expectations and variances of relative returns on the position of the stock price towards their long-run equilibrium levels. Thus, our model is consistent with their argument that valuation ratios (fundamental bounds in our case) can be used to predict stock price changes.

Liu and Longstaff (2004) study a similar problem. In their setting, an investor has perfect knowledge about the future value of the security at time T, and thus faces an arbitrage opportunity related to a security, which is modeled as a Brownian Bridge process.

This assumption is consistent with the 3-factor pricing model of Fama and French (1993) in which the factors HML or SMB are functions of the price of the security.


see Black and Scholes (1973), and Merton (1973).

Black and Litterman (see Black and Litterman (1990)) were one of the first to use investors’ subjective views along with CAPM as a benchmark in portfolio management context. Their model uses a Bayesian approach to combine the subjective views of an investor regarding the expected returns of assets with the market equilibrium expected returns (via the prior distribution) to form a new estimate of expected returns. The resulting new set of returns (the posterior distribution) leads to portfolios with more stable weights and thus result in a better asset allocation.
relative to the believed boundaries. We finish with some conclusions and possible extensions.

2 RR Model: “Range Reversion”

In the range reversion model the active investor has a belief regarding an approximate lower bound $L_t$ and an approximate upper bound $U_t$ of a risky asset with price $S_t$. We call asset $S$ the stock. The investor believes that the price will revert back into the range $[L_t, U_t]$ with a certain speed over a given time interval $(0,T)$. We allow $L_t$ and $U_t$ to vary with time, which is more realistic if $T$ is long. This type of modeling is similar to the mean reversion models, except in those models the stock price reverts back to a single value. Consequently, mean reversion models are a special case of our model in which $L_t = U_t$.

Mathematically, the range reversion model can be written as follows. Denote

$$\tilde{L}_t = \log L_t, \quad \tilde{U}_t = \log U_t, \quad Y_t = \log S_t$$

We then model the log-asset price using the Stochastic Differential Equation (SDE)

$$dY_t = \left[\mu_t - \frac{\sigma_t^2}{2} + n_L \max\{0, \tilde{L}_t - Y_t\} - n_U \max\{0, Y_t - \tilde{U}_t\}\right] dt + \sigma_t dB_t \quad (2.1)$$

Here, $B$ is a one-dimensional Brownian motion, and $n_L$ and $n_U$ are the “speeds of range reversion” from the lower bound and the upper bound respectively. If stock price $S$ is below $L$ (log-stock price $Y$ is below $\tilde{L}$) the term containing $n_L$ pushes up the price $S$ back towards $L$. Similarly if $S$ is above $U$ it reverts back toward $U$ at speed $n_U$. Setting $L = -\infty$, $U = \infty$, gives the standard generalized Black-Scholes-Merton model, henceforth BSM model. In the computations we will take $\mu, \sigma$ to be deterministic.

2.1 Optimal investing in the RR model

In this section we compare optimal investments of an investor who uses a BSM model with an investor who uses a RR model. Consider a BSM model for the stock price

$$\hat{S}_t = \hat{S}_0 e^{(\hat{\mu} - \frac{\hat{\sigma}^2}{2})t + \hat{\sigma}B_t}$$

Also assume that there exists a market index $I$, modeled as a geometric Brownian Motion process:

$$I_t = I_0 e^{(\mu_I - \frac{\sigma_I^2}{2})t + \sigma_I B_t + \sqrt{1-\rho^2}W_t}$$

where $W$ is a Brownian Motion independent of $B$. Moreover, assume that there is a risk-free asset with constant interest rate $r \geq 0$.
We assume that both the BSM investor and the RR investor have the same beliefs regarding the index \( I \) and the risk-free asset, as described above. They differ in their beliefs about \( S \).

For the given time horizon \( T > 0 \), we assume that \( \sigma, \mu \) in the RR model (2.1) are constant and that

\[
\tilde{L}_t = \tilde{L} - r(T - t) \quad \tilde{U}_t = \tilde{U} - r(T - t)
\]

where \( \tilde{L}, \tilde{U} \) are constant. In other words, the likely range for the final stock price \( S_T \) is \([L, U] \), and the likely range for the value of \( S_t \) for \( t < T \) is \([L e^{-r(T-t)}, U e^{-r(T-t)}] \).

As the measure of risk we compute the beta of asset \( S^{12} \), defined by

\[
\beta = \frac{Cov[I_T/I_0, S_T/S_0]}{Var[I_T/I_0]} (2.2)
\]

To solve the portfolio optimization problems, we assume that the investors have logarithmic preference in order to simplify the calculations. In practical terms this implies that the investors will be relatively aggressive. Given this framework, we compute optimal investment weights for both a static, Markowitz Mean-Variance portfolio, and for a dynamic, Merton log-optimal portfolio.

To compare our results with those of the standard model, we denote by \( \pi_S, \pi_I \) the optimal weight proportions in assets \( S \) and \( I \) respectively, for the RR model. The optimal weights for the BSM investor are denoted by \( \hat{\pi}_S, \hat{\pi}_I \). The value of the investor's portfolio at time \( T \) is denoted by \( X_T \). Finally, in order to simplify notation, we define the quantities representing return in excess of the risk free return,

\[
\bar{I}_T = I_T - I_0 e^{rT} \quad \bar{S}_T = S_T - S_0 e^{rT} \quad (2.3)
\]

We now first derive formulas for optimal investment strategies, and later we compare numerical results in different models.

### 2.1.1 Optimal mean-variance investment

Consider the Markowitz Mean-Variance setting in which \( \log(X_T/X_0) = \log(1 + (X_T/X_0 e^{rT} - 1)) \) is approximated by

\[
\frac{X_T}{X_0 e^{rT}} - 1 - \frac{\gamma}{2} (\frac{X_T}{X_0 e^{rT}} - 1)^2 \quad (2.4)
\]

where \( \gamma \) is a risk aversion parameter. The investor maximizes expected value of utility of terminal wealth at time \( T \), approximated by the quantity in (2.4).

\(^{12}\)We only consider here partial equilibrium, in which investors take prices as given and cannot influence them; thus, it is not a priori clear that beta is the right measure of risk in this model. We will consider full equilibrium implications of heterogeneous beliefs on the range of prices in future work.
Proposition 2.1 For the investor maximizing expected value of the expression in (2.4), the optimal static portfolio proportions are given by

\[
\pi^M_S = \frac{2 - \gamma}{\gamma} S_0 e^{rT} \frac{E[\bar{S}_T]E[I_T^2] - E[I_T \bar{S}_T]E[I_T]}{E[I_T^2]E[\bar{S}_T^2] - (E[\bar{S}_T I_T])^2} 
\]

(2.5)

\[
\pi^M_I = \frac{2 - \gamma}{\gamma} S_0 e^{rT} \frac{E[I_T]E[\bar{S}_T^2] - E[I_T \bar{S}_T]E[\bar{S}_T]}{E[I_T^2]E[\bar{S}_T^2] - (E[\bar{S}_T I_T])^2} 
\]

(2.6)

and similarly for \( \hat{\pi}^M_S \) and \( \hat{\pi}^M_I \).

**Proof:** Straightforward, and available from authors.

The main reason why the optimal strategies will differ in the RR and the BSM model is that in the BSM model we have, for example, that the expected relative return, is given by

\[
E[\bar{S}_T / S_0] = e^{\hat{\mu}T}
\]

thus independent of the current stock price. This is not the case for the RR model, where the expected return will differ with the location of the stock price relative to the boundaries. Similarly for the correlation of stock \( S \) with index \( I \). We provide numerical results below.

### 2.1.2 Optimal dynamic investment

We now consider proportions \( \pi, \hat{\pi} \) that are optimal for the investor who can rebalance continuously his portfolio (without transaction costs or other frictions).

For the RR model, in the notation

\[
dS_t = \mu^S_t S_t dt + \sigma^S_t S_t dB_t
\]

we have

\[
\sigma^S_t = \sigma_t \\
\mu^S_t = \mu_t + n_L \max\{0, \bar{L}_t - \log S_t\} - n_U \max\{0, \log S_t - \bar{U}_t\}
\]

Introduce the volatility matrix

\[
\Sigma = \begin{pmatrix} \sigma^S & 0 \\ \rho \sigma_I & \sqrt{1 - \rho^2} \sigma_I \end{pmatrix}
\]

and the vector of excess expected return rates

\[
R = \{\mu^S - r, \mu_I - r\}
\]

Then, from the classical Merton’s problem (see Merton (1969, 1971)), we know that the optimal vector of proportions \( \pi \) of wealth to be held in stock for an investor with logarithmic utility is given by

\[
\pi_t = (\Sigma \Sigma')^{-1} R_t 
\]

(2.7)
Similarly for the optimal \( \hat{\pi} \) in the BSM model with \( \mu^S, \sigma^S \) replaced by \( \hat{\mu}, \hat{\sigma} \). We use (2.7) to compute optimal dynamic portfolios.

Again, the main difference is that the parameters entering the computation in (2.7) are independent of the stock price for the BSM model, while this is not true for the RR model.

2.1.3 Comparative Statics

To assess the economic significance of the difference between the RR and BSM models we use the following benchmark parameters which approximate those observed for individual common stocks and the market index:

\[
T = 1, \ r = 0.03, \ I_0 = 1, \ S_0 = 1, \ X_0 = 1, \ \mu = 0.08, \ \sigma = 0.3, \ \mu_I = 0.05, \ \sigma_I = 0.25, \ \rho = 0.2,
\]

\[
\tilde{U} = \log S_0 + \mu T + 2\sigma \sqrt{T}, \ \tilde{L} = \log S_0 + \mu T - 2\sigma \sqrt{T}
\]

We choose \( \hat{\mu}, \hat{\sigma} \) so that the mean and the variance of the stock at time \( T \) are the same in the two models for \( S_0 = \hat{S}_0 = 1 \), that is, so that

\[
E[S_T] = E[\hat{S}_T] = e^{\hat{\mu}T}
\]

\[
E[S_T^2] = E[\hat{S}_T^2] = e^{(2\hat{\mu} + \hat{\sigma}^2)T}
\]

In the calculations, we keep \( \hat{\mu} \) and \( \hat{\sigma} \) fixed at the level corresponding to \( S_0 = 1 \), but allow \( S_0 \) to vary. The interpretation of this procedure is as follows. Suppose the stock price is low today. For the BSM investor (since BSM is a stationary model) this does not change the long-run mean \( \hat{\mu} \) and volatility \( \hat{\sigma} \) (and thus the risk-premium on the stock). Consequently, the optimal investment weights will not change with the stock price. For the RR investor, if the stock price is close to what he believes is its approximate lower bound, the risk premium should be higher than predicted by Merton, and the optimal weight should be greater.

Figures 2.1 to 2.3 present the results of the calculations in graphical form comparing the RR model with the BSM model.\(^\text{13}\) The reader should note that the graphs in this paper often have different scales for X and for Y axis.

Figure 2.1 shows that the investment weights for the RR investor change markedly as the stock price approaches either of the bounds. When the stock is near the bottom boundary the RR investor greatly increases his relative holding in the stock and the reverse is true for the upper boundary.\(^\text{14}\)

\(^{13}\)The portfolio weights for the RR model are actually expected values of the portfolio weights at time \( t = T/2 \). We do this because at time \( t = 0 \) the stock price is inside the believed bounds, and not much interesting happens.

\(^{14}\)This may help explain why fund managers may want to hold less-diversified portfolios. A fund manager that has superior skills or information to estimate the boundaries accurately, may decide to hold more concentrated and less-diversified portfolios. Coval and Moskowitz (1999, 2001) show that mutual funds have strong preferences for investing in local firms where they might have informational advantages. Our findings are consistent with those results.
In order to compensate for this behavior, the RR investor also takes more extreme positions in the index than the BSM investor, but not as extreme as in the stock, because the index has unlimited range in the model\textsuperscript{15}.

Figure 2.2. shows the Merton’s optimal dynamic weights for the log investors of the RR type and the BSM type. The quantitative results are even more dramatic than those for the mean-variance calculation, but the qualitative behavior is the same.

Figure 2.3 shows how the weights change with the speed of range reversion. For low speeds, less than 0.1, the weights are very close to BSM weights. As the RR speeds rise, the behavior becomes more extreme. However, after the value of 100, further variation in the weights is limited.

3 RD Model: Modeling return distribution at a future time

We now present a class of models in which the active investor has a strong belief he places on the future stock price distribution, at a fixed future time $T$. For example, this is consistent with the view that over the long run the investor believes that the stock price will be constrained to be in the vicinity of its fundamental value. We refer to this scenario as the RD model\textsuperscript{16}.

More specifically, given a future time $T$, let $S_T$ be a random variable which represents the active investor’s beliefs regarding the distribution of the stock price at time $T$. We assume that $S_T$ is of the form

$$S_T = f(B^Q_T)$$

where $f$ is a deterministic function and $B^Q$ is a Brownian Motion under a risk-neutral probability. Brownian Motion $B$ under the physical probability is such that

$$B^Q(t) = B(t) + \theta^Q t$$

where the risk premium $\theta^Q$ is assumed to be constant, for simplicity. As usual, under risk-neutral probability the stock price is the expected value of its discounted future value. That

\textsuperscript{15}These findings may be used to explain the investment strategies of corporate executives. Under the assumption that executives are more knowledgeable about the their company’s future prospects, they can better assess the "closeness" of their company’s stock price to fundamental bounds and make investment decisions based on that assessment. This has the potential to explain the stock holding behavior of corporate executives and shed light on the under-diversification phenomena. For example, if an executive believes his firm’s stock is close to the lower bound (or is undervalued relative to the assessed fundamentals), he will hold more shares and thus, be under-diversified.

\textsuperscript{16}Liu and Longstaff (2004) study the investment decisions of an active investor who faces an arbitrage opportunity regarding the future value of a security about which the investor has perfect foresight.
is, we have

\[ S_t = E^Q_t [e^{-r(T-t)} f(B^Q_T)] \] (3.8)

Denote by \( N \) the standard normal cumulative distribution. We have the following useful result, part (ii) of which is based on the Feynman-Kac theorem.

**Proposition 3.2** (i) For an arbitrary continuous distribution \( F \), if we set

\[ S_T = F^{-1}(N(B_T/\sqrt{T})) \] (3.9)

then \( S_T \) will have \( F \) as its distribution.

(ii) Furthermore, \( S_t = V(t, B^Q_t) \), where the function \( V \) is a solution to the following PDE:

\[ \partial_t V + \frac{1}{2} \partial_{xx} V - rV = 0 \]

with the boundary condition

\[ V(T, x) = f(x), \quad \forall x \]

**Proof:** Straightforward, and available from authors.

The focus here is on modeling the stock price at a fixed future time, and calibrating the risk premium \( \theta^Q \) to the current stock price using (3.8) with \( t = 0 \).

### 3.1 BSMCT Model

To account for the bounds on the stock price in the active investor’s beliefs, we consider a special case of RD models, a model in which the log-price \( Y_T = \log S_T \) has a conditionally truncated normal distribution. More precisely, the distribution of \( Y = Y_T \) is the distribution of the normal random variable with mean

\[ m = \log S_0 + (\mu - \sigma^2/2)T \] (3.10)

and variance

\[ \tilde{\sigma}^2 = \sigma^2 T \]

conditional on taking values in the interval with endpoints

\[ \tilde{L} = \log L, \quad \tilde{U} = \log U. \]

We call this model BSMCT (for Black-Scholes-Merton Conditionally Truncated).

It can be shown that the distribution function of \( Y \) is given by

\[ F_Y(y) = \frac{N \left( \frac{y-m}{\sigma} \right) - N \left( \frac{\tilde{L}-m}{\sigma} \right)}{N \left( \frac{\tilde{U}-m}{\sigma} \right) - N \left( \frac{\tilde{L}-m}{\sigma} \right)} \]
Computing $F_Y^{-1}$ from this, we see from (3.9) that the corresponding model for the stock price at time $T$ is

$$log(S_T) = m + \bar{\sigma}N^{-1}\left\{ N\left(\bar{L} - m\right) + N\left(\frac{B^Q_T - \theta^Q T}{\sqrt{T}}\right)\left[ N\left(\bar{U} - m\right) - N\left(\frac{\bar{L} - m}{\bar{\sigma}}\right)\right]\right\}$$

(3.11)

Recall that $\theta^Q$ is not a free parameter – it has to be chosen so that

$$S_0 = E^Q[e^{-rT}S_T]$$

Denote

$$U^* = \frac{\bar{U} - m}{\bar{\sigma}} , \quad L^* = \frac{\bar{L} - m}{\bar{\sigma}}$$

The formulas needed for computations are based on the following proposition.

**Proposition 3.3** In the BSMCT model the stock price at time $T$ can be written as

$$S_T = e^{m + \bar{\sigma}N^{-1}\left\{ N(L^*) + N\left(\frac{B^Q_T - \theta^Q T}{\sqrt{T}}\right)\left[ N(U^*) - N(L^*)\right]\right\}}$$

and the stock price for $t < T$ is given by $S_t = V(t, B^Q_t)$ where

$$V(t, x) = e^{-r(T-t)}E^Q\left[ e^{m + \bar{\sigma}N^{-1}\left\{ N(L^*) + N\left(\frac{\bar{U} - m}{\bar{\sigma}}\right)\left[ N(U^*) - N(L^*)\right]\right\}}\right]$$

(3.12)

We also have

$$V_x(0, 0) = e^{-rT}E^Q\left[ \frac{\bar{\sigma}}{\sqrt{T}} S_T \frac{n\left(\frac{B^Q_T - \theta^Q T}{\sqrt{T}}\right)}{n\left(\frac{\log S_T - m}{\bar{\sigma}}\right)}\right]$$

(3.13)

**Proof**: Available from authors.

If we write

$$dS_t = S_t[\mu_t^S dt + \sigma_t^S dB_t]$$

in the terminology of CAPM, we can define the “instantaneous beta” of asset $S$ by

$$\beta_S(t) = \frac{\rho \sigma_S(t)}{\sigma_I}$$

(3.14)

where $\sigma_I$ is the volatility of the index. In order to compute the instantaneous beta of $S$ we need the following lemma, which follows directly from Ito’s rule:

**Lemma 3.1** In BSMCT model, we have

$$\sigma_S(t) = \frac{V_x(t, B^Q_t)}{S_t}$$
3.2 Optimal investing in the BSMCT model

In this section, we compare optimal investments of an investor who uses a BSM model or an RR model, and an investor who uses a BSMCT model. The BSM model for $S$ is the same as in the previous section, as is the model for the market index $I$. As before, we also assume that both the BSM investor and the BSMCT investor have the same beliefs regarding the index $I$ and the risk-free asset, but they differ in their beliefs about $S$. Note that since $S_T$ takes values in $[L, U]$, the stock price values before $T$ are contained in the corresponding discounted interval $[Le^{-r(T-t)}, Ue^{-r(T-t)}]$. We again assume that the investors have logarithmic utility and we compute optimal investment weights for a static, Markowitz Mean-Variance portfolio, and for a dynamic, Merton log-optimal portfolio. The qualitative behavior of the dynamic portfolios is the same as for the static portfolios, but more extreme.

3.2.1 Mean-Variance portfolio

The formulas here are the same in the RR case, From Proposition (2.1):

$$\pi^M_S = \frac{2 - \gamma S_0 e^{rT}}{\gamma} \frac{E[S_T]E[I^2_T] - E[I_T S_T]E[I_T]}{E[I^2_T]E[S^2_T] - (E[S_T I_T])^2}$$

(3.15)

$$\pi^M_I = \frac{2 - \gamma S_0 e^{rT}}{\gamma} \frac{E[I_T]E[S^2_T] - E[I_T S_T]E[S_T]}{E[I^2_T]E[S^2_T] - (E[S_T I_T])^2}$$

(3.16)

and similarly for $\hat{\pi}^M_S$ and $\hat{\pi}^M_I$.

3.2.2 Dynamic portfolios

We know that $S_t = V(t, B^Q_t)$ satisfies

$$dS_t = rS_t dt + V_x(t, B^Q_t) dB^Q_t$$

It follows from this that

$$dS_t = \mu^S S_t dt + \sigma^S S_t dB_t$$

with

$$\mu^S = r + \theta^Q V_x/S, \quad \sigma^S = V_x/S$$

Introduce the volatility matrix

$$\Sigma = \begin{pmatrix} \sigma^S & 0 \\ \rho \sigma_I & \sqrt{1 - \rho^2} \sigma_I \end{pmatrix}$$

and the vector of excess returns

$$R = \{\mu^S - r, \mu_I - r\}$$
Then, from the classical Merton problem, we know that the optimal proportion \( \pi \) of wealth to be held in stock for an investor with logarithmic utility is given by

\[
\pi_t = (\Sigma \Sigma')^{-1} R_t
\]  

(3.17)

Similarly for the optimal \( \hat{\pi} \) in the Black-Scholes model with \( \mu^S, \sigma^S \) replaced by \( \hat{\mu}, \hat{\sigma} \). We use (3.17) to compute optimal dynamic portfolios.

3.2.3 Comparative Statics

In the computations, we again use these benchmark parameters:

\[ T = 1, \ r = 0.03, \ I_0 = 1, \ S_0 = 1, \ X_0 = 1, \ \mu = 0.08, \ \sigma = 0.3, \ \mu_t = 0.05, \ \sigma_t = 0.25, \ \rho = 0.2, \]

\[ L = 0.65, \ U = 2 \]

We choose \( \hat{\mu}, \hat{\sigma} \) so that the mean and the variance of the stock at time \( T \) are the same in the two models for \( S_0 = \hat{S}_0 = 1 \), that is, so that

\[
E[S_T] = E[\hat{S}_T] = e^{\hat{\mu}T}
\]

\[
E[S_T^2] = E[\hat{S}_T^2] = e^{(2\hat{\mu}+\hat{\sigma}^2)T}
\]

As before, we keep \( \hat{\mu} \) and \( \hat{\sigma} \) fixed at this level corresponding to \( S_0 = 1 \), but vary initial stock price \( S_0 \), by varying risk premium \( \theta^Q \).

We consider three different BSMCT models. In the first, we fix the value \( m \) of (3.10) at the value corresponding to \( S_0 = 1 \). This corresponds to the case in which the investor has a fixed belief about the distribution of the final stock price \( S_T \), no matter what the current stock price value \( S_0 \) is. In the second case, we will let \( m \) change with \( S_0 \). This reflects an active investor whose belief about \( S_T \) changes with the current value of \( S_0 \), and is, in this sense, close to the BSM investor, except that the distribution of the log-price for him is a conditionally truncated normal distribution. In the third model, \( m \) is fixed, but with \( L = 0 \) and \( U = \infty \). That is, there are no bounds on the distribution, as in the BSM model, but the investor does not adapt his beliefs about \( S_T \) as the value of \( S_0 \) changes, unlike in the BSM model.

Figure 3.1 shows how the risk premium \( \theta^Q \) changes with the stock price for those three models. Overall, the least extreme is the model with no bounds, although the adaptive, \( m \) floating model changes less in the middle range.

Figure 3.2 shows expected returns \( E[S_T/S_0] \) for all four non-standard models. The \( m \) fixed models with and without the bounds cannot be distinguished, indicating that the bounds are not that restrictive with respect to the expected returns. They are also the most extreme models, as the investor sticks to a fixed distribution of \( S_T \), independent of the value.
of $S_0$. The RR returns are least extreme, as the bounds in that model are only likely bounds, and not strict bounds.

Figure 3.3 presents the variances $Var[S_T/S_0]$ of returns. Again, the two $m$ fixed models are similar, and have more extreme variance changes than the RR and $m$ floating models. This is partly because we divide by $S_0$, and the numerator does not change in the $m$ fixed models.

Figure 3.4 presents the results for the static betas as defined in (2.2), which mirror the results for the variances across the different models. On the other hand, the instantaneous betas in Figure 3.5, as defined in (3.14), are quite different from the static betas, but very similar in all three BSCT models: the instantaneous betas are low when the stock price is close to the lower and upper bounds, and higher in the middle, but always lower than the stationary beta in the BSM model. The more extreme behavior is due to the fact that this is a local beta, which does not take into account possible future changes of the stock price dynamics, and the fact that the local volatility is lower when closer to the bounds. The instantaneous beta of the RR model is not presented, but it is the same as the stationary BSM beta, since the volatility is fixed in the RR model.

Figures 3.6, 3.7, and 3.8 present optimal Markowitz weights in the stock and the index for the $m$ fixed, $m$ floating and the unbounded BSMCT models, respectively. Figure 3.9 shows the stock investment weight as a function of the stock price for all four models. Once again the two $m$ fixed cases, with and without bounds are very similar, and the most extreme in the middle. The RR case shows the least variation, and the $m$ floating case is the most extreme at the boundaries of the range. Figure 3.10 is similar to Figure 3.9, but for optimal Merton weights. They are more variable than Markowitz weights, so we only show them on a smaller range of $S_0$, as they become extreme at wider range.

4 Conclusions and Extensions

From the standpoint of active investors, the standard equilibrium asset pricing models cannot be correct. As Warren Buffett notes, the very reason for being an active investor is the belief that certain securities are not valued appropriately. If that is the case, the active investor cannot rely on the risk and expected return results derived from equilibrium models. This leads to the question of how the active investor should assess risk and return of an individual security. There is no general answer to that question. It depends on the type of misvaluation that the active investor believes exists. In this paper, we analyze a basic form of mispricing. In particular, we assume that via the application of fundamental analysis the active investor can estimate bounds on the stock price. The expected return and the optimal investment policies depend on the current price of the stock relative to these bounds. We consider two
types of such boundaries, one with reversion from the boundaries, and another in which the investor, in addition to the boundaries, has a belief on the risk premium of the stock. We derive optimal investment strategies for such models, when investing in a single stock and a market index. Our results show the extent to which an investor who places fundamental boundaries on a stock price at certain intervals will alter his investment strategy when the stock price is close to those boundaries.

One extension of our models would be to consider multiple stocks. More precisely, in the case of $d$ stocks, we can replace $\sigma B$ in stock $i$ with

$$\sum_{j=1}^{d} \sigma_{ij} B_T^{(j)}, \quad i = 1, \ldots, d$$

for a given $d$–dimensional Brownian Motion $B = (B^{(1)}, \ldots, B^{(d)})$. Appropriately choosing $\sigma_{ij}$’s produces various correlation structures on the stocks. We can also replace $\sigma^2$ for stock $i$ with

$$\sum_{j=1}^{d} \sigma_{ij}^2$$

and since we can write

$$\sum_{j=1}^{d} \sigma_{ij} B_T^{(j)} = \Sigma_i W_T^{(i)}$$

for some one-dimensional Brownian Motion $W^{(i)}$, we can still use the RD model as before, for individual stocks. To model a multi-dimensional distribution of the vector $(S_T^1, \ldots, S_T^d)$, a copula approach could be employed.

From the theoretical standpoint, it would also be of interest to analyze full equilibrium models in the presence of investors with RR or RD beliefs. As Fama (1970, 1998) states it, the explanation of the effect of extra factors as pricing of risk or as abnormal returns to mispricing cannot be resolved without the specification of an asset pricing model. Our model seems to have the potential to be extended to a full equilibrium model consistent with empirical findings. We will explore these in a near-future project.
References


RR Markowitz portfolios, $n_L=n_U=1$

Table: Portfolio weight vs. Stock Price for RR and BSM models.

Figure 2.1: Static optimal portfolio weights for RR model with $n_L=n_U=1$ and BSM model.

RR Dynamic portfolios, $n_L=n_U=1$

Table: Portfolio weight vs. Stock Price for RR and BSM models.

Figure 2.2: Dynamic optimal portfolio weights for RR model, $n_L=1$, $n_U=1$, and BSM model.

Various RR speeds

Table: Weight in stock vs. Stock Price for RR models with different RR speeds.

Figure 2.3: Optimal portfolio weights in stock for RR models with different RR speeds.

Expected returns

Table: Expected returns vs. Stock Price for RR and BSM models.

Figure 3.2: Expected returns. BSMCT m fixed and BSMCT L=0, U=∞ cannot be distinguished.

Risk Premia

Table: Risk premium vs. Stock Price for BSM models.

Figure 3.1: Risk premia for BSMCT models.

Variances of returns

Table: Variance vs. Stock Price for all four models.

Figure 3.3: Variances of returns for all four models.
Figure 3.4: Betas for the whole period for all four models.

Figure 3.5: Instantaneous betas for all BSMCT models.

Figure 3.6: Optimal Portfolio weights for RR model and BSM model.

Figure 3.7: Optimal Portfolio weights for BSMCT m floating model and BSM model.

Figure 3.8: Optimal Portfolio weights for BSMCT L=0, U=infinity model and BSM model.

Figure 3.9: Optimal weights for all four models.