Supporting Information:

Candidate Entry and Political Polarization:

An Anti-Median Voter Theorem

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Note: For convenience, equation numbers continue with the numbering in the article.

Proof of Lemma 1

Proof. We need to prove that the function

$$B(x_i) = Q_{ne}(n,q)\int_{-1}^{1} f_{ne}(x|\sigma)\ U(x_i,x)dx\ \ + Q_e(n,q)\int_{-1}^{1} f_e(x|\sigma)\ U(x_i,x)dx \tag{15}$$

is strictly convex in x_i (i.e., $B''(x_i) > 0$) with $B'(x_i) < 0$ for $x_i = -1$ and $B'(x_i) > 0$ for $x_i = 1$ (cf. best response condition (8)). We prove it for the case in which U is twice continuously differentiable and satisfies three properties:

$$\begin{aligned}(i)\ \frac{\partial^2\ U(x_i,\gamma)}{\partial x_i^2} &> 0\\ (ii)\ \frac{\partial^2\ U(x_i,\gamma)}{\partial \gamma^2} &> 0\\ (iii)\ \frac{\partial\ U(x_i,\gamma)}{\partial x_i} &= 0 \text{ if } x_i = \gamma.\end{aligned}$$

In other words, utility functions $-U$ are smooth, strictly concave and single peaked. The proof for linear $-U$ ("tent preferences") follows a similar logic and we omit the details.

(Convexity) First, it is straightforward to see that $B''(x_i) > 0$:

$$B''(x_i) = Q_{ne}(n,q)\int_{-1}^{1} f_{ne}(x|\sigma)\frac{\partial^2\ U(x_i,x)}{\partial x_i^2}dx + Q_e(n,q)\int_{-1}^{1} f_e(x|\sigma)\frac{\partial^2\ U(x_i,x)}{\partial x_i^2}dx > 0,$$

because $\frac{\partial^2\ U(x_i,x)}{\partial x_i^2} > 0$.

(Relative maximum and unique minimum) Next, for $x_i = -1$ we have

$$B'(-1) = Q_{ne}(n,q)\int_{-1}^{1} f_{ne}(x|\sigma)\frac{\partial\ U(x_i,x)}{\partial x_i}|_{x_i=-1}dx + Q_e(n,q)\int_{-1}^{1} f_e(x|\sigma)\frac{\partial\ U(x_i,x)}{\partial x_i}|_{x_i=-1}dx < 0$$

where the strict inequality holds for a -1-type because our assumptions on U imply that

$\frac{\partial\ U(x_i,x)}{\partial x_i} < 0$ if $x_i < x$. Similarly, for $x_i = 1$ we have

$$B'(1) = Q_{ne}(n,q)\int_{-1}^{1} f_{ne}(x|\sigma)\frac{\partial\ U(x_i,x)}{\partial x_i}|_{x_i=1}dx + Q_e(n,q)\int_{-1}^{1} f_e(x|\sigma)\frac{\partial\ U(x_i,x)}{\partial x_i}|_{x_i=1}dx > 0$$

where the strict inequality holds for a 1-type because our assumptions on U imply that $\frac{\partial\ U(x_i,x)}{\partial x_i} > 0$ if $x_i > x$.

Thus, the net-benefits of entering are always U-shaped, for any strategy used by the other citizens. This means that for any symmetric (type-dependent) mixed entry strategy, $\sigma_j(x)$, played by all citizens $j \neq i$ there is a unique interior minimum, which we will call at $x_{\min}$—i.e., B is strictly decreasing for $x_i < x_{\min}$, has a derivative of 0 at $x_i = x_{\min}$, and is strictly increasing for $x_i > x_{\min}$—and two relative maxima at $x_i = -1$ and $x_i = 1$.[1] ■

Proof of Proposition 1

Proof. We first rewrite citizen i's best response entry strategy (7) in order to characterize the different equilibrium cases depending on the entry cost. Thereafter, we use Envelope Theorem and Intermediate Value Theorem to prove that a symmetric entry equilibrium always exists and, finally, show that it is unique.

Recall our assumptions $n \geq 2$, $c > 0$, and $b \geq 0$ and that the citizen types, $x_i \in [-1,1] \subset \mathbb{R}$, are distributed according to any continuous cumulative probability function, $F(x)$, strictly increasing and twice differentiable on $[-1,1]$ and with $F(-1) = 0$, $F(1) = 1$, and density $f(x)$ ($A1$-$A3$ in section "General Model" in the article).

First, use expressions (4) and (6) to rewrite the best response entry condition (7):

$$(1-p)^{n-1}\left(\tfrac{n-1}{n}\right)\left[b + \frac{\int_{\check{x}_l}^{\check{x}_r} f(x)\ U(x_i,x)dx}{1-p}\right] + \sum_{m=2}^{n}\binom{n-1}{m-1}p^{m-1}(1-p)^{n-m}\frac{1}{m}$$

$$\times\left[b + \frac{\int_{-1}^{\check{x}_l} f(x)\ U(x_i,x)dx + \int_{\check{x}_r}^{1} f(x)\ U(x_i,x)dx}{p}\right] \geq c. \qquad (16)$$

[1] The cutpoint best response property is even more general, and applies even if other citizens are not all using the same strategy. However, here we are only interested in symmetric equilibria.

For this and subsequent proofs it is helpful to separate the "integral"- and "probability"-terms in condition (16). This yields the following modified best response condition:[2]

$$LHS(7') = P_{ne}(n,p)\int_{\check{x}_l}^{\check{x}_r} f(x)\ U(x_i,x)dx \tag{17}$$
$$+P_e(n,p)\left(\int_{-1}^{\check{x}_l} f(x)\ U(x_i,x)dx + \int_{\check{x}_r}^{1} f(x)\ U(x_i,x)dx\right) + P_b(n,p)b \geq c.$$

The subscript 'ne' in the P_{ne}-term refers to the situation where none of the other citizens enters, and

$$P_{ne}(n,p) \equiv \frac{(n-1)(1-p)^{n-2}}{n} > 0 \ \text{ for } p \in [0,1). \tag{18}$$

And, the subscript 'e' in the P_e-term refers to the situation where at least one other citizen enters, and

$$\begin{aligned}
P_e(n,p) &\equiv \frac{1}{p}\left[\sum_{m=2}^{n}\binom{n-1}{m-1}p^{m-1}(1-p)^{n-m}\frac{1}{m} + (1-p)^{n-1} - (1-p)^{n-1}\right] \\
&= \frac{1}{p}\left[\sum_{m=1}^{n}\binom{n-1}{m-1}p^{m-1}(1-p)^{n-m}\frac{1}{m} - (1-p)^{n-1}\right] \\
&= \frac{1}{p}\left[\frac{1}{np}\sum_{m=1}^{n}\binom{n}{m}p^{m}(1-p)^{n-m} - (1-p)^{n-1}\right] \\
&= \frac{1}{np^2}\left[\sum_{m=0}^{n}\binom{n}{m}p^{m}(1-p)^{n-m} - (1-p)^{n}\right] - \frac{(1-p)^{n-1}}{p} \\
&= \frac{1-(1-p)^n}{np^2} - \frac{(1-p)^{n-1}}{p} > 0 \ \text{ for } p \in (0,1].
\end{aligned} \tag{19}$$

Finally, the subscript 'b' in the P_b-term refers to the benefits from holding office and, using

[2] In section "Extensions" in the article, we compare our stochastic default policy, d, with a fixed default policy $\overline{d} \in [-1,1]$. In this case, the best response entry condition (14) can be rewritten as:

$$(1-p)^{n-1}(\tfrac{n-1}{n})U(x_i,\overline{d}) + P_e(n,p)\left(\int_{-1}^{\check{x}_l} f(x)U(x_i,x)dx + \int_{\check{x}_r}^{1} f(x)U(x_i,x)dx\right) + P_b(n,p)b \geq c.$$

The first term on the left-hand side is U-shaped in x_i with a minimum at $\overline{d}$, the second term is U-shaped in x_i (cf. the proof of Lemma 1), and the third term is constant in x_i. Thus, the left-hand side is overall U-shaped with a unique minimum value at $\overline{x}_{\min}(\check{x}_l, \check{x}_r, \overline{d})$. In other words, while $\overline{d}$ affects $\overline{x}_{\min}$, it does not change the U-shape in x_i of the left-hand side of the best response entry condition.

similar rearrangements as for expression (19),

$$P_b(n,p) \equiv \frac{1-(1-p)^n}{np} - \frac{(1-p)^{n-1}}{n} > 0 \text{ for } p \in [0,1]. \tag{20}$$

We continue by using condition (17) to specify the two best response conditions for citizen types $x_i = \check{x}_l$ and $x_i = \check{x}_r$. To avoid abundant equilibrium characterization, we introduce the notation $\delta \in \{l, r\}$ and the indicator functions

$$F_\delta(x) = \begin{cases} F(x) & if \quad \delta = r \\ F(-x) & if \quad \delta = l \end{cases} \quad \text{and} \quad f_\delta(x) = \begin{cases} f(x) & if \quad \delta = r \\ f(-x) & if \quad \delta = l \end{cases},$$

for $x \in [-1,1] \subseteq \mathbb{R}$. Thus, we consider the mirror images $F(-x)$ and $f(-x)$ of $F(x)$ and $f(x)$, respectively, with $F_{\delta=r}(-1) = F(-1) = 0$ and $F_{\delta=r}(1) = F(1) = 1$, and with $F_{\delta=l}(-1) = F(1) = 1$ and $F_{\delta=l}(1) = F(-1) = 0$.

Using this, we can modify the best response entry condition (17) as follows: if all other citizens $j \neq i$ are using a cutpoint strategy $\check{e}_j$ as defined in expression (2) (see Lemma 1), the best response entry strategy of a citizen type $x_i = \check{x}_\delta$, for $\delta = l, r$ and $\delta \neq -\delta$, is to enter if and only if:

$$\begin{aligned} &P_{ne}(n,p) \int_{\check{x}_{-\delta}}^{\check{x}_\delta} f_\delta(x) U(\check{x}_\delta, x) dx \\ &+ P_e(n,p) \left(\int_{-1}^{\check{x}_{-\delta}} f_\delta(x) U(\check{x}_\delta, x) dx + \int_{\check{x}_\delta}^{1} f_\delta(x) U(\check{x}_\delta, x) dx \right) + P_b(n,p) b \geq c, \end{aligned} \tag{21}$$

where $p = p_{-\delta} + p_\delta$, $p_{-\delta} = F_\delta(\check{x}_{-\delta})$, and $p_\delta = 1 - F_\delta(\check{x}_\delta)$.

(Necessary and sufficient conditions) We can use this best response entry strategy to characterize two necessary and sufficient conditions for a cutpoint equilibrium, $(\check{x}^*_{-\delta}, \check{x}^*_\delta)$, to exist, which must hold simultaneously for types $\check{x}_{-\delta}$ and $\check{x}_\delta$. First, note the important relationship between $LHS(17)$, $LHS(21)$, and Lemma 1. When the "c-line" on $RHS(17)$ intersects the net-benefits curve on $LHS(17)$ at $x_i = \check{x}_l$ and $x_i = \check{x}_r$, it must hold that $x_{\min} \in [\check{x}_l, \check{x}_r]$. Because the net-benefits curve is U-shaped in x_i, this means that the cutpoint strategy $\check{e}$ (see expression (2)) fulfills a necessary condition for the existence of a cutpoint equilibrium. Then, using the two best response strategies (21) for $\delta = l, r$, the following equilibrium characterizations do indeed constitute necessary and sufficient conditions for an entry equilibrium in cutpoint strategies to exist.

There are four different equilibrium cases:

Case (i)**:** If $c \leq \underline{c} \equiv \frac{1}{n}\left[b + \int_{-1}^{1} f(x)U(\check{x}^*_{\min}, x)dx\right]$, then $\check{e}^*_i = 1, \forall i$ ("everybody enters"), where $\check{x}^*_l = \check{x}^*_r = \check{x}^*_{\min} \in (-1, 1)$ is determined by $\int_{-1}^{\check{x}^*_{\min}} f(x)\frac{\partial U(x_i,x)}{\partial x_i}|_{x_i=\check{x}^*_{\min}} dx = \int_{\check{x}^*_{\min}}^{1} f(x)\frac{\partial U(x_i,x)}{\partial x_i}|_{x_i=\check{x}^*_{\min}} dx$. This case is derived as follows: if $LHS(17)$ is greater than or equal to c for all values of $x_i, \check{x}_l$, and $\check{x}_r$, then the unique equilibrium is for all n citizens to enter. This corresponds to an equilibrium cutpoint $x_{\min} = \check{x}_{\min} = \check{x}_l = \check{x}_r$.[3] Thus, for this to hold, in $LHS(21)$ we simply set $x_{\min} = \check{x}_{\min} = \check{x}_\delta = \check{x}_{-\delta}$ and only consider the case $m = n$ in the P-terms. Then, as stated above, the inequality condition (21) reduces to:

$$\frac{1}{n}\left[b + \int_{-1}^{1} f_\delta(x)U(\check{x}_{\min}, x)dx\right] \equiv \underline{c} \geq c \quad \text{for } \delta = l, r, \tag{22}$$

because $p = F_\delta(\check{x}_{\min}) + 1 - F_\delta(\check{x}_{\min}) = 1$, and therefore, $P_e\left[n, p(\check{x}_{\min}) = 1\right] = \frac{1}{n}$ and $P_b\left[n, p(\check{x}_{\min}) = 1\right] = \frac{1}{n}$ (see expressions (19) and (20)). Thus, there is universal entry if and only if $c \leq \underline{c}$.[4] Finally, knowing $x_{\min} = \check{x}_{\min} = \check{x}_l = \check{x}_r$, we can determine $\check{x}_{\min}$ by using the first derivative of the left-hand side of $LHS(17)$ with respect to x_i, setting this equal to zero, and replacing $\check{x}_l, \check{x}_r$, and x_i with $\check{x}_{\min}$. This gives:

$$\frac{\partial LHS(7')\big|_{x_i=\check{x}_{\min}=\check{x}_l=\check{x}_r}}{\partial x_i} =$$

$$P_e(n, p = 1)\left[\int_{-1}^{\check{x}_{\min}} f_\delta(x)\frac{\partial U(x_i, x)}{\partial x_i}|_{x_i=\check{x}_{\min}} dx - \int_{\check{x}_{\min}}^{1} f_\delta(x)\frac{\partial U(x_i, x)}{\partial x_i}|_{x_i=\check{x}_{\min}} dx\right] = 0$$

[3]The specification of cutpoints is arbitrary when there is universal entry. Any $\widehat{x}$ such that $\check{x}_l = \check{x}_r = \widehat{x}$ implies universal entry.

[4]Note that condition (22) implies that if $c = 0$, there is always universal entry because the left-hand side is greater than or equal to zero for any feasible combination of n and b.

$$\Leftrightarrow \int_{-1}^{\check{x}_{\min}} f_\delta(x)\frac{\partial U(x_i,x)}{\partial x_i}|_{x_i=\check{x}_{\min}}dx = \int_{\check{x}_{\min}}^{1} f_\delta(x)\frac{\partial U(x_i,x)}{\partial x_i}|_{x_i=\check{x}_{\min}}dx, \tag{23}$$

which implicitly determines $\check{x}_{\min}$, as stated above.

Case (ii)**:** If $c \geq \bar{\bar{c}} \equiv \max[\bar{c}_l, \bar{c}_r]$, where $\bar{c}_l \equiv \frac{n-1}{n}\left[b + \int_{-1}^{1} f(x)U(-1,x)dx\right]$ and $\bar{c}_r \equiv \frac{n-1}{n}\left[b + \int_{-1}^{1} f(x)U(1,x)dx\right]$, then $\check{x}_l^* = -1$, $\check{x}_r^* = 1$, and $\check{e}_i^* = 0, \forall i$ ("nobody enters").

This case is derived as follows: if $LHS(17)$ is smaller than or equal to c for all values of $x_i, \check{x}_l$, and $\check{x}_r$, then the unique equilibrium is for no citizen to enter. This corresponds to an equilibrium pair of cutpoints $(\check{x}_l = -1, \check{x}_r = 1)$. Thus, for this to hold, we reverse the inequality sign of condition (21), simply set $\check{x}_{-\delta} = -1$ and $\check{x}_\delta = 1$ in $LHS(21)$, and only consider the case $m = 0$ in the P-terms. Then, as stated above, condition (21) reduces to

$$\bar{c}_\delta \equiv \frac{n-1}{n}\left[b + \int_{-1}^{1} f_\delta(x)U(1,x)dx\right] \leq c \ \text{ for } \delta = l, r, \tag{24}$$

because $p = p_{-\delta} + p_\delta = F_\delta(\check{x}_{-\delta} = -1) + 1 - F_\delta(\check{x}_\delta = 1) = 0$, and therefore, $P_{ne}(n, p = 0) = \frac{n-1}{n}$ and $P_b(n, p = 0) = \frac{n-1}{n}$ (see expressions (18) and (20)). Thus, there is zero entry if and only if $c \geq \bar{\bar{c}} \equiv \max[\bar{c}_l, \bar{c}_r]$ (note that the probability of any citizen having type $x_i = -1$ or $x_i = 1$ is equal to zero).

Case (iii)**:** If $\widetilde{c} \equiv \min[\widetilde{c}_l, \widetilde{c}_r] \leq c < \bar{\bar{c}}$, where $\widetilde{c}_{-\delta} \equiv c(\check{x}_{-\delta} = -1, \check{x}_\delta = \overline{x}_\delta)$ and $\overline{x}_\delta = \check{x}_\delta^*(\check{x}_{-\delta}^* = -1) \in (\check{x}_{\min}, 1)$ for $\delta = l, r$, then there is a unique cutpoint equilibrium where only some more extreme citizen types in one direction are expected to enter. Specifically, if $\widetilde{c} = \widetilde{c}_{-\delta}$ then citizen types $x_i \geq \overline{x}_\delta$ enter, that is, $[\check{x}_{-\delta}^* = -1, \check{x}_\delta^* \in [\overline{x}_\delta, 1)]$, and all other types $(x_i < \overline{x}_\delta)$ do not enter. This unique equilibrium is characterized by the two best response conditions

$$P_{ne}(n, p_\delta)\int_{-1}^{\check{x}_\delta^*} f_\delta(x)U(\check{x}_\delta^*, x)dx + P_e(n, p_\delta)\int_{\check{x}_\delta^*}^{1} f_\delta(x)U(\check{x}_\delta^*, x)dx + P_b(n, p_\delta)b = c \tag{25}$$

and

$$P_{ne}(n,p_\delta)\int_{-1}^{\check{x}^*_\delta} f_\delta(x)U(-1,x)dx + P_e(n,p_\delta)\int_{\check{x}^*_\delta}^{1} f_\delta(x)U(-1,x)dx + P_b(n,p_\delta)b \leq c, \tag{26}$$

where $\widetilde{c} = \widetilde{c}_{-\delta}$ and $p_\delta = 1 - F_\delta(\check{x}^*_\delta)$. Note that the probability of any citizen type -1 or $\overline{x}_\delta$ occurring is equal to zero. Moreover, $\overline{x}_\delta = \check{x}^*_\delta(\check{x}^*_{-\delta} = -1)$ is implicitly determined by

$$P_{ne}(n,p_\delta)\left[\int_{-1}^{\overline{x}_\delta} f_\delta(x)U(\overline{x}_\delta,x)dx - \int_{-1}^{\overline{x}_\delta} f_\delta(x)U(-1,x)dx\right] + P_e(n,p_\delta)\left[\int_{-1}^{\overline{x}_\delta} f_\delta(x)U(\overline{x}_\delta,x)dx - \int_{\overline{x}_\delta}^{1} f_\delta(x)U(-1,x)dx\right] = 0, \tag{27}$$

and $\widetilde{c}_{-\delta}$ is determined by replacing $\check{x}^*_\delta$ with $\overline{x}_\delta$ on the left-hand side of condition (25).

This case is derived as follows: if $LHS(17)$ is greater than or equal to c for all values $x_i \geq \overline{x}_\delta$ and smaller than c for all values of $x_i < \overline{x}_\delta$, then the unique equilibrium is for all citizen types equal to or larger than $\overline{x}_\delta$ to enter, and for all other types not to enter. This corresponds to a cutpoint equilibrium $[\check{x}_{-\delta} = -1, \check{x}_\delta \in [\overline{x}_\delta, 1)]$. Thus, for this to hold, for a type $\check{x}_\delta$ we state condition (21) as equality and simply set $\check{x}^*_{-\delta} = -1$ in $LHS(21)\,|_{\check{x}_\delta}$, where the subscript denotes the citizen type whose strategy we investigate (see condition (25)), and for a type $\check{x}_{-\delta}$ we reverse the inequality sign and set $\check{x}^*_{-\delta} = -1$ in $LHS(21)\,|_{\check{x}_{-\delta}}$ (see condition (26)). In condition (27), we determine the boundary case $\overline{x}_\delta = \check{x}_\delta(\check{x}_{-\delta} = -1)$—where a type $\check{x}_{-\delta} = -1$ is just indifferent between entering and not entering as a candidate (note that the probability of this type occurring is equal to zero)—by setting the left-hand sides of conditions (25) and (26) equal and making simple rearrangements. Importantly, below we use Envelope Theorem to show that a citizen type $x_{-\delta} = -1$ always prefers not to enter if $c > \widetilde{c}$. Therefore, for $\widetilde{c} \leq c < \overline{\widetilde{c}}$ we only need condition (25) to compute the interior cutpoint policy $\check{x}_\delta(\check{x}_{-\delta} = -1)$.

Case (iv)**:** If $\underline{c} < c < \widetilde{c}$, then there is a unique equilibrium pair of interior cutpoints, $(\check{x}^*_l, \check{x}^*_r)$, where some more extreme citizen types in both directions are expected to enter. Specifically, for $\delta = l, r$, if $\widetilde{c} = \widetilde{c}_{-\delta}$ then $[\check{x}^*_{-\delta} \in (-1, \check{x}^*_{\min}),\ \check{x}^*_{\delta} \in (\check{x}^*_{\min}, \overline{x}_{\delta})]$ and if $\widetilde{c} = \widetilde{c}_{-\delta} = \widetilde{c}_{\delta}$ then $[\check{x}^*_{-\delta} \in (-1, \check{x}^*_{\min}),\ \check{x}^*_{\delta} \in (\check{x}^*_{\min}, 1)]$, and in all these cases some citizen types in both directions who are more extreme than or equal to $\check{x}^*_{-\delta}$ or $\check{x}^*_{\delta}$ are expected to enter. This unique interior equilibrium is characterized by the equality condition

$$
\begin{aligned}
&P_{ne}(n,p)\int_{\check{x}^*_{-\delta}}^{\check{x}^*_{\delta}} f_{\delta}(x)U(\check{x}^*_{\delta},x)dx \qquad (28)\\
&\quad + P_e(n,p)\left[\int_{-1}^{\check{x}^*_{-\delta}} f_{\delta}(x)U(\check{x}^*_{\delta},x)dx + \int_{\check{x}^*_{\delta}}^{1} f_{\delta}(x)U(\check{x}^*_{\delta},x)dx\right] + P_b(n,p)b = c,
\end{aligned}
$$

which must hold simultaneously for $\delta = l, r$, where $p = F_{\delta}(\check{x}^*_{-\delta}) + 1 - F_{\delta}(\check{x}^*_{\delta})$.

This case is derived as follows: if $LHS(17)$ is greater than or equal to c for all values of $x_i \leq \check{x}^*_{-\delta}$ and $x_i \geq \check{x}^*_{\delta}$ and smaller than c for all values of $x_i \in (\check{x}^*_{-\delta}, \check{x}^*_{\delta})$, then the unique equilibrium is for all citizen types who are more extreme than or equal to $\check{x}^*_{-\delta}$ and $\check{x}^*_{\delta}$ to enter, and for all other more moderate types not to enter. This corresponds to a cutpoint equilibrium $[\check{x}_{-\delta} \in (-1, \check{x}_{\min}),\ \check{x}_{\delta} \in (\check{x}_{\min}, \overline{x}_{\delta})]$ for $\widetilde{c} = \widetilde{c}_{-\delta}$. Thus, for this to hold, we simply have to state (21) as equality for both $\delta = l, r$ (see condition (28)) and simultaneously compute the values of the interior cutpoints $\check{x}_{\delta}$ and $\check{x}_{-\delta}$.

(Existence) Next, we prove that a cutpoint equilibrium, $(\check{x}^*_{-\delta}, \check{x}^*_{\delta})$, always exist and is always unique for any cumulative probability distribution of ideal points, $F(x)$, satisfying $A1$-$A3$ (see section "General Model" in the article). We proceed in the following steps. Here, we use Envelope Theorem and Intermediate Value Theorem to show existence. Thereafter, we prove uniqueness using our result that for any given entry probability, p, there is a unique cutpoint equilibrium.

We begin by using Envelope Theorem. Recall that $LHS(17)$ is a continuous function of $x_i, \check{x}_{-\delta}$, and $\check{x}_{\delta}$. Now consider the following value function:

$$\upsilon[\check{x}_{-\delta}, \check{x}_{\delta}\,|\,f_{\delta}(x), n, c, b] = \int_{-1}^{1} f_{\delta}(x)\left[c - LHS(17)[x, \check{x}_{-\delta}, \check{x}_{\delta}\,|\,f_{\delta}(x), n, c, b]\right]dx$$

and the maximization problem

$$\upsilon^*[\check{x}^*_{-\delta}, \check{x}^*_{\delta} \,|\, f_\delta(x), n, c, b] \equiv \max_{\check{x}_{-\delta}, \check{x}_\delta} \upsilon[\check{x}_{-\delta}, \check{x}_\delta \,|\, f_\delta(x), n, c, b].$$

However, if both cutpoints are interior (the case with one interior cutpoint will be discussed below), we know from equilibrium condition (28) that a solution to this problem—i.e., a cutpoint equilibrium, $(\check{x}^*_{-\delta}, \check{x}^*_\delta)$—is implicitly determined by

$$LHS(17)[\check{x}^*_{-\delta}, \check{x}^*_\delta \,|\, f_\delta(x), n, c, b] \Big|_{x_i=\check{x}^*_{-\delta}} = LHS(17)[\check{x}^*_{-\delta}, \check{x}^*_\delta \,|\, f_\delta(x), n, c, b] \Big|_{x_i=\check{x}^*_\delta} = c$$

which, using Lemma 1, gives:

$$\upsilon^*[\check{x}^*_{-\delta}, \check{x}^*_\delta \,|\, f_\delta(x), n, c, b] = \int_{\check{x}^*_{-\delta}}^{\check{x}^*_\delta} f_\delta(x) \left[c - LHS(17)[x, \check{x}^*_{-\delta}, \check{x}^*_\delta \,|\, f_\delta(x), n, c, b]\right] dx. \tag{29}$$

Here, we are interested in the effects of a marginal change in the entry costs, c, on $\upsilon^*[.]$, and on the equilibrium cutpoints in particular. Since the two cutpoints are mutually dependent, let us write the pair as $[\check{x}^*_{-\delta}(\check{x}^*_\delta, c), \check{x}^*_\delta(\check{x}^*_{-\delta}, c)]$. Then, by the chain rule we have

$$\begin{aligned}\frac{d\upsilon^*[\check{x}^*_{-\delta}(\check{x}^*_\delta, c), \check{x}^*_\delta(\check{x}^*_{-\delta}, c), c \,|\, f_\delta(x), n, b]}{dc} &= \frac{\partial \upsilon^*[.]}{\partial c} \\ &+ \frac{\partial \upsilon^*[.]}{\partial \check{x}_{-\delta}(\check{x}_\delta, c)} \left[\frac{d\check{x}_{-\delta}(\check{x}_\delta, c)}{dc} + \frac{\partial \check{x}_{-\delta}(\check{x}_\delta, c)}{\partial \check{x}_\delta(\check{x}_{-\delta}, c)} \frac{d\check{x}_\delta(\check{x}_{-\delta}, c)}{dc}\right] \\ &+ \frac{\partial \upsilon^*[.]}{\partial \check{x}_\delta(\check{x}_{-\delta}, c)} \left[\frac{d\check{x}_\delta(\check{x}_{-\delta}, c)}{dc} + \frac{\partial \check{x}_\delta(\check{x}_{-\delta}, c)}{\partial \check{x}_{-\delta}(\check{x}_\delta, c)} \frac{d\check{x}_{-\delta}(\check{x}_\delta, c)}{dc}\right],\end{aligned}$$

which, using the first-order equilibrium condition $\frac{\partial \upsilon^*[.]}{\partial \check{x}_{-\delta}(\check{x}_\delta, c)} = \frac{\partial \upsilon^*[.]}{\partial \check{x}_\delta(\check{x}_{-\delta}, c)} = 0$, yields

$$\frac{d\upsilon^*[\check{x}^*_{-\delta}(\check{x}^*_\delta, c), \check{x}^*_\delta(\check{x}^*_{-\delta}, c), c \,|\, f_\delta(x), n, b]}{dc} = \frac{\partial \upsilon^*[.]}{\partial c} = \int_{\check{x}^*_{-\delta}}^{\check{x}^*_\delta} f_\delta(x)dx > 0 \text{ for } p \in [0, 1). \tag{30}$$

Therefore, a marginal change in c affects $\upsilon^*[.]$ only directly, but not indirectly through changes in $\check{x}_{-\delta}(\check{x}_\delta, c)$ and $\check{x}_\delta(\check{x}_{-\delta}, c)$ (in other words, the effects on $LHS(17)[.]$ in $\upsilon^*[.]$ are negligible and marginal changes in the cutpoints are independent from each other).

This is an important result, and it also informs us about how $\check{x}_{-\delta}$ and $\check{x}_{\delta}$ change when c changes marginally. Expression (30) shows that an increase in $v^*[.]$ through a marginal increase from c to c' is entirely due to the higher entry costs of each potential citizen type $x \in [\check{x}^*_{-\delta}, \check{x}^*_{\delta}]$. Among these citizens, for c only types $\check{x}^*_{-\delta}(c)$ and $\check{x}^*_{\delta}(c)$ enter and all other, moderate types $x \in (\check{x}^*_{-\delta}(c), \check{x}^*_{\delta}(c))$ abstain. By contrast, for c' the entry costs exceed the net-benefits also for types $\check{x}^*_{-\delta}(c)$ and $\check{x}^*_{\delta}(c)$, who now abstain too. Therefore, if both equilibrium cutpoints $\check{x}^*_{-\delta}$ and $\check{x}^*_{\delta}$ are interior (see Proposition 1 *(iv)*), marginally increasing c to c' yields more extreme equilibrium cutpoints, or $\check{x}'^*_{-\delta} < \check{x}^*_{-\delta}$ and $\check{x}^*_{\delta} < \check{x}'^*_{\delta}$. As a consequence, the entry probability decreases in both directions (i.e., $p_{-\delta} > p'_{-\delta}$ and $p_{\delta} > p'_{\delta}$) and hence decreases overall (i.e., $p > p'$). If only one equilibrium cutpoint is interior, $\check{x}^*_{\delta}$, and the other is at the boundary, $\check{x}^*_{-\delta} = -1$ (see Proposition 1 *(iii)*), it is readily verified that the value function (29) and its derivative (30) can be used by simply setting $\check{x}^*_{-\delta} = -1$. Then, marginally increasing c to c' yields the interior cutpoint to become more extreme, or $\check{x}^*_{\delta} < \check{x}'^*_{\delta}$, while the boundary cutpoint remains unchanged, or $\check{x}^*_{-\delta} = \check{x}'^*_{-\delta} = -1$. As a consequence, the entry probability only decreases in the direction of the interior cutpoint (i.e., $p_{\delta} > p'_{\delta}$ and $p_{-\delta} = p'_{-\delta} = 0$) and hence decreases overall (i.e., $p > p'$).

Moreover, importantly, for interior equilibrium cutpoints the net-benefits of citizen types with exactly these cutpoints are larger for c' than for c, respectively. This is because from expression (30) we know that, on the margin, for c' the new equilibrium cutpoints are simply reached by moving upwards along the U-shaped net-benefit curve for c, that is, along $LHS(17)[x_i, \check{x}^*_{-\delta}, \check{x}^*_{\delta} \,|n, c, b]$. Finally, note that our results also establish that in any equilibrium it must hold that $\check{x}^*_l \in [-1, \check{x}^*_{\text{min}}]$ and $\check{x}^*_r \in [\check{x}^*_{\text{min}}, 1]$, because cutpoints never get more moderate if c increases. In summary, continuously increasing c creates one or more continuous equilibrium paths $[\check{x}^*_{-\delta}(\check{x}^*_{\delta}, c),\ \check{x}^*_{\delta}(\check{x}^*_{-\delta}, c)]$ with the following properties: *(i)* the interior cutpoints get more extreme (at the boundary, $\check{x}^*_{-\delta} = -1$ remains), *(ii)* $p_{-\delta}, p_{\delta}$, and p decrease; and *(iii)* $LHS(21)$ increases. The endpoints of any path are at $\underline{c}$ ($p = 1$, where $\check{x}^*_{-\delta} = \check{x}^*_{\delta} = \check{x}_{\text{min}}$) and at $\overline{\overline{c}}$ ($p = 0$, where $\check{x}^*_{-\delta} = -1$ and $\check{x}^*_{\delta} = 1$). Using expressions (22) and (24) we have:

$$\underline{c} \;<\; \overline{\overline{c}} \Rightarrow \frac{1}{n}\left[b + \int_{-1}^{1} f_{\delta}(x)U(\check{x}_{\min}, x)dx\right] < \frac{n-1}{n}\left[b + \max\left[\int_{-1}^{1} f_{\delta}(x)U(-1,x)dx, \int_{-1}^{1} f_{\delta}(x)U(1,x)dx\right]\right],$$

where the strict inequality holds because $\frac{1}{n} < \frac{n-1}{n}$ for $n > 2$[5] and $\int_{-1}^{1} f_{\delta}(x)U(\check{x}_{\min}, x)dx < \max\left[\int_{-1}^{1} f_{\delta}(x)U(-1,x)dx, \int_{-1}^{1} f_{\delta}(x)U(1,x)dx\right]$. Therefore, by the Intermediate Value Theorem, at least one equilibrium path $[\check{x}^*_{-\delta}(\check{x}^*_{\delta}, c),\ \check{x}^*_{\delta}(\check{x}^*_{-\delta}, c)]$ must exist. Finally, for $c \leq \underline{c}$ and $\overline{\overline{c}} \leq c$, existence (and uniqueness) is readily verified for universal entry and universal abstention, respectively. This completes our proof of existence.

(Uniqueness) Next, we prove uniqueness of $(\check{x}^*_{-\delta}, \check{x}^*_{\delta})$ when there is at least one interior cutpoint. To do so, we show that for any given entry probability $\overline{p} \in [0,1]$ at most one pair of cutpoints can simultaneously fulfill the best response condition (28) for $\delta = l, r$ (see Proposition 1 (*iv*)), or conditions (25) and (26) (see Proposition 1 (*iii*)). The main idea of the proof is that any continuous equilibrium path must use all $p \in [0,1]$, and thus, if there is only one cutpoint equilibrium for $\overline{p}$, this would mean there is a unique equilibrium path. Note that keeping $\overline{p}$ constant means that the three $P(n, \overline{p})$-terms in these conditions are not affected when $\check{x}_{-\delta}$ and $\check{x}_{\delta}$ change (see expressions (18) to (20)). It also means that it can neither be a unilateral change in one cutpoint only, nor a simultaneous change in both cutpoints in opposite directions (i.e., jointly more extreme or jointly less extreme). Note that for a fixed $\overline{p}$ this also holds for equilibria with only one interior cutpoint. Thus, by keeping $\overline{p}$ constant, we need to analyze changes in $\check{x}_{-\delta}$ and $\check{x}_{\delta}$ in the same direction. Without loss of generality, we focus on increases from $\check{x}$ to $\check{x}'$, that is, $\check{x}_{-\delta} < \check{x}'_{-\delta} \leq \check{x}_{\min}$ and $\check{x}_{\min} \leq \check{x}_{\delta} < \check{x}'_{\delta}$, under the constraint that $\overline{p}(\check{x}_{-\delta}, \check{x}_{\delta}) = \overline{p}(\check{x}'_{-\delta}, \check{x}'_{\delta})$. We account for these increases by modifying the partition of the

[5] For $n = 2$, there are special cases where $\underline{c} = \overline{\overline{c}}$ (e.g., for the utility function $-\left|\frac{1}{2}(x_i - \gamma)\right|^{\alpha}$ used in subsection "Example: Expected Net-Benefits, Entry Costs, and Unique Path of Entry Equilibria" in the article, when $\alpha = 1$ and citizen ideal points are uniformly distributed), in which case only two possible equilibria exist: either both citizens enter or both abstain.

integrals in $LHS\,(21)$. Then, before the change is implemented, for a $\check{x}_\delta$-type this gives:

$$
\begin{aligned}
& LHS\,(21)\,\big|_{(\check{x}_{-\delta},\check{x}_\delta),\check{x}_\delta} \\
= \; & P_e(n,\overline{p})\int_{-1}^{\check{x}_{-\delta}} f_\delta(x)U(\check{x}_\delta,x)dx \\
& +P_{ne}(n,\overline{p})\int_{\check{x}_{-\delta}}^{\check{x}'_{-\delta}} f_\delta(x)U(\check{x}_\delta,x)dx + P_{ne}(n,\overline{p})\int_{\check{x}'_{-\delta}}^{\check{x}_\delta} f_\delta(x)U(\check{x}_\delta,x)dx \\
& +P_e(n,\overline{p})\int_{\check{x}_\delta}^{\check{x}'_\delta} f_\delta(x)U(\check{x}_\delta,x)dx + P_e(n,\overline{p})\int_{\check{x}'_\delta}^{1} f_\delta(x)U(\check{x}_\delta,x)dx \;+\; P_b(n,\overline{p})b.
\end{aligned} \tag{31}
$$

Next, we rewrite this expression for a $\check{x}'_\delta$-type, after increasing both cutpoints. Compared to expression (31), note that besides replacing $\check{x}_\delta$ with $\check{x}'_\delta$ in the utility function, $U(.)$, also the P-terms of the second and fourth terms are affected. This gives:

$$
\begin{aligned}
& LHS\,(21)\,\Big|_{(\check{x}'_{-\delta},\check{x}'_\delta),\check{x}'_\delta} \\
= \; & P_e(n,\overline{p})\int_{-1}^{\check{x}_{-\delta}} f_\delta(x)U(\check{x}'_\delta,x)dx \\
& +P_{ne}(n,\overline{p})\int_{\check{x}_{-\delta}}^{\check{x}'_{-\delta}} f_\delta(x)U(\check{x}'_\delta,x)dx + P_{ne}(n,\overline{p})\int_{\check{x}'_{-\delta}}^{\check{x}_\delta} f_\delta(x)U(\check{x}'_\delta,x)dx \\
& +P_e(n,\overline{p})\int_{\check{x}_\delta}^{\check{x}'_\delta} f_\delta(x)U(\check{x}'_\delta,x)dx + P_e(n,\overline{p})\int_{\check{x}'_\delta}^{1} f_\delta(x)U(\check{x}'_\delta,x)dx \;+\; P_b(n,\overline{p})b \\
& +\left[P_e(n,\overline{p}) - P_{ne}(n,\overline{p})\right]\int_{\check{x}_{-\delta}}^{\check{x}'_{-\delta}} f_\delta(x)U(\check{x}'_\delta,x)dx \\
& -\left[P_e(n,\overline{p}) - P_{ne}(n,\overline{p})\right]\int_{\check{x}_\delta}^{\check{x}'_\delta} f_\delta(x)U(\check{x}'_\delta,x)dx,
\end{aligned} \tag{32}
$$

where the last two terms are used to make the first six terms comparable to the six terms in expression (31). Importantly, these six terms are strictly larger in expression (32) than in (31). This follows from Lemma 1 by setting $x_i = \check{x}_\delta$ and $x_i = \check{x}'_\delta$, respectively, and using $x_{\min}(\check{x}_{-\delta},\check{x}_\delta) \leq \check{x}_\delta < \check{x}'_\delta$ (because $\check{x}'_\delta$ moves on the same net-benefits curve as $\check{x}_\delta$).[6]

[6] To see this, we simplify expression (31) and the first six terms of expression (32), the latter of which are equivalent to $LHS\,(21)\,\Big|_{(\check{x}_{-\delta},\check{x}_\delta),\check{x}'_\delta}$. This gives:

Next, we examine the last two terms of expression (32). If it holds that

$$[P_e(n,\overline{p}) - P_{ne}(n,\overline{p})] \int_{\check{x}_{-\delta}}^{\check{x}'_{-\delta}} f_\delta(x) U(\check{x}'_\delta, x) dx$$
$$\geq [P_e(n,\overline{p}) - P_{ne}(n,\overline{p})] \int_{\check{x}_\delta}^{\check{x}'_\delta} f_\delta(x) U(\check{x}'_\delta, x) dx$$

$$\Leftrightarrow \int_{\check{x}_{-\delta}}^{\check{x}'_{-\delta}} f_\delta(x) U(\check{x}'_\delta, x) dx \geq \int_{\check{x}_\delta}^{\check{x}'_\delta} f_\delta(x) U(\check{x}'_\delta, x) dx, \tag{33}$$

where $P_e(n,\overline{p}) \geq P_{ne}(n,\overline{p})$ for $p \in (0,1]$,[7] then we have shown that $LHS(21)$ always strictly increases for a given $\overline{p}$ if both cutpoints increase. Note that for $\overline{p}=0$, the only feasible pair is $(\check{x}_{-\delta}=-1, \check{x}_\delta=1)$. To see that condition (33) indeed holds, it is sufficient to show that the minimal gain on the left-hand side, $\int_{\check{x}_\delta}^{\check{x}'_\delta} f_\delta(x) dx U(\check{x}'_\delta, \check{x}'_{-\delta})$ (using $\int_{\check{x}_{-\delta}}^{\check{x}'_{-\delta}} f_\delta(x)dx = \int_{\check{x}_\delta}^{\check{x}'_\delta} f_\delta(x)dx$, since $\overline{p}$ is held constant), is equal to or larger than the maximal loss on the right-hand side, $\int_{\check{x}_\delta}^{\check{x}'_\delta} f_\delta(x) dx U(\check{x}'_\delta, \check{x}_\delta)$. That is, we set the most extreme values constant and multiply them by the equal probabilities. This gives:

$$\int_{\check{x}_\delta}^{\check{x}'_\delta} f_\delta(x) dx U(\check{x}'_\delta, \check{x}'_{-\delta}) \geq \int_{\check{x}_\delta}^{\check{x}'_\delta} f_\delta(x) dx U(\check{x}'_\delta, \check{x}_\delta) \tag{34}$$
$$\Leftrightarrow U(\check{x}'_\delta, \check{x}'_{-\delta}) \geq U(\check{x}'_\delta, \check{x}_\delta),$$

$$LHS\,(21)\big|_{(\check{x}_{-\delta},\check{x}_\delta),\check{x}_\delta} = P_e(n,\overline{p}) \int_{-1}^{\check{x}_{-\delta}} f_\delta(x) U(\check{x}_\delta, x) dx + P_{ne}(n,\overline{p}) \int_{\check{x}_{-\delta}}^{\check{x}_\delta} f_\delta(x) U(\check{x}_\delta, x) dx$$
$$+P_e(n,\overline{p}) \int_{\check{x}_\delta}^{1} f_\delta(x) U(\check{x}_\delta, x) dx + P_b(n,\overline{p}) b$$
$$< LHS\,(21)\Big|_{(\check{x}_{-\delta},\check{x}_\delta),\check{x}'_\delta} = P_e(n,\overline{p}) \int_{-1}^{\check{x}_{-\delta}} f_\delta(x) U(\check{x}'_\delta, x) dx + P_{ne}(n,\overline{p}) \int_{\check{x}_{-\delta}}^{\check{x}_\delta} f_\delta(x) U(\check{x}'_\delta, x) dx$$
$$+P_e(n,\overline{p}) \int_{\check{x}_\delta}^{1} f_\delta(x) U(\check{x}'_\delta, x) dx + P_b(n,\overline{p}) b.$$

for $\check{x}_\delta < \check{x}'_\delta$ (see Lemma 1).

[7] Using expressions (18) and (19), this is derived as follows: $P_e(n,p) \geq P_{ne}(n,p)$ for $p \in (0,1]$

$$\Leftrightarrow \quad p\sum_{k=1}^{n}(1-p)^{k-1} \geq np(1-p)^{n-1} + (n-1)\,p^2(1-p)^{n-2} \Leftrightarrow \sum_{k=1}^{n} \frac{1}{(1-p)^{n-k-1}} \geq n-p$$
$$\Leftrightarrow \quad \frac{1}{(1-p)^{n-2}} + \frac{1}{(1-p)^{n-3}} + ... + 1 + (1-p) \geq n-p, \text{ since } \frac{1}{(1-p)^{n-k-1}} \geq 1 \text{ for } k=1,...,n-1$$

if $p \in (0,1)$ and $\frac{1}{(1-p)^{n-k-1}} = 1-p$ if $k=n$. Note that $P_e(n,\overline{p}) = P_{ne}(n,\overline{p})$ if $n=2$.

which always holds because $\check{x}'_{-\delta} \leq \check{x}_{\min} \leq \check{x}_{\delta} < \check{x}'_{\delta}$. Note that the same things hold when there is one boundary cutpoint, $\check{x}'_{-\delta} = -1$ (this is readily verified by replacing $\check{x}'_{-\delta}$ with -1 in expressions (32) to (34)). Therefore, increasing $\check{x}_{-\delta}$ and $\check{x}_{\delta}$ while keeping $\overline{p}$ constant yields $LHS\,(21)\left|_{(\check{x}_{-\delta},\check{x}_{\delta}),\check{x}_{\delta}} < LHS\,(21)\right|_{(\check{x}'_{-\delta},\check{x}'_{\delta}),\check{x}'_{\delta}}$, and also $LHS\,(21)\left|_{(\check{x}_{-\delta},\check{x}_{\delta}),\check{x}_{-\delta}} > LHS\,(21)\right|_{(\check{x}'_{-\delta},\check{x}'_{\delta}),\check{x}'_{-\delta}}$ (to understand the latter inequality, consider the reverse decreases from $\check{x}'_{-\delta}$ to $\check{x}_{-\delta}$ and $\check{x}'_{\delta}$ to $\check{x}_{\delta}$, which is analyzed analogous to the increases above). However, in equilibrium it must hold for $(\check{x}_{-\delta}, \check{x}_{\delta})$ that $LHS\,(21)\left|_{(\check{x}_{-\delta},\check{x}_{\delta}),\check{x}_{-\delta}} = LHS\,(21)\right|_{(\check{x}_{-\delta},\check{x}_{\delta}),\check{x}_{\delta}}$, and thus, given the two inequalities it cannot hold simultaneously for $(\check{x}'_{-\delta}, \check{x}'_{\delta})$ that $LHS\,(21)\left|_{(\check{x}'_{-\delta},\check{x}'_{\delta}),\check{x}'_{-\delta}} = LHS\,(21)\right|_{(\check{x}'_{-\delta},\check{x}'_{\delta}),\check{x}'_{\delta}}$. Thus, for any given entry probability $\overline{p} \in [0, 1]$, there is a unique cutpoint equilibrium. Given the properties of the equilibrium path derived above, this also means that there is a unique cutpoint equilibrium for any given $c > 0$, which completes our proof Proposition 1. ■

Proof of Proposition 2

Proof. We begin by analyzing the comparative statics effects of changes in the costs of entry, c, and the benefits from holding office, b on the equilibrium cutpoints, $(\check{x}_l^*, \check{x}_r^*)$, that use at least one interior cutpoint. Thereafter, we derive the cutpoints for very large n, that is, $\lim_{n\to\infty} \check{x}_l^*(n)$ and $\lim_{n\to\infty} \check{x}_r^*(n)$.

The proof uses the best response entry strategy (21). First, note that the three $P(n, p)$-terms (see expressions (18) to (20)) in this condition are not directly affected by a change in c or b, and the three integral terms are not directly affected by a change c, b, or n. Importantly, if $\check{x}_{-\delta}$ and $\check{x}_{\delta}$ are interior, we know from the proof of Proposition 1 that there is a unique equilibrium path where $LHS(21)\left|_{\check{x}_{-\delta}} = LHS(21)\right|_{\check{x}_{\delta}} = c$ and both cutpoints simultaneously get more extreme if c increases. Moreover, if $\check{x}_{-\delta} = -1$ and $\check{x}_{\delta}$ is interior, there is a unique equilibrium path where $LHS(21)\left|_{\check{x}_{-\delta}=-1} \leq LHS(21)\right|_{\check{x}_{\delta}} = c$ and $\check{x}_{-\delta} = -1$ remains and $\check{x}_{\delta}$ gets more extreme if c increases. These results can be used to derive the following comparative statics effects:

(Costs of entry) $LHS(21)$ is constant in c while $RHS(21)$ is strictly increasing in c for

$\delta = l, r$. Because on the unique equilibrium path $LHS(21)$ is strictly increasing if both interior cutpoints $\check{x}_{-\delta}$ and $\check{x}_{\delta}$ get more extreme (if $\check{x}_{-\delta} = -1$ remains and the interior cutpoint $\check{x}_{\delta}$ increases), this implies that on this path $\check{x}_{-\delta}$ strictly decreases (remains) and $\check{x}_{\delta}$ strictly increases if c increases. This implies less entry, in the sense of stochastic dominance, and therefore the expected number of candidates decreases. It also implies that candidates and policy outcomes are more extreme, on average.

(Benefits from holding office) $LHS(21)$ is strictly increasing in b (since $P_b(n, p) > 0$) while $RHS(21)$ is constant in b for $\delta = l, r$. Because on the unique equilibrium path $LHS(21)$ is strictly decreasing if both interior cutpoints $\check{x}_{-\delta}$ and $\check{x}_{\delta}$ get more moderate (if $\check{x}_{-\delta}(\check{x}_{\delta} \geq \overline{x}_{\delta}) = -1$ remains or increases and the interior cutpoint $\check{x}_{\delta}$ decreases), this implies that on this path $\check{x}_{-\delta}$ strictly increases (remains or strictly increases) and $\check{x}_{\delta}$ strictly decreases if b increases. This implies more entry, in the sense of stochastic dominance, and therefore the expected number of entrants increases. It also implies that candidates and policy outcomes are less extreme, on average.

(Community size) Here we show that $\lim_{n \to \infty} \check{x}_l^*(\check{x}_r^*, n) = -1$. The proof that $\lim_{n \to \infty} \check{x}_r^*(\check{x}_l^*, n) = 1$ is identical. Because we are looking at infinite sequences on a compact set, there must exist at least one convergent subsequence so we only need to show $\liminf_{n \to \infty} \check{x}_l^*(\check{x}_r^*, n) = -1$. Suppose to the contrary that $\liminf_{n \to \infty} \check{x}_l^*(\check{x}_r^*, n) > -1$. Then there exists an ϵ and a subsequence $\{n_k\} \to \infty$ and an integer $\overline{k}$ such that for $k > \overline{k}$ the probability a randomly selected citizen enters equals $p_k > \epsilon$. This implies that the equilibrium probability of winning along this subsequence goes to zero. But this in turn implies that nobody will enter, which implies $\check{x}_l^*(\check{x}_r^*, n_k) = -1$, a contradiction. ■

Proof of Proposition 3

Proof. To show (i), we already showed in Proposition 1 that no entry is an equilibrium in the finite n case if and only if $c \geq \frac{n-1}{n} \max[b + v_l, b + v_r]$. The result follows immediately. Therefore, in both cases (ii) and (iii) we must have $\tau > 0$. The best response condition (21) for $\delta = l, r$

is:

$$P_{ne}(n,p)\int_{\check{x}_{-\delta}}^{\check{x}_{\delta}} f_\delta(x)U(\check{x}_\delta,x)dx$$
$$+P_e(n,p)\left(\int_{-1}^{\check{x}_{-\delta}} f_\delta(x)U(\check{x}_\delta,x)dx+\int_{\check{x}_\delta}^{1} f_\delta(x)U(\check{x}_\delta,x)dx\right)+P_b(n,p)b\geq c.$$

For $n\to\infty$, and using $\lim_{n\to\infty}\check{x}^*_{-\delta}(n)=-1$ and $\lim_{n\to\infty}\check{x}^*_{\delta}(n)=1$ (see Proposition 2) and $\lim_{n\to\infty}c=c$, the best response condition for a citizen type $\check{x}_\delta$ can be reduced to:

$$\lim_{n\longrightarrow\infty}\left[P_{ne}(n,p)\int_{-1}^{1} f_\delta(x)U(1,x)dx+P_e(n,p)p_{-\delta}(n)U(1,-1)+P_b(n,p)b\right]\geq c, \tag{35}$$

since $U(1,-1)=\lim_{n\to\infty}\frac{\int_{-1}^{\check{x}^*_{-\delta}(n)} f_\delta(x)U(1,x)dx}{\int_{-1}^{\check{x}^*_{-\delta}(n)} f_\delta(x)dx}$ and $U(1,1)=\lim_{n\to\infty}\frac{\int_{\check{x}^*_{\delta}(n)}^{1} f_\delta(x)U(1,x)dx}{\int_{\check{x}^*_{\delta}(n)}^{1} f_\delta(x)dx}=0.$

Moreover, using expressions (18) and (20) gives:

$$\lim_{n\longrightarrow\infty}P_{ne}(n,p)=\lim_{n\longrightarrow\infty}\frac{(n-1)(1-p)^{n-2}}{n}=\lim_{n\longrightarrow\infty}(1-p)^{n-2},$$

$$\lim_{n\longrightarrow\infty}P_e(n,p)p_{-\delta}(n)=\lim_{n\longrightarrow\infty}\frac{p_{-\delta}}{p}\left[\frac{1-(1-p)^n}{np}-(1-p)^{n-1}\right],$$

and

$$\lim_{n\longrightarrow\infty}P_b(n,p)=\lim_{n\longrightarrow\infty}\left[\frac{1-(1-p)^n}{np}-\frac{(1-p)^{n-1}}{n}\right].$$

For large N, since p is close to 0,[8] we can approximate the binomial distribution by the Poisson distribution using $\binom{N}{k}p^k(1-p)^{N-k}\approx\frac{(Np)^k}{k!}e^{-np}$. Moreover, let us denote $\tau\equiv\lim_{n\to\infty}E(m)$ $=\lim_{n\to\infty}np$ and $\tau_\delta\equiv\lim_{n\to\infty}E(m_\delta)=\lim_{n\to\infty}np_\delta$, where $p=p_{-\delta}+p_\delta$, $m=m_{-\delta}+m_\delta$, and $\tau=\tau_{-\delta}+\tau_\delta$ for $\delta=l,r$ and $\delta\neq-\delta$. Then, setting $k=0$ and $N=n-2$ in the Poisson approximation yields:

$$(1-p)^{n-2}\approx\frac{[(n-2)p]^0}{0!}e^{-(n-2)p}=e^{-(n-2)p}\approx e^{-\tau},$$

[8]Note that p must converge to zero, and np must be bounded. If not, then the expected number of entrants would be infinite, and therefore nobody will enter because the probability of winning is zero and $c>0$.

where $e^{-(n-2)p_\delta} \approx e^{-np_\delta}$ and $e^{-(n-2)p} \approx e^{-np}$. Similarly, setting $k = 0$ and $N = n$ ($N = n-1$) in the Poisson approximation yields:

$$\begin{aligned}
&\frac{p_{-\delta}}{p}\left[\frac{1-(1-p)^n}{np} - (1-p)^{n-1}\right] \\
\approx\ & \frac{\tau_{-\delta}}{\tau}\left[\frac{1-\frac{(np)^0}{0!}e^{-np}}{\tau} - \frac{[(n-1)p]^0}{0!}e^{-(n-1)p}\right] \approx \frac{\tau_{-\delta}}{\tau}\left[\frac{1-e^{-\tau}}{\tau} - e^{-\tau}\right]
\end{aligned}$$

and

$$\frac{1-(1-p)^n}{np} - \frac{(1-p)^{n-1}}{n} \approx \frac{1-\frac{(np)^0}{0!}e^{-np}}{\tau} - \frac{\frac{[(n-1)p]^0}{0!}e^{-(n-1)p}}{n} = \frac{1-e^{-\tau}}{\tau},$$

where $e^{-(n-1)p_\delta} \approx e^{-np_\delta}$ and $e^{-(n-1)p} \approx e^{-np}$ and $\lim_{n\to\infty} \frac{e^{-(n-1)p}}{n} = 0$ since $e^{-(n-1)p} \leq 1$ for $p \in [0,1]$. Thus we can rewrite the best response condition (35) as

$$e^{-\tau}v_\delta + \frac{\tau_{-\delta}}{\tau}\left[\frac{1-e^{-\tau}}{\tau} - e^{-\tau}\right]U(1,-1) + \frac{1-e^{-\tau}}{\tau}b \geq c. \tag{36}$$

To prove case (*ii*), observe that $\tau_\delta > 0$ for both $\delta = l, r$ implies that (36) holds with equality for both $\delta = l, r$. Rearranging terms yields

$$\tau_\delta = -\tau_{-\delta} + \frac{1}{c}\left[\tau e^{-\tau}v_\delta + \frac{\tau_{-\delta}}{\tau}\left[1-(\tau+1)e^{-\tau}\right]U(1,-1) + (1-e^{-\tau})b\right], \tag{37}$$

Thus equations (12) and (13) admit strictly positive solutions to τ_l and τ_r. To show the opposite direction, suppose that equations (12) and (13) admit strictly positive solutions to τ_l and τ_r. Then for large n there must be interior equilibrium cutpoints $\{\check{x}_l^*(n), \check{x}_r^*(n)\}$ whose expected entry rates from the left and right are arbitrarily close to τ_l and τ_r, respectively.

Case (*iii*) follows immediately from the first two cases. We can say a bit more about the range of costs where case (*iii*) holds. Specifically, there will be entry only from the left if $c \in [\widetilde{c}_r, b + v_l)$, with $\widetilde{c}_r$ being the entry cost at which equations (12) and (13) both hold with equality and $\tau_r = 0$. Similarly, there will be entry only from the right if $c \in [\widetilde{c}_l, b + v_r)$, with $\widetilde{c}_l$ being the entry cost at which equations (12) and (13) both hold with equality and $\tau_l = 0$. For example,

$$\widetilde{c}_r = e^{-\tau_l(\widetilde{c}_r)}v_r + \left[\frac{1-e^{-\tau_l(\widetilde{c}_r)}}{\tau_l(\widetilde{c}_r)} - e^{-\tau_l(\widetilde{c}_r)}\right]U(1,-1) + \frac{1-e^{-\tau_l(\widetilde{c}_r)}}{\tau_l(\widetilde{c}_r)}b.$$

where $\tau_l(\widetilde{c}_r)$ solves (12) at $\widetilde{c}_r$, with $\tau_r = 0$.

Moreover, if $v_l > v_r$ we can only have the case of one-sided entry from the left, and if $v_r > v_l$ we can only have the case of one-sided entry from the right.

To see this, suppose that $v_l > v_r$ (the case of $v_r > v_l$ is proved similarly). From (*i*) we know that if $c = b + v_l$ then $\tau_l = \tau_r = \tau = 0$. Moreover, equation (12), with $\frac{\tau_r}{\tau}$ replaced by 0 (because there is no entry) is satisfied with equality at $c_l \equiv v_l + b$. To see this, note that when $\tau_l = \tau_r = \tau = 0$ equation (12) can be written as:

$$c_l = \lim_{\tau \longrightarrow 0}\left[\tau e^{-\tau}v_l + \frac{(1-e^{-\tau})}{\tau}b\right] = v_l + b \tag{38}$$

by l'Hôpital's rule. Hence at this cost, a citizen with $\check{x}_l = -1$ is *indifferent between entering and not entering.* But since $v_l > v_r$, we have $c_l > v_r + b$ so a citizen with $\check{x}_r = 1$ is *strictly better off not entering.* Since equations (12) and (13) vary continuously in c, τ_l, and τ_r it must be the case that for all costs $c < c_l$ in a small enough neighborhood of c_l the equilibrium will have $\tau_l > 0$ and $\tau_r = 0$. It is straightforward to prove that $\widetilde{c}_r$ exists. It cannot be the case that $\tau_r = 0$ for all values of c because $U(1,-1) > \max\{v_l, v_r\}$, and if equations (12) and (13) *both* hold with equality for some c such that $\tau_l > 0$ and $\tau_r = 0$, then equations (12) and (13) *both* hold with equality with $\tau_l > 0$ and $\tau_r > 0$ for all $c' < c$. To see that entry from only one direction must be in the l-direction if and only if $v_l > v_r$, we only need to show that whenever equations (12) and (13) both hold and $\tau > 0$ we must have $\tau_l \geq \tau_r$. This follows because a necessary condition for equations (12) and (13) to both hold and $\tau > 0$ is:

$$\tau e^{-\tau}v_l + \frac{\tau_r}{\tau}\left[1-(\tau+1)e^{-\tau}\right]U(1,-1) = \tau e^{-\tau}v_r + \frac{\tau_l}{\tau}\left[1-(\tau+1)e^{-\tau}\right]U(1,-1),$$

which holds if and only if $\tau_l \geq \tau_r$. This completes the proof of Proposition 3 (*iii*) and hence Proposition 3. ■

Proof of Proposition 4

Proof. The proof uses the same logic as the proof of Proposition 1 for private information. First, we derive citizen i's best response entry strategy with directional information about each entrant's ideal points (which is revealed via nomination conventions of party L and party R), and show that it is always a cutpoint strategy (cf. Lemma 1). Thereafter, we use Envelope Theorem and Intermediate Value Theorem to prove that a symmetric entry equilibrium always exists for any cumulative probability distribution of ideal points, $F(x)$, satisfying $A1$-$A3$ (see section "General Model" in the article). Finally, we argue why the equilibrium characterization and comparative statics results with directional information are very similar to the case with private information (cf. Propositions 2 and 3).

(Best response entry strategy) Consider citizen i. Suppose all citizens $j \neq i$ are using an entry strategy defined by two cutpoints:

$$\breve{e}_j = \begin{cases} 0 & if \quad x_j \in (\breve{x}_l, \breve{x}_r) \\ 1 & if \quad x_j \in [-1, \breve{x}_l] \cup [\breve{x}_r, 1], \end{cases} \tag{39}$$

where $(\breve{x}_l, \breve{x}_r)$ is some pair of ideal points with $-1 \leq \breve{x}_l \leq \breve{x}_r \leq 1$. Hence, j is a contender for the nomination of party L (R) if $x_j \in [-1, \breve{x}_l]$ ($x_j \in [\breve{x}_r, 1]$). We assume that voters can verify an entrant's political leaning, and hence, we rule out the possibility that she competes in the nominating convention of the opposing party for strategic reasons. Moreover, let s_δ denote the probability that a randomly selected citizen $j \neq i$ enters to seek the nomination of party δ, with $\delta = l, r$, where $s_{-\delta} \equiv \Pr(x_j \leq \breve{x}_{-\delta}) = F(\breve{x}_{-\delta})$, $s_\delta \equiv \Pr(x_j \geq \breve{x}_\delta) = 1 - F(\breve{x}_\delta)$, and $s \equiv s_{-\delta} + s_\delta$ for our $F(x)$, $x \in [-1, 1] \subset \mathbb{R}$, and $\delta \neq -\delta$. Recall the voting cutpoint $\breve{x}_\phi \equiv \frac{\overline{x}_{-\delta} + \overline{x}_\delta}{2}$, with $\overline{x}_{-\delta} \equiv \frac{\int_{-1}^{\breve{x}_{-\delta}} f_\delta(x) x dx}{s_{-\delta}}$ and $\overline{x}_\delta \equiv \frac{\int_{\breve{x}_\delta}^{1} f_\delta(x) x dx}{s_\delta}$. Then, party $-\delta$'s and party δ's expected vote share is given by $\phi_{-\delta}(\breve{x}_\phi) \equiv \int_{-1}^{\breve{x}_\phi} f_\delta(x) dx$ and $\phi_\delta(\breve{x}_\phi) \equiv 1 - \phi_{-\delta}(\breve{x}_\phi) = \int_{\breve{x}_\phi}^{1} f_\delta(x) dx$, respectively.

To derive the equilibrium cutpoints, $\left[(\breve{x}_l^*, \breve{x}_r^*), \breve{x}_\phi^*\right]$, we must compare citizen i's expected payoffs as both an entrant and a non-entrant, given the equilibrium decisions in subsequent stages (i.e., the *Nomination*, *Voting*, and *Policy decision* stages). Then, $\left[\left(\breve{x}_l^*, \breve{x}_r^*\right), \breve{x}_\phi^*\right]$ is a symmetric equilibrium if and only if $\breve{e}_i\left(\breve{x}_l^*, \breve{x}_r^*\right)$ is a best response for citizen i when $\breve{e}_j\left(\breve{x}_l^*, \breve{x}_r^*\right)$ is the entry strategy of all $j \neq i$.

The expected payoff of a type $x_i \in [\breve{x}_\phi, 1]$ for *entering* as a δ-candidate, $\breve{e}_i = 1$, when all other citizens $j \neq i$ use the entry strategy $\breve{e}_j$, is:

$$
\begin{aligned}
&E[\pi_i \mid x_i \in [\breve{x}_\phi, 1], \breve{e}_i = 1] = (1-s)^{n-1}\, b \\
&\quad + \sum_{m_\delta=2}^{n} \binom{n-1}{m_\delta - 1} (s_\delta)^{m_\delta - 1} (1-s)^{n-m_\delta} \left[\frac{b}{m_\delta} - \frac{m_\delta - 1}{m_\delta}\, E\left[U(x_i,\gamma) \mid \gamma \in [\breve{x}_\delta, 1]\right]\right] \qquad (40) \\
&\quad + \sum_{m_{-\delta}=1}^{n-1} \binom{n-1}{m_{-\delta}} (s_{-\delta})^{m_{-\delta}} (1-s)^{n-m_{-\delta}-1} \left[\rho_\delta b - (1-\rho_\delta)\; E\left[U(x_i,\gamma) \mid \gamma \in [-1, \breve{x}_{-\delta}]\right]\right] \\
&\quad + \sum_{m_\delta=2}^{n-1} \sum_{m_{-\delta}=1}^{n-m_\delta} \binom{n-1}{m_\delta - 1} \binom{n-m_\delta}{m_{-\delta}} (s_\delta)^{m_\delta - 1} (s_{-\delta})^{m_{-\delta}} (1-s)^{n-m_\delta-m_{-\delta}} \\
&\qquad \times \left[\frac{\rho_\delta}{m_\delta} b - (1-\rho_\delta)\, E\left[U(x_i,\gamma) \mid \gamma \in [-1, \breve{x}_{-\delta}]\right] - \frac{m_\delta - 1}{m_\delta} \rho_\delta E\left[U(x_i,\gamma) \mid \gamma \in [\breve{x}_\delta, 1]\right]\right] - c.
\end{aligned}
$$

The first term gives the case where i receives b since she is the only entrant, which occurs with probability $(1-s)^{n-1}$. The second term gives the cases where, in addition to herself, there are $m_\delta - 1$ other contenders for the nomination of party δ, but there is no party $-\delta$ because nobody enters in this direction ($m_{-\delta} = 0$). This occurs with probability $\binom{n-1}{m_\delta - 1} (s_\delta)^{m_\delta - 1} (1-s)^{n-m_\delta}$, where the summation accounts for all possible $m_\delta - 1 \geq 1$, and the probability that i will be nominated and hence lead the community is $1/m_\delta$. Then, her expected benefits from holding office are b/m_δ and her expected payoff loss in case a like-minded entrant will lead the community is $\frac{m_\delta - 1}{m_\delta}\, E\left[U(x_i,\gamma) \mid \gamma \in [\breve{x}_\delta, 1]\right]$ (see expression (42) below). The third term gives the cases where i is the only contender for the nomination of party δ ($m_\delta = 1$), and there are $m_{-\delta} = 1, ..., n-1$ contenders in party $-\delta$, which occurs with probability $\binom{n-1}{m_{-\delta}} (s_{-\delta})^{m_{-\delta}} (1-s)^{n-m_{-\delta}-1}$. Then, i will lead the community and gain b with probability ρ_δ, that is, she is the nominee with probability 1 and her "party" will win with probability ρ_δ (see expression (43), below) and her expected payoff loss is $(1-\rho_\delta)\; E\left[U(x_i,\gamma) \mid \gamma \in [-1, \breve{x}_{-\delta}]\right]$ (see expression (41), below). Finally, the fourth term gives the cases with at least two contenders for nomination of party δ ($m_\delta - 1 \geq 1$, including i) and at least one contender in party $-\delta$ ($m_{-\delta} \geq 1$), which occurs with probability $\binom{n-1}{m_\delta-1}\binom{n-m_\delta}{m_{-\delta}}(s_\delta)^{m_\delta-1}(s_{-\delta})^{m_{-\delta}}(1-s)^{n-m_\delta-m_{-\delta}}$ (the summations account for all possible $m_\delta + m_{-\delta} = 3, ..., n$ with $m_\delta \geq 2$ and $m_{-\delta} \geq 1$). Then, citizen i's expected benefit from holding office is $\frac{\rho_\delta}{m_\delta} \cdot b$ and her expected payoff loss if she will not be the community leader is $(1-\rho_\delta) E[U(x_i,\gamma) \mid \gamma \in [-1, \breve{x}_{-\delta}]] + \frac{m_\delta-1}{m_\delta}\rho_\delta E[U(x_i,\gamma) \mid \gamma \in [\breve{x}_\delta, 1]]$ (i.e., some $j \neq i$ will lead the community with probability $1 - \frac{\rho_\delta}{m_\delta}$). Specifically, the expected payoff loss if party $-\delta$ wins and implements a policy $\gamma \in [-1, \breve{x}_{-\delta}]$, which occurs with probability $1-\rho_\delta$, is:

$$E[U(x_i,\gamma) \mid \gamma \in [-1, \breve{x}_{-\delta}]] = \frac{\int_{-1}^{\breve{x}_{-\delta}} f_\delta(x) U(x_i, x) dx}{s_{-\delta}} \quad for \;\; \breve{x}_{-\delta} \neq -1, \tag{41}$$

and if another δ-party contender will be the nominee and implements a policy $\gamma \in [\breve{x}_\delta, 1]$, which occurs with probability $\frac{m_\delta-1}{m_\delta}\rho_\delta$, the expected payoff loss is:

$$E[U(x_i,\gamma) \mid \gamma \in [\breve{x}_\delta, 1]] = \frac{\int_{\breve{x}_\delta}^{1} f_\delta(x) U(x_i, x) dx}{s_\delta} \quad for \;\; \breve{x}_\delta \neq 1. \tag{42}$$

Moreover, the probability that party δ will win is given by

$$\begin{cases} 0 & \text{if } m_\delta = 0 \wedge m_{-\delta} \geq 1 \\ \rho_\delta(\phi_\delta) \equiv \sum_{n_\delta = \lfloor n/2 \rfloor + 1}^{n} \binom{n}{n_\delta} (\phi_\delta)^{n_\delta} (1-\phi_\delta)^{n-n_\delta} & \\ \qquad + \begin{cases} \frac{1}{2}\binom{n}{n/2}(\phi_\delta)^{n/2}(1-\phi_\delta)^{n/2} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases} & \text{if } m_\delta \geq 1 \wedge m_{-\delta} \geq 1 \;. \\ 1 & \text{if } m_\delta \geq 1 \wedge m_{-\delta} = 0 \end{cases} \tag{43}$$

The summation in the first term of $\rho_\delta(\phi_\delta)$ accounts for all possible victories of party δ, that is for $n_\delta = \left\lfloor \frac{n}{2} \right\rfloor + 1, ..., n$, where n_δ ($n_{-\delta} \equiv n - n_\delta$) denotes the vote count of party δ ($-\delta$). The probability of the n_δ-events are given by $\binom{n}{n_\delta}(\phi_\delta)^{n_\delta}(1-\phi_\delta)^{n-n_\delta}$ (recall that the probability that a randomly selected citizen votes for party δ is ϕ_δ). Moreover, the second term of $\rho_\delta(\phi_\delta)$

accounts for a tie, $n_{-\delta} = n_{\delta}$, which can only occur for n even and in which case a random tie-breaking determines the winner with probability one half for each party.

Finally, i bears the entry costs, c, independent of how many other citizens enter as a left or right candidate, which gives the fifth term in expression (40).

By contrast, the expected payoff of a type $x_i \in [\breve{x}_{\phi}, 1]$ for *not entering* as a δ-candidate, $\breve{e}_i = 0$, is:

$$E[\pi_i \mid x_i \in [\breve{x}_{\phi}, 1], \breve{e}_i = 0] = (1-s)^{n-1}\left[\frac{b}{n} - \frac{n-1}{n}E[U(x_i, d) \mid d \in (\breve{x}_{-\delta}, \breve{x}_{\delta})] \right]$$
$$- \sum_{m_\delta=2}^{n} \binom{n-1}{m_\delta - 1} (s_\delta)^{m_\delta - 1} (1-s)^{n-m_\delta} \; E\left[U(x_i, \gamma) \mid \gamma \in [\breve{x}_\delta, 1]\right] \tag{44}$$
$$- \sum_{m_{-\delta}=1}^{n-1} \binom{n-1}{m_{-\delta}} (s_{-\delta})^{m_{-\delta}} (1-s)^{n-m_{-\delta}-1} \; E\left[U(x_i, \gamma) \mid \gamma \in [-1, \breve{x}_{-\delta}]\right]$$
$$- \sum_{m_\delta=2}^{n-1} \sum_{m_{-\delta}=1}^{n-m_\delta} \binom{n-1}{m_\delta - 1} \binom{n-m_\delta}{m_{-\delta}} (s_\delta)^{m_\delta - 1} (s_{-\delta})^{m_{-\delta}} (1-s)^{n-m_\delta-m_{-\delta}}$$
$$\times \left[(1-\rho_\delta) E\left[U(x_i, \gamma) \mid \gamma \in [-1, \breve{x}_{-\delta}]\right] + \rho_\delta E\left[U(x_i, \gamma) \mid \gamma \in [\breve{x}_\delta, 1]\right] \right].$$

The first term corresponds to the event where, like herself, no other citizen enters, which occurs with probability $(1-s)^{n-1}$. In this case the stochastic default policy, d, takes effect (i.e., one citizen is randomly selected as the new community leader). Then, citizen i's expected benefits from holding office is b/n (we assume that d does not invoke any entry costs in this event; see footnote 8 in the article) and with probability $(n-1)/n$ she will not be selected which yields her an expected payoff loss of:

$$E[U(x_i, d) \mid d \in (\breve{x}_{-\delta}, \breve{x}_\delta)] = \frac{\int_{\breve{x}_{-\delta}}^{\breve{x}_\delta} f_\delta(x) \; U(x_i, x) dx}{1-s} \quad \textit{for } \breve{x}_{-\delta} \neq \breve{x}_\delta. \tag{45}$$

Observe that if $\breve{x}_{-\delta} = \breve{x}_\delta$, the default policy is irrelevant because all citizens enter. The other three terms in expression (44) are very similar to those in expression (40), except that i does not enter and thus cannot gain b.

Finally, relating expressions (40) and (44) and rearranging yields the *best response* entry strategy for a citizen with ideal point $x_i \in [\breve{x}_\phi, 1]$, if all other citizens are using cutpoint strategy $\breve{e}_j$, $j \neq i$, which is to enter as a δ-candidate if and only if[9]

$$
\begin{aligned}
&(1-s)^{n-1}\left(\tfrac{n-1}{n}\right)[b + E[\,U(x_i,d) \mid d \in (\breve{x}_{-\delta}, \breve{x}_\delta)]] \\
&\quad + \sum_{m_\delta=2}^{n} \binom{n-1}{m_\delta - 1} (s_\delta)^{m_\delta - 1} (1-s)^{n-m_\delta} \frac{1}{m_\delta} \left[b + E\left[U(x_i,\gamma) \mid \gamma \in [\breve{x}_\delta, 1]\right] \right] \qquad (46) \\
&\quad + \sum_{m_{-\delta}=1}^{n-1} \binom{n-1}{m_{-\delta}} (s_{-\delta})^{m_{-\delta}} (1-s)^{n-m_{-\delta}-1} \; \rho_\delta \left[b + E\left[U(x_i,\gamma) \mid \gamma \in [-1, \breve{x}_{-\delta}]\right] \right] \\
&\quad + \sum_{m_\delta=2}^{n-1} \sum_{m_{-\delta}=1}^{n-m_\delta} \binom{n-1}{m_\delta - 1} \binom{n-m_\delta}{m_{-\delta}} (s_\delta)^{m_\delta - 1} (s_{-\delta})^{m_{-\delta}} (1-s)^{n-m_\delta-m_{-\delta}} \\
&\qquad \times \frac{\rho_\delta}{m_\delta} \left[b + E\left[U(x_i,\gamma) \mid \gamma \in [\breve{x}_\delta, 1]\right] \right] \;\geq\; c,
\end{aligned}
$$

where $LHS(46)$ and $RHS(46)$ give citizen i's expected net-benefits and costs from running for office, respectively. Note that the expected payoff loss $(1-\rho_\delta)\, E\left[U(x_i,\gamma) \mid \gamma \in [-1, \breve{x}_{-\delta}]\right]$ does no longer appear in the fourth term of condition (46), because the winning probability of party $-\delta$ is unaffected by citizen i's decision to enter when $m_\delta \geq 2 \wedge m_{-\delta} \geq 1$.

(U-shaped net-benefits curve) Next, we show that a symmetric entry equilibrium of our citizen candidate model with directional information is always in cutpoint strategies. Similar to subsection "Cutpoint Strategies and Best Response Condition" in the article, we analyze the best response of a citizen type $x_i \in [\breve{x}_\phi, 1]$ for any arbitrary symmetric entry strategy, $\sigma(x) : [-1,1] \to [0,1]$, played by all $j \neq i$, where $\sigma(x)$ denotes the probability of entering for a citizen with ideal point x. Then, the left-hand side of best response condition (46) can be

[9] Without loss of generality, we assume that indifferent citizen types choose to enter.

written more generally as:

$$\widetilde{Q}_{ne}(n,q)\int_{-1}^{1} f_{\delta,ne}(x|\sigma)\; U(x_i,x)dx + \widetilde{Q}_{e,-\delta}(n,q_{-\delta},q_\delta)\int_{-1}^{\breve{x}_\phi} f_{\delta,e}(x|\sigma)\; U(x_i,x)dx \qquad (47)$$

$$+\,\widetilde{Q}_{e,\delta}(n,q_{-\delta},q_\delta)\int_{\breve{x}_\phi}^{1} f_{\delta,e}(x|\sigma)\; U(x_i,x)dx + \widetilde{Q}_b(n,q_{-\delta},q_\delta)b \;\geq\; c,$$

where, $q_{-\delta} \equiv \int_{-1}^{\breve{x}_\phi} \sigma(x) f_\delta(x)dx$ and $q_\delta \equiv \int_{\breve{x}_\phi}^{1} \sigma(x) f_\delta(x)dx$ give the probabilities that a randomly selected $j \neq i$ enters as a $-\delta$-candidate and δ-candidate, respectively, with $q \equiv q_{-\delta} + q_\delta$, and $\breve{x}_\phi$ is implicitly determined by $\int_{-1}^{\breve{x}_\phi} f_{\delta,e}(x|\sigma)\; U(\breve{x}_\phi,x)dx = \int_{\breve{x}_\phi}^{1} f_{\delta,e}(x|\sigma)\; U(\breve{x}_\phi,x)dx$. The conditional distribution of types in the no entry ('ne') and some entry ('e') events are $f_{\delta,ne}(x|\sigma) = \frac{[1-\sigma(x)]f_\delta(x)}{1-q}$ and $f_{\delta,e}(x|\sigma) = \frac{\sigma(x)f_\delta(x)}{q}$, assuming $q \in (0,1)$ (see expressions (9) and (10) in the article). Then, $\widetilde{Q}_{ne}(n,q) \equiv (1-q)^{n-1}\binom{n-1}{n}$ corresponds to the case where no $j \neq i$ enters; $\widetilde{Q}_{e,-\delta}(n,q_{-\delta},q_\delta) \equiv \sum_{m_{-\delta}=1}^{n-1}\binom{n-1}{m_{-\delta}}(q_{-\delta})^{m_{-\delta}}(1-q)^{n-m_{-\delta}-1}\rho_\delta$ corresponds to the cases where at least one $j \neq i$ enters as a $-\delta$-candidate but no like-minded δ-candidate of i enters; $\widetilde{Q}_{e,\delta}(n,q_{-\delta},q_\delta) \equiv \sum_{m_\delta=2}^{n}\binom{n-1}{m_\delta-1}(q_\delta)^{m_\delta-1}(1-q)^{n-m_\delta}\frac{1}{m_\delta} + \sum_{m_\delta=1}^{n}\sum_{m_{-\delta}=1}^{n-m_\delta}\binom{n-1}{m_\delta-1}\binom{n-m_\delta}{m_{-\delta}}$ $(q_\delta)^{m_\delta-1}(q_{-\delta})^{m_{-\delta}}(1-q)^{n-m_\delta-m_{-\delta}}\frac{\rho_\delta}{m_\delta}$ corresponds to the cases where at least one $j \neq i$ enters as a δ-candidate but nobody enters in the $-\delta$-direction and hence there is no party $-\delta$ (first term) and to the cases with entry in both directions (second term); and $\widetilde{Q}_b(n,q_{-\delta},q_\delta) \equiv \widetilde{Q}_{ne}(n,q) + \widetilde{Q}_{e,-\delta}(n,q_{-\delta},q_\delta) + \widetilde{Q}_{e,\delta}(n,q_{-\delta},q_\delta)$.

It is straightforward to see that the result of Lemma 1 also applies to our model with directional information: that is, for any symmetric entry strategy, $\sigma(x)$, played by all other citizens, $LHS\,(47)$ is a U-shaped function in x_i. This is because all $\widetilde{Q}$-terms, b, $f_{\delta,ne}(x|\sigma)$ and $f_{\delta,e}(x|\sigma)$ and thus $\breve{x}_\phi$ are fixed, so that the best response of a citizen type $x_i \in [\breve{x}_\phi, 1]$ depends only on the utility loss function, $U(x_i,x)$, which we assume is convex (i.e., U-shaped, or V-shaped in case of linear preferences). Therefore, $LHS\,(47)$ is U-shaped in x_i and, in equilibrium, any symmetric entry strategy must be a cutpoint strategy also in our model with directional information.

(Existence) Next, we prove that a symmetric entry equilibrium in a pair of cutpoints,

$(\breve{x}^*_{-\delta}, \breve{x}^*_{\delta})$, and a voting cutpoint $\breve{x}_{\phi}(\breve{x}_l, \breve{x}_r)$ always exists for any cumulative probability distribution of ideal points, $F(x)$, satisfying $A1$-$A3$ (see section "General Model" in the article). As in the proof of Proposition 1, we do so using Envelope Theorem and Intermediate Value Theorem. Recall that $LHS(46)$ is a continuous function of x_i, the entry cutpoints $(\breve{x}_{-\delta}, \breve{x}_{\delta})$, and the voting cutpoint $\breve{x}_{\phi} \in (\breve{x}_{-\delta}, \breve{x}_{\delta})$. Now consider the following value function:

$$\vartheta[\breve{x}_{-\delta}, \breve{x}_{\delta}, \breve{x}_{\phi} \,|f_{\delta}(x), n, c, b] = \int_{-1}^{1} f_{\delta}(x) \left[c - LHS(46)[x, \breve{x}_{-\delta}, \breve{x}_{\delta}, \breve{x}_{\phi} \,|f_{\delta}(x), n, c, b]\right] dx$$

and the maximization problem

$$\vartheta^*[\breve{x}^*_{-\delta}, \breve{x}^*_{\delta}, \breve{x}^*_{\phi} \,|f_{\delta}(x), n, c, b] \equiv \max_{\breve{x}_{-\delta}, \breve{x}_{\delta}, \breve{x}_{\phi}} \vartheta[\breve{x}_{-\delta}, \breve{x}_{\delta}, \breve{x}_{\phi} \,|f_{\delta}(x), n, c, b].$$

If both entry cutpoints are interior, then equilibrium condition (46) must hold as an equality for both $x_i = \breve{x}_{-\delta}$ and $x_i = \breve{x}_{\delta}$ (cf. the proof of Proposition 1 (iv)) and a solution to this problem—i.e., a cutpoint equilibrium, $(\breve{x}^*_{-\delta}, \breve{x}^*_{\delta})$—is implicitly determined by

$$LHS(46)[\breve{x}^*_{-\delta}, \breve{x}^*_{\delta}, \breve{x}^*_{\phi} \,|f_{\delta}(x), n, c, b] \Big|_{x_i = \breve{x}^*_{-\delta}} = LHS(46)[\breve{x}^*_{-\delta}, \breve{x}^*_{\delta}, \breve{x}^*_{\phi} \,|f_{\delta}(x), n, c, b] \Big|_{x_i = \breve{x}^*_{\delta}} = c$$

which, using our previous result that $LHS\,(47)$ is U-shaped in x_i, gives:

$$\vartheta^*[\breve{x}^*_{-\delta}, \breve{x}^*_{\delta}, \breve{x}^*_{\phi} \,|f_{\delta}(x), n, c, b] = \int_{\breve{x}^*_{-\delta}}^{\breve{x}^*_{\delta}} f_{\delta}(x) \left[c - LHS(46)[x, \breve{x}^*_{-\delta}, \breve{x}^*_{\delta}, \breve{x}^*_{\phi} \,|f_{\delta}(x), n, c, b]\right] dx \qquad (48)$$

(the case with one interior cutpoint will be explained below). We are interested in how a marginal change in the entry costs, c, affects $\vartheta^*[.]$ and, in particular, $\left(\breve{x}^*_{-\delta}, \breve{x}^*_{\delta}\right)$. Because of the mutual dependence of the cutpoints, we can write the triple as $\left([\breve{x}^*_{-\delta}(\breve{x}^*_{\delta}, \breve{x}^*_{\phi}, c), \breve{x}^*_{\delta}(\breve{x}^*_{-\delta}, \breve{x}^*_{\phi}, c)],\right.$ $\left.\breve{x}^*_{\phi}(\breve{x}^*_{-\delta}, \breve{x}^*_{\delta})\right)$. Note that while $\breve{x}^*_{-\delta}$ and $\breve{x}^*_{\delta}$ depend directly on c, $\breve{x}_{\phi}$ only depends indirectly on

c via both entry cutpoints (i.e., $\frac{d\breve{x}^*_\phi(\breve{x}^*_{-\delta},\breve{x}_\delta)}{dc}=0$). Then, by the chain rule we have

$$\begin{aligned}\frac{d\vartheta^*[\breve{x}^*_{-\delta}(\breve{x}^*_\delta,\breve{x}^*_\phi,c),\breve{x}^*_\delta(\breve{x}^*_{-\delta},\breve{x}^*_\phi,c),\breve{x}^*_\phi(\breve{x}^*_{-\delta},\breve{x}^*_\delta)\,|\,f_\delta(x),n,b]}{dc}&=\frac{\partial\vartheta^*[.]}{\partial c}\\ &+\frac{\partial\vartheta^*[.]}{\partial\breve{x}_{-\delta}(\breve{x}_\delta,\breve{x}_\phi,c)}\left[\frac{d\breve{x}_{-\delta}}{dc}+\frac{\partial\breve{x}_{-\delta}}{\partial\breve{x}_\delta}\frac{d\breve{x}_\delta}{dc}+\frac{\partial\breve{x}_{-\delta}}{\partial\breve{x}_\phi}\left(\frac{\partial\breve{x}_\phi}{\partial\breve{x}_\delta}\frac{d\breve{x}_\delta}{dc}+\frac{\partial\breve{x}_\phi}{\partial\breve{x}_\delta}\frac{\partial\breve{x}_\delta}{\partial\breve{x}_{-\delta}}\frac{d\breve{x}_{-\delta}}{dc}\right)\right]\\ &+\frac{\partial\vartheta^*[.]}{\partial\breve{x}_\delta(\breve{x}_{-\delta},\breve{x}_\phi,c)}\left[\frac{d\breve{x}_\delta}{dc}+\frac{\partial\breve{x}_\delta}{\partial\breve{x}_{-\delta}}\frac{d\breve{x}_{-\delta}}{dc}+\frac{\partial\breve{x}_\delta}{\partial\breve{x}_\phi}\left(\frac{\partial\breve{x}_\phi}{\partial\breve{x}_{-\delta}}\frac{d\breve{x}_{-\delta}}{dc}+\frac{\partial\breve{x}_\phi}{\partial\breve{x}_{-\delta}}\frac{\partial\breve{x}_{-\delta}}{\partial\breve{x}_\delta}\frac{d\breve{x}_\delta}{dc}\right)\right]\\ &+\frac{\partial\vartheta^*[.]}{\partial\breve{x}_\phi(\breve{x}_{-\delta},\breve{x}_\delta)}\left[\frac{\partial\breve{x}_\phi}{\partial\breve{x}_{-\delta}}\left(\frac{d\breve{x}_{-\delta}}{dc}+\frac{\partial\breve{x}_{-\delta}}{\partial\breve{x}_\delta}\frac{d\breve{x}_\delta}{dc}+\frac{\partial\breve{x}_{-\delta}}{\partial\breve{x}_\phi}\frac{\partial\breve{x}_\phi}{\partial\breve{x}_\delta}\frac{d\breve{x}_\delta}{dc}\right)\right.\\ &\left.\qquad+\frac{\partial\breve{x}_\phi}{\partial\breve{x}_\delta}\left(\frac{d\breve{x}_\delta}{dc}+\frac{\partial\breve{x}_\delta}{\partial\breve{x}_{-\delta}}\frac{d\breve{x}_{-\delta}}{dc}+\frac{\partial\breve{x}_\delta}{\partial\breve{x}_\phi}\frac{\partial\breve{x}_\phi}{\partial\breve{x}_{-\delta}}\frac{d\breve{x}_{-\delta}}{dc}\right)\right].\end{aligned}$$

Next, using the first-order equilibrium condition $\frac{\partial\vartheta^*[.]}{\partial\breve{x}_{-\delta}(\breve{x}_\delta,\breve{x}_\phi,c)}=\frac{\partial\vartheta^*[.]}{\partial\breve{x}_\delta(\breve{x}_{-\delta},\breve{x}_\phi,c)}=\frac{\partial\vartheta^*[.]}{\partial\breve{x}_\phi(\breve{x}_{-\delta},\breve{x}_\delta)}=0$ yields

$$\frac{d\vartheta^*[\breve{x}^*_{-\delta}(\breve{x}^*_\delta,\breve{x}^*_\phi,c),\breve{x}^*_\delta(\breve{x}^*_{-\delta},\breve{x}^*_\phi,c),\breve{x}^*_\phi(\breve{x}^*_{-\delta},\breve{x}^*_\delta)\,|\,f_\delta(x),n,b]}{dc}=\frac{\partial\vartheta^*[.]}{\partial c}=\int\limits_{\breve{x}^*_{-\delta}}^{\breve{x}^*_\delta}f_\delta(x)dx>0\text{ for }s\in[0,1).$$

As in the proof of Proposition 1, a marginal change in c affects $\vartheta^*[.]$ only directly, but not indirectly through changes in $\breve{x}_{-\delta}(\breve{x}_\delta,\breve{x}_\phi,c)$, $\breve{x}_\delta(\breve{x}_{-\delta},\breve{x}_\phi,c)$, and $\breve{x}_\phi(\breve{x}_{-\delta},\breve{x}_\delta)$. Consequently, the effects on $LHS(46)[.]$ in $\vartheta^*[.]$ are negligible and marginal changes in the three cutpoints are independent from each other. Then, following along the same lines as the existence proof of Proposition 1, it is straightforward to see that continuously increasing c creates one or more continuous equilibrium paths of entry cutpoints $[\breve{x}^*_{-\delta}(\breve{x}^*_\delta,\breve{x}^*_\phi,c),\breve{x}^*_\delta(\breve{x}^*_{-\delta},\breve{x}^*_\phi,c)]$ with the following properties (the case with one interior cutpoint is derived by simply setting $\breve{x}^*_{-\delta}=-1$): (i) the interior cutpoints get more extreme (at the boundary, $\breve{x}^*_{-\delta}=-1$ remains), (ii) $s_{-\delta},s_\delta$, and thus s decreases; and (iii) $LHS(46)$ increases. Moreover, the endpoints of any path are at $\underline{c}'$ ($s=1$, where $\breve{x}^*_{-\delta} = \breve{x}^*_{\delta} = \breve{x}^*_{\min} = \breve{x}^*_{\phi}$) and at $\bar{\bar{c}}$ ($s = 0$, where $\breve{x}^*_{-\delta} = -1$ and $\breve{x}^*_{\delta} = 1$), with

$$\begin{aligned}
\underline{c}' \equiv\; & (s_\delta)^{n-1} \frac{1}{n} \left[b + \frac{\int_{\breve{x}_{\min}}^{1} f_\delta(x) U(\breve{x}_{\min}, x) dx}{\int_{\breve{x}_{\min}}^{1} f_\delta(x) dx} \right] + (s_{-\delta})^{n-1} \;\; \rho_\delta \left[b + \frac{\int_{-1}^{\breve{x}_{\min}} f_\delta(x) U(\breve{x}_{\min}, x) dx}{\int_{-1}^{\breve{x}_{\min}} f_\delta(x) dx} \right] \\
& + \sum_{m_\delta = 2}^{n-1} \binom{n-1}{m_\delta - 1} (s_\delta)^{m_\delta - 1} (1 - s_\delta)^{n - m_\delta} \frac{\rho_\delta}{m_\delta} \left[b + \frac{\int_{\breve{x}_{\min}}^{1} f_\delta(x) U(\breve{x}_{\min}, x) dx}{\int_{\breve{x}_{\min}}^{1} f_\delta(x) dx} \right]
\end{aligned}$$

and

$$\underline{c}' \leq \bar{\bar{c}} \equiv \frac{n-1}{n} \left[b + \max \left[\int_{-1}^{1} f_\delta(x) U(-1, x) dx, \int_{-1}^{1} f_\delta(x) U(1, x) dx \right] \right].$$

Note that while $\underline{c}'$ is typically different from the $\underline{c}$ in expression (22) for private information, we can use $\bar{\bar{c}}$ in expression (24) as the cost boundary for "no entry", because in this case one citizen is randomly determined to lead the community and it is straightforward to see that the first term on the left-hand side of the best response entry strategy $LHS(46)$ is again equal to $\bar{\bar{c}}$. Then, by the Intermediate Value Theorem, at least one equilibrium path $\left([\breve{x}^*_{-\delta}(\breve{x}^*_{\delta}, \breve{x}^*_{\phi}, c), \breve{x}^*_{\delta}(\breve{x}^*_{-\delta}, \breve{x}^*_{\phi}, c)], \breve{x}^*_{\phi}(\breve{x}^*_{-\delta}, \breve{x}^*_{\delta}) \right)$ must exist. Finally, for $c \leq \underline{c}'$ and $\bar{\bar{c}} \leq c$, existence (and uniqueness) is readily verified for universal entry and universal abstention, respectively. This completes our proof of existence for our citizen candidate model with directional information about candidates' ideal points.

Finally, because for a given equilibrium path $\left[\left(\breve{x}^*_{-\delta}, \breve{x}^*_{\delta} \right), \breve{x}^*_{\phi} \right]$ properties (i) to (iii) hold, an increase in the entry cost, c, or a decrease in the benefits from holding office, b, will result in more extreme entry cutpoints (for only one interior cutpoint, $\breve{x}^*_{-\delta} = -1$ will remain) (cf. the proof of Proposition 2). Similarly, it is readily verified that if the community size, n, approaches infinity, only citizens with ideal points at the boundaries will enter as a candidate (cf. the proof of Proposition 3). Note that $\lim_{n \to \infty} \breve{x}^*_{\phi} = 0$ in this limit case. ■