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Dynamic Free Riding with Irreversible Investments: On-line Appendix

Abstract

In this appendix we present the proofs omitted in "Dynamic Free Riding with Irreversible Investments" by Marco Battaglini, Salvatore Nunnari and Thomas Palfrey.

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1 Proof of Proposition 1

Proposition 1. *For any d, δ, n and $y^o \in \left[[u']^{-1}(1-\delta(1-d)), [u']^{-1}(1-\delta(1-\frac{d}{n}))\right]$, there is an equilibrium with steady state y^o in an irreversible economy. In all these equilibria convergence is monotonic and gradual.*

Define $y^*(\delta, d, n) = [u']^{-1}(1-\delta(1-d)/n)$ and $y^{**}(\delta, d, n) = [u']^{-1}(1-\delta(1-d/n))$: these are the points at which

$$y'(g) = \frac{1-d-\frac{n\left(1-u'(g)\right)}{\delta}}{1-n} \tag{1}$$

is, respectively, zero and one. Define $\overline{y}(d,\delta) = [u']^{-1}(1-\delta(1-d))$: this is the point at which (1) is equal to $1-d$. Note that $y^*(\delta,d,n) < \overline{y}(d,\delta)$ and $\overline{y}(d,\delta) < y^{**}(\delta,d,n)$. Moreover, since we are assuming that the planner interior solution is feasible ($y^*_P(\delta,d,n) < W/d$), we have $y^{**}(\delta,d,n) < W/d$. To construct an equilibrium with steady state $y^o \in [\overline{y}(\delta,d), y^{**}(\delta,d,n)]$ we proceed in 3 steps.

Step 1. We first construct the strategies associated to a generic y^o. For a generic $y^o \in [\overline{y}(\delta,d), y^{**}(\delta,d,n)]$, let $\widetilde{y}(g\,|y^o)$ be the solution of the differential equation (1) when we require the initial condition: $\widetilde{y}(y^o\,|y^o) = y^o$. Given y^o, moreover, let us define the two thresholds $g^3(y^o) = y^o/(1-d)$ and $g^2(y^o) = \max\{\min_{g\geq 0}\{g\,|\widetilde{y}(g\,|y^o) \leq W+(1-d)g\}, y^*(\delta,d,n)\}$. In words, the second threshold is the largest point between the point at which $\widetilde{y}(g\,|y^o)$ crosses from below $W+(1-d)g$, and $y^*(\delta,d,n)$ (see Figure 1 in the paper for an example). It is easy to verify that, by construction, $g^3(y^o) \geq \overline{y}(\delta,d)$; moreover, $\widetilde{y}(g\,|y^o) \in ((1-d)g, W+(1-d)g)$ with $\widetilde{y}'(g\,|y^o) \in [0,1]$ and $\widetilde{y}''(g\,|y^o) \geq 0$ in $\left[g^2(y^o), y^o\right]$. For any $y^o \in [\overline{y}(\delta,d), y^{**}(\delta,d,n)]$, we now define the investment function as follows:

$$y(g\,|y^o) = \begin{cases} \min\left\{W+(1-d)g, \widetilde{y}\left(g^2(y^o)\,|y^o\right)\right\} & g \leq g^2(y^o) \\ \widetilde{y}(g\,|y^o) & g^2(y^o) < g \leq y^o \\ y^o & y^o < g \leq g^3(y^o) \\ (1-d)\,g & g > g^3(y^o) \end{cases} \tag{2}$$

Note that when depreciation is zero, then $g^3(y^o) = y^o$ and $y'(g\,|y^o) = 1$ at $g = y^o$: so (2) coincides exactly with the investment function illustrated in Figure 1 in the paper. For future reference, define $g^1(y^o) = \max\left\{0, \left(\widetilde{y}\left(g^2(y)\,|y^o\right) - W\right)/(1-d)\right\}$. This is the point at which $W+(1-d)g = \widetilde{y}\left(g^2(y^o)\,|y^o\right)$, if positive. Since $\widetilde{y}\left(g^2(y)\,|y^o\right) < W+(1-d)g^2(y^o)$, $g^1(y^o) \in \left[0, g^2(y^o)\right]$. We have:

Lemma A.1. $y(g\,|y^o) \in \left[g^2(y^o), y^o\right]$ *for* $g \in \left[g^2(y^o), y^o\right]$.

Proof. Because $y\left(g\left|y^{o}\right.\right)$ is monotonic non-decreasing in $g \in \left[g^{2}(y^{o}), y^{o}\right]$, for any $g \in \left[g^{2}(y^{o}), y^{o}\right]$ we have $y\left(g\left|y^{o}\right.\right) \in \left[y\left(g^{2}(y^{o})\left|y^{o}\right.\right), y^{o}\right]$. Since $y\left(g\left|y^{o}\right.\right)$ has slope lower than one in $\left[g^{2}(y^{o}), y^{o}\right]$ and $y\left(y^{o}\left|y^{o}\right.\right) = y^{o}$ for $y^{o} \geq g^{2}(y^{o})$, we must have $y\left(g^{2}(y^{o})\left|y^{o}\right.\right) \geq g^{2}(y^{o})$, so $y\left(g\left|y^{o}\right.\right) \geq g^{2}(y^{o})$ *for* $g \in \left[g^{2}(y^{o}), y^{o}\right]$. Similarly, $y\left(y^{o}\left|y^{o}\right.\right) = y^{o}$ implies $y\left(g\left|y^{o}\right.\right) \leq y^{o}$ *for* $g \in \left[g^{2}(y^{o}), y^{o}\right]$. ■

Step 2. We now construct the value functions corresponding to each steady state y^{o}. For $g \in \left[g^{2}(y^{o}), y^{o}\right]$ define the value function recursively as

$$v(g|y^{o}) = \frac{W + (1-d)g - y\left(g\left|y^{o}\right.\right)}{n} + u(y\left(g\left|y^{o}\right.\right)) + \delta v(y\left(g\left|y^{o}\right.\right)). \tag{3}$$

By Theorem 3.3 in Stokey, Lucas, and Prescott (1989), the right hand side of (3) is a contraction: it defines a unique, continuous and differentiable value function $v(g\left|y^{o}\right.)$ for this interval of g. (Differentiability follows from the differentiability of $y\left(g\left|y^{o}\right.\right)$). Note that $y\left(g\left|y^{o}\right.\right) = \widetilde{y}\left(g\left|y^{o}\right.\right)$ for any $g \in \left[g^{2}(y^{o}), y^{o}\right]$ and, by Lemma A.1, $\widetilde{y}\left(g\left|y^{o}\right.\right) \in \left[g^{2}(y^{o}), y^{o}\right]$ *for* $g \in \left[g^{2}(y^{o}), y^{o}\right]$. From the definition of $\widetilde{y}\left(g\left|y^{o}\right.\right)$ and the discussion in Section 4 in the paper, it follows that $u'(g) + \delta v'(g; y^{o}) = 1$ for any $g \in \left[g^{2}(y^{o}), y^{o}\right]$. In the rest of the state space we define the value function recursively. In $\left[g^{1}(y^{o}), g^{2}(y^{o})\right]$, if $g^{1}(y^{o}) < g^{2}(y^{o})$, the value function is defined as:

$$v(g\left|y^{o}\right.) = \frac{W + (1-d)g - y\left(g^{2}(y^{o})\left|y^{o}\right.\right)}{n} + u(y\left(g^{2}(y^{o})\left|y^{o}\right.\right)) + \delta v(y\left(g^{2}(y^{o})\left|y^{o}\right.\right)) \tag{4}$$

where $v(y\left(g^{2}(y^{o})\left|y^{o}\right.\right))$ is well defined since $y\left(g^{2}(y^{o})\left|y^{o}\right.\right) \in \left[g^{2}(y^{o}), y^{o}\right]$.

Lemma A.2. *For* $g \in \left[g^{1}(y^{o}), y^{o}\right]$, $u(g) + \delta v(g\left|y^{o}\right.)$ *is concave with slope larger or equal than* 1.

Proof. If $g^{1}(y^{o}) = g^{2}(y^{o})$, the result is immediate. Assume therefore, $g^{1}(y^{o}) < g^{2}(y^{o})$. In this case $g^{2}(y^{o}) = y^{*}(\delta, d, n)$. For any $g \in \left[g^{1}(y^{o}), g^{2}(y^{o})\right]$, $y(g; y^{o}) = y\left(y^{*}(\delta, d, n)\left|y^{o}\right.\right)$. So we have $v'(g\left|y^{o}\right.) = (1-d)/n$ implying: $u'(g) + \delta v'(g\left|y^{o}\right.) = u'(g) + \delta(1-d)/n > 1$ since $g \leq g^{2}(y^{o}) = y^{*}(\delta, d, n)$. ■

Consider $g < g^{1}(y^{o})$. In $\left[g_{-1}, g^{1}(y^{o})\right]$ the value function is defined as:

$$v(g\left|y^{o}\right.) = u(W + (1-d)g) + \delta v(W + (1-d)g\left|y^{o}\right.) \tag{5}$$

where $g_{-1} = \max\left\{0, \left[g^{1}(y^{o}) - W\right]/(1-d)\right\}$. Assume that we have defined the value function in $g \in \left[g_{-t}, g_{-(t-1)}\right]$ as v_{-t}, for all t such that $g_{-(t-1)} > 0$. Then we can define $v_{-(t+1)}$ as (5) in $\left[g_{-(t+1)}, g_{-t}\right]$ with $g_{-(t+1)} = \left[g_{-t} - W\right]/(1-d)$.

Lemma A.3. *For* $g \in [0, y^{o}]$, $u(g) + \delta v(g\left|y^{o}\right.)$ *is concave with slope greater than or equal than* 1.

Proof. We prove this by induction on t. Consider now the interval $\left[\left[g^{1}(y^{o}) - W\right]/(1-d), g^{1}(y^{o})\right]$. In this range we have $v'(g\left|y^{o}\right.) = \left[u'(W + (1-d)g) + \delta v'(W + (1-d)g\left|y^{o}\right.)\right](1-d) \geq 1-d$, since $W + (1-d)g \in \left[g^1(y^o), y^o\right]$. It follows that for $g \in \left[\left[g^1(y^o) - W\right]/(1-d), g^1(y^o)\right]$: $u'(g) + \delta v'(g\,|y^o\,) \geq u'(g) + \delta(1-d) > 1$. Where the last inequality follows from the fact that $g \leq g^1(y^o) < \overline{y}(\delta, d)$. We conclude that $u'(g) + \delta v'_{-1}(g\,|y^o\,)$ is concave, it has derivative larger than 1. Assume that we have shown that for $g \in \left[g_{-t}, g^3(y^o)\right]$, $u(g) + \delta v_{-t}(g\,|y^o\,)$ is concave and $u'(g) + \delta v'_{-t}(g\,|y^o\,) > 1$. Consider in $g \in \left[g_{-(t+1)}, g_{-t}\right]$. We have:

$$v'(g\,|y^o\,) = [u'(W + (1-d)g) + \delta v'(W + (1-d)g\,|y^o\,)]\,(1-d) \geq 1-d$$

since $W + (1-d)g \geq \left[g_{-t}, g^3(y^o)\right]$. So $u'(g) + \delta v'(g\,|y^o\,) \geq u'(g) + \delta(1-d) \geq 1$. By the same argument as above, moreover, v is concave at g_{-t}. We conclude that for any $g \leq g^1$, $u(g) + \delta v(g\,|y^o\,)$ is concave and it has slope larger than 1. ■

For $g \in \left(y^o, g^3(y^o)\right]$ we define the value function as: $v(g\,|y^o\,) = \frac{W+(1-d)g-y^o}{n} + u(y^0) + \delta v(y^o\,|y^o\,)$.

Lemma A.4. *For $g \leq g^3(y^o)$, $u(g) + \delta v(g\,|y^o\,)$ is concave with slope less than or equal than 1.*

Proof. For $g \in \left(y^o, g^3(y^o)\right]$, $v'(g\,|y^o\,) = (1-d)/n$. Since $g \geq y^o \geq y^*(\delta, d, n)$, we have $u'(g) + \delta v'(g\,|y^o\,) = u'(g) + \delta\,(1-d)/n < 1$. Previous lemmas imply $u(g) + \delta v(g|y^o)$ is concave and has slope greater than or equal than 1 for $g \leq g^3(y^o)$. ■

Finally consider $g > g^3(y^o)$.

Lemma A.5. *For any $g \geq g^3(y^o)$, $u(g) + \delta v(g\,|y^o\,)$ has slope less than or equal than 1.*

Proof. In $g > g^3(y^o)$, we must have $(1-d)g \in \left[y^o, g^3(y^o)\right]$. From the proof of Lemma A.5 we know that $u'(g) + \delta v'(g) < 1$ for $g \in \left[y^o, g^3(y^o)\right]$, so we have:

$$v'(g) = (1-d)\,[u'((1-d)g) + \delta v'((1-d)g)] < 1-d$$

for $g > g^3(y^o)$. This fact implies that $u'(g) + \delta v'(g) < u'(g) + \delta\,(1-d)$ for any $g > g^3(y^o)$. Since $g^3(y^o) > \overline{y}(\delta, d)$ we have $u'(g) + \delta\,(1-d) < u'(\overline{y}(\delta, d)) + \delta\,(1-d) = 1$ for $g > g^3(y^o)$. It follows that $v^*(g)$ is has slope lower than 1 in $g > g^3(y^o)$. ■

From Lemmata A1-A5 we conclude that $u(g) + \delta v(g\,|y^o\,)$ has a global maximum at any $g \in \left[g^3(y^o), y^o\right]$.

Step 3. Define $x(g\,|y^o\,) = [W + (1-d)g - y(g\,|y^o\,)]/n$ and $i(g\,|y^o\,) = [y(g\,|y^o\,) - (1-d)g]/n$ as the levels of per capita private consumption and investment, respectively. Note that by construction, $x(g\,|y^o\,) \in [0, W/n]$. We now establish that $y(g\,|y^o\,)$, $x(g\,|y^o\,)$ and the associated value function $v\,(g\,|y^o\,)$ defined in the previous steps constitute an equilibrium. The fact that $v\,(g\,|y^o\,)$ describes the expected continuation value to an agent follows by construction. To see that $y(g\,|y^o\,)$ is an optimal reaction function given $v(g\,|y^o\,)$, note that an agent solves the following problem:

$$\max_y \left\{ \begin{array}{c} u(y) - y + \delta v(y) \\ y \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n} y(g),\ y \geq \frac{(1-d)g}{n} + \frac{n-1}{n} y(g) \end{array} \right\} \tag{6}$$

where $y(g) = y(g\,|y^o\,)$. The investment function $y(g\,|y^o\,)$ satisfies the constraints of this problem if $y(g\,|y^o\,) \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n} y(g\,|y^o\,)$, so if $y(g\,|y^o\,) \leq W + (1-d)g$; and if $y(g\,|y^o\,) \geq \frac{(1-d)g}{n} + \frac{n-1}{n} y(g\,|y^o\,)$, so if $y(g\,|y^o\,) \geq (1-d)\,g$. Both conditions are automatically satisfied by construction. If $g < g^1(y^o)$, we have $u'(y) + \delta v'(y) \geq 1$ for all $y \in [(1-d)g, W + (1-d)g]$, so $y(g\,|y^o\,) = W + (1-d)g$ is optimal. If $g \geq g^3(y^o)$, $u'(y) + \delta v'(y) < 1$ for all $y \in [(1-d)g, W + (1-d)g]$, so $y(g\,|y^o\,) = (1-d)g$. In $g \in \left(g^1(y^o), g^3(y^o)\right]$ a point maximizing $u(y) + \delta v(y)$ is feasible and chosen, so again $y(g\,|y^o\,)$ is an optimal choice. ■

2 Proof of Proposition 2

Propositon 2. *For any δ and n, we have that $\left|\overline{y}_{IR}(\delta, d, n) - \underline{y}_{IR}(\delta, d, n)\right| \to 0$ as $d \to 0$. Moreover, there is $\overline{d} > 0$ a such that for $d < \overline{d}$, all equilibrium paths are gradual.*

Consider a sequence $d^m \to 0$. For each d^m there is at least an associated equilibrium $y_m(g)$, $v_m(g)$ with steady state y_m^o. To prove the result we proceed in two steps. In Section 2.1 we prove that for any $\xi > 0$, there is a $\widetilde{m}$ such that for $m > \widetilde{m}$, $\underline{y}_{IR}(\delta, d^m, n) \geq [u']^{-1}(1-\delta) - \xi$. In Section 2.2 we prove that the steady state of any equilibrium can not be larger than $[u']^{-1}(1 - \delta(1 - d/n))$. Since, as shown in Proposition 1, $[u']^{-1}(1 - \delta(1 - d/n))$ is an equilibrium steady state for any $d \geq 0$ and it converges to $[u']^{-1}(1-\delta)$, we must have $\left|\overline{y}_{IR}(\delta, d, n) - \underline{y}_{IR}(\delta, d, n)\right| \to 0$ as $d \to 0$. In Lemmata A.6 and A.7 presented in Section 2.2 we show that $y'(g) \in (0, 1)$ in a left neighborhood of the steady state y^o if $y^o > [u']^{-1}(1 - \delta(1-d)/n)$. Since all equilibrium steady states converge to $[u']^{-1}(1-\delta) > [u']^{-1}(1-\delta/n)$, this implies that that convergence of g to the steady state is gradual in all equilibria if d is sufficiently small.

2.1 The lower bound

We prove the result by contradiction. Suppose to the contrary there is a sequence of steady states y_m^0, with associated equilibrium investment and value functions $y_m(g)$, $v_m(g)$, and an $\xi > 0$ such that $y_m^0 < \overline{y}(0) - \xi$ for any arbitrarily large m, where $\overline{y}(d) = [u']^{-1}(1 - \delta(1-d))$. Define $y_m^0(g) = y_m(g)$, and $y_m^j(g) = y_m(y_m^{j-1}(g))$ and consider a marginal deviation from the steady state from y_m^0 to $y_m^0 + \Delta$. By the irreversibility constraint we have $y_m(g) \geq (1 - d^m)\, g$. Using this property and the fact that y_m^0 is a steady state, so $y_m^j(y_m^0) = y_m^0$, we have:

$$y_m(y_m^0 + \Delta) - y_m(y_m^0) \geq (1 - d^m)\left(y_m^0 + \Delta\right) - y_m^0 = (1 - d^m)\,\Delta - d^m y_m^0$$

This implies that, as $m \to \infty$, for any given Δ: $\left[y_m(y_m^0 + \Delta) - y_m^0\right]/\Delta \geq 1 + o_1(d^m)$ where $o_1(d^m) \to 0$ as $m \to 0$. We now show with an inductive argument that a similar property holds for all iterations $y_m^j(y_m^0)$. Assume we have shown that: $\left[y_m^{j-1}(y_m^0 + \Delta) - y_m^0\right]/\Delta \geq 1 + o_{j-1}(d^m)$ where $o_{j-1}(d^m) \to 0$ as $m \to 0$. We must have: $y_m(y_m^{j-1}(y_m^0 + \Delta)) - y_m^j(y_m^0) \geq (1 - d^m)\, y_m^{j-1}(y_m^0 + \Delta) - y_m^0$. We therefore have: $y_m(y_m^{j-1}(y_m^0 + \Delta)) - y_m^0 \geq y_m^{j-1}(y_m^0 + \Delta) - y_m^0 - d^m y_m^{j-1}(y_m^0 + \Delta)$ so we have:

$$\frac{y_m(y_m^{j-1}(y_m^0 + \Delta)) - y_m^0}{\Delta} \geq \frac{y_m^{j-1}(y_m^0 + \Delta) - y_m^0}{\Delta} - \frac{d^m y_m^{j-1}(y_m^0 + \Delta)}{\Delta} \geq 1 + o_j(d^m) \qquad (7)$$

where $o_j(d^m) = o_{j-1}(d^m) - \frac{d^m y_m^{j-1}(y_m^0+\Delta)}{\Delta}$, so $o_j(d^m) \to 0$ as $m \to 0$.

We can write the value function after the deviation to $y_m^0 + \Delta$ as:

$$V(y_m^0 + \Delta) = \sum_{j=0}^{\infty} \delta^{j-1} \left[\frac{W + (1 - d^m)\, y_m^{j-1}(y_m^0 + \Delta) - y_m^j(y_m^0 + \Delta)}{n} + u(y_m^j(y_m^0 + \Delta))\right]$$

For any given function $f(x)$, define $\Delta f(x) = f(x + \Delta) - f(x)$. We can write:

$$\begin{aligned} \Delta V(y_m^0)/\Delta &= \sum_{j=0}^{\infty} \delta^{j-1} \left[\begin{array}{c} \frac{(1-d^m)\Delta y_m^{j-1}(y_m^0)/\Delta - \Delta y_m^j(y_m^0)/\Delta}{n} \\ + \left[u'(y_m^0) + o(\Delta)\right] \Delta y_m^j(y_m^0)/\Delta \end{array}\right] \\ &\geq \sum_{j=0}^{\infty} \delta^{j-1} \left[\begin{array}{c} \frac{(1-d^m)(1+o_{j-1}(d^m)) - (1+o_j(d^m))}{n} \\ + \left[u'(y_m^0) + o(\Delta)\right](1 + o_j(d^m)) \end{array}\right] \end{aligned} \qquad (8)$$

where $o(\Delta) \to 0$ as $\Delta \to 0$. In the first equality we use the fact that if we choose Δ small, since $y_m(g)$ is continuous, $\Delta y_m^j(y_m^0)$ is small as well. This implies that

$$\left(u(y_m^j(y_m^0 + \Delta)) - u(y_m^j(y_m^0))\right) / \left[y_m^j(y_m^0 + \Delta) - y_m^j(y_m^0)\right]$$

converges to $u'(y_m^j(y_m^0))$ as $\Delta \to 0$. The inequality in 8 follows from (7). Given Δ, as $m \to \infty$, we therefore have $\lim_{m\to\infty} \Delta V(y_m^0)/\Delta \geq \frac{u'(y_m^0)+o(\Delta)}{1-\delta}$. We conclude that for any $\varepsilon > 0$, there must be a Δ_ε such that for any $\Delta \in (0, \Delta_\varepsilon)$ there is a m_Δ guaranteeing that $\Delta V(y_m^0)/\Delta \geq \frac{u'(y_m^0)}{1-\delta} - \varepsilon$ for $m > m_\Delta$. After a marginal deviation to $y_m^0 + \Delta$, therefore, the change in agent's objective function is:

$$u'(y_m^0) + \delta \Delta V(y_m^0)/\Delta - 1 \geq \frac{u'(y_m^0)}{1 - \delta} - \delta\varepsilon - 1$$

for m sufficiently large. A necessary condition for the un-profitability of a deviation from y^0_m to $y^0_m + \Delta$ is therefore: $y^0_m \geq [u']^{-1}(1 - \delta + \delta\varepsilon(1-\delta))$. Since ε can be taken to be arbitrarily small, for an arbitrarily large m, this condition implies $y^0_m \geq \overline{y}(0) - \xi/2$, which contradicts $y^0_m < \overline{y}(0) - \xi$. We conclude that $\underline{y}_{IR}(\delta, d, n) \to \overline{y}(0)$ as $d \to 0$.

2.2 The upper bound

Suppose to the contrary that there is stable steady state at $y^o > [u']^{-1}(1 - \delta(1 - d/n))$. We must have $y^o \in \left([u']^{-1}(1 - \delta(1 - d/n)), W/d\right]$, since it is not feasible for a steady state to be larger than W/d. Consider a left neighborhood of y^o, $N_\varepsilon(y^o) = (y^o - \varepsilon, y^o)$. The value function can be written in $g \in N_\varepsilon(y^o)$ as:

$$v(g) = u(y(g)) + \delta v(y(g)) - y(g) + \frac{W + (1-d)g}{n} + (1 - 1/n)\, y(g) \tag{9}$$

where $y(g)$ is the equilibrium strategy associated to y^o. In $N_\varepsilon(y^o)$ the constraint $y \geq \frac{1-d}{n} g + \frac{n-1}{n} y(g)$ cannot be binding (else we would have $y(g) = (1-d)g$, but this is not possible in a neighborhood of $y^o > 0$). We consider two cases.

Case 1. Suppose first that $y^o < W/d$. We must therefore have that $y(g) < W + (1-d)g$ in $N_\varepsilon(y^o)$, so the constraint $y \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n} y$ is not binding. The solution is in the interior of the constraint set of (6), and the objective function $u(y(g)) + \delta v(y(g)) - y(g)$ is constant for $g \in N_\varepsilon(y^o)$.

Lemma A.6. *If $y^o > [u']^{-1}(1 - \delta(1-d)/n)$, then there is a left neighborhood $N_\varepsilon(y^o)$ in which $y(g)$ is not constant.*

Proof. Suppose to the contrary that, for any $N_\varepsilon(y^o)$, there is an interval in $N_\varepsilon(y^o)$ in which $y(g)$ is constant. Using the expression for $v(g)$ presented above, we must have $v'(g) = (1-d)/n$ for any g in this interval. Since $N_\varepsilon(y^o)$ is arbitrary, then we must have a sequence $g^m \to y^o$ such that $v'(g^m) = (1-d)/n \; \forall m$. We can therefore write:

$$\begin{aligned} \lim_{\Delta \to 0} \frac{v(y^o) - v(y^o - \Delta)}{\Delta} &= \lim_{\Delta \to 0} \lim_{m \to \infty} \frac{v(g^m) - v(g^m - \Delta)}{\Delta} \\ &= \lim_{m \to \infty} \lim_{\Delta \to 0} \frac{v(g^m) - v(g^m - \Delta)}{\Delta} = \frac{1-d}{n} \end{aligned}$$

where the second equality follows from the continuity of $v(g)$. This implies that $v^-(y^o)$, left derivative of $v(g)$ at y^o, is well defined and equal to $\frac{1-d}{n}$. Consider now a marginal reduction of g at y^o. The change in utility is (as $\Delta \to 0$):

$$\begin{aligned} \Delta U(y^o) &= u(y^o - \Delta) - u(y^o) + \delta\left[v(y^o - \Delta) - v(y^o)\right] + \Delta \\ &= \left[1 - \left(u'(y^o) + \delta \frac{1-d}{n}\right)\right] \Delta \end{aligned}$$

In order to have $\Delta U(y^o) \leq 0$, we must have $u'(y^o) + \delta(1-d)/n \geq 1$. This implies $y^o \leq [u']^{-1}(1-\delta(1-d)/n)$, a contradiction. Therefore, if there is stable steady state at $y^o > [u']^{-1}(1-\delta(1-d)/n)$, then $y(g)$ is not constant in $N_\varepsilon(y^o)$. ■

Lemma A.6 implies that there is a left neighborhood $N_\varepsilon(y^o)$ in which $u(g)+\delta v(g)-g$ is constant if $y^o > [u']^{-1}(1-\delta(1-d)/n)$ (since otherwise $y(g)$ would be constant). Moreover, since y^o is a stable steady state and $y(g)$ is strictly increasing, $g \in N_{\varepsilon'}(y^o)$ implies $y(g) \in N_{\varepsilon'}(y^o))$ for any open left neighborhood $N_{\varepsilon'}(y^o) = (y^o - \varepsilon', y^o) \subset N_\varepsilon(y^o)$. These observations imply:

Lemma A.7. *If $y^o > [u']^{-1}(1-\delta(1-d)/n)$, then there is a left neighborhood $N_\varepsilon(y^o)$ in which*

$$y'(g) = \frac{n}{n-1}\left(\frac{1-u'(g)}{\delta} - \frac{1-d}{n}\right) \tag{10}$$

Proof. There is a $N_\varepsilon(y^o)$ and a constant K such that $\delta v(g) = K + g - u(g)$ for $g \in N_\varepsilon(y^o))$. Hence $v(g)$ is differentiable in $N_\varepsilon(y^o)$. Moreover, $y(g) \in N_\varepsilon(y^o)$ for all $g \in N_\varepsilon(y^o)$. Hence $u(y(g)) + \delta v(y(g)) - y(g)$ is constant in $g \in N_\varepsilon(y^o)$ as well. These observations and the definition of $v(g)$ imply that $v'(g) = \frac{1-d}{n} + \left(1-\frac{1}{n}\right)y'(g)$ in $N_\varepsilon(y^o)$. Given that $u'(g)+\delta v'(g) = 1$ in $g \in N_\varepsilon(y^o)$, we must have: $u'(g)+\delta v'(g) = u'(g) + \delta\left[\frac{1-d}{n} + \left(1-\frac{1}{n}\right)y'(g)\right] = 1$ which implies (10) for any $g \in N_\varepsilon(y^o)$. ■

Let g^m be a sequence in $N_\varepsilon(y^o)$ such that $g^m \to y^o$. We must have

$$\begin{aligned} y^-(y^o) &= \lim_{\Delta\to 0}\frac{y(y^o)-y(y^o-\Delta)}{\Delta} = \\ &= \lim_{\Delta\to 0}\lim_{m\to\infty}\frac{y(g^m)-y(g^m-\Delta)}{\Delta} = \\ &= \lim_{m\to\infty}\lim_{\Delta\to 0}\frac{y(g^m)-y(g^m-\Delta)}{\Delta} = \frac{n}{n-1}\left(\frac{1-u'(y^o)}{\delta} - \frac{1-d}{n}\right) \end{aligned} \tag{11}$$

where $y^-(y^o)$ is the left derivative of $y(g)$ at y^o, the second equality follows from continuity and the last equality follows from Lemma A.7 and the fact that under the starting assumption we have $y^o > [u']^{-1}(1-\delta(1-d/n)) > [u']^{-1}(1-\delta(1-d)/n)$. Consider a state $(y^o - \Delta)$. For y^o to be stable we need that for any small Δ:

$$y(y^o - \Delta) \geq y^o - \Delta = y(y^o) + (y^o-\Delta) - y^o$$

where the equality follows from the fact that $y(y^o) = y^o$. As $\Delta \to 0$, this implies $y^-(y^o) \leq 1$ in $N_\varepsilon(y^o)$. By (11), we must therefore have: $\frac{n}{n-1}\left(\frac{1-u'(y^o)}{\delta} - \frac{1-d}{n}\right) \leq 1$. This implies: $y^o \leq [u']^{-1}(1-\delta(1-d)/n)$, a contradiction.

Case 2. Assume now that $y^o = W/d$ and consider first the case in which it is a strict local maximum of the objective function $u(y)+\delta v(y) - y$. In this case in a left neighborhood $N_\varepsilon(y^o)$, we have that the upper bound $y \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n}y(g)$ is binding: implying $y(g) = W + (1-d)g$

in $N_\varepsilon(y^o)$. We must therefore have a sequence of points $g^m \to y^o$ such that $g^m = y(g^{m-1})$ and $y(g^m) = W + (1-d)g^m \ \forall m$. Given this, we can write:

$$\begin{aligned} v(g^m) &= u(g^{m+1}) + \delta v(g^{m+1}) = u(g^{m+1}) + \delta \left[u(g^{m+2}) + \delta v(g^{m+2})\right] \\ &= \sum_{j=0}^{\infty} \delta^j u(W + (1-d)g^{m+j}) \end{aligned}$$

We therefore must have that $v(g^m)$ is differentiable and $\delta v'(g^m) = \sum_{j=0}^{\infty} [\delta(1-d)]^{j+1} u'(W+(1-d)g^{m+j})$. Since $u'(g^m)+\delta v'(g^m) \geq 1$, we have $u'(g^m)+\sum_{j=0}^{\infty} [\delta(1-d)]^{j+1} u'(W+(1-d)g^{m+j}) \geq 1$ for all m. Consider the limit as $m \to \infty$. Since $u'(g)$ is continuous and $g^m \to y^o$, we have:

$$\begin{aligned} 1 &\leq \lim_{m\to\infty} \left[u'(g^m) + \sum_{j=0}^{\infty} [\delta(1-d)]^{j+1} u'(W + (1-d)g^{m+j})\right] \\ &= u'(y^o) + \sum_{j=0}^{\infty} [\delta(1-d)]^{j+1} u'(y^o) = \frac{u'(y^o)}{1-\delta(1-d)} \end{aligned}$$

This implies $y^o \leq [u']^{-1}(1-\delta(1-d)) < [u']^{-1}(1-\delta(1-d/n))$, a contradiction. Assume now that $y^o = W/d$, but it is not a strict maximum of $u(y)+\delta v(y) - y$ in any left neighborhood. It must be that $u(y)+\delta v(y)-y$ is constant in some left neighborhood $N_\varepsilon(y^o)$. If this were not the case, then in any left neighborhood we would have an interval in which $y(g)$ is constant, but this is impossible by Lemma A.6. But then if $u(y)+\delta v(y)-y$ is constant in some $N_\varepsilon(y^o)$, the same argument as in Step 1 implies a contradiction. ■

3 Proof of Proposition 4

Proposition 4. For any $d > 0$ and n, there is a $\overline{\delta} < 1$ *such that the most efficient SPE path in a RIE and the most efficient SPE path in a IIE coincide with the Pareto efficient investment path for any* $\delta > \overline{\delta}$. *Hence, neither the most efficient SPE path in a RIE nor the most efficient SPE path in a IIE are characterized by gradualism for any* $\delta > \overline{\delta}$.

We first show that there is a $\delta_1 < 1$, such that for $\delta > \delta_1$ the efficient path is a SPE path in a irreversible investment economy. To this goal, we first define the equilibrium strategies and establish some key properties. Let $y^M(g;d,\delta)$, $v^M(g;d,\delta)$ be, respectively, the investment function and the value function of the Markov equilibrium with the lowest steady state characterized in Proposition 2 when the discount factor is δ and the rate of depreciation is d. Let $g^M(d,\delta) = [u']^{-1}(1-\delta(1-d)/n)$ be the associated steady state. It is easy to see that, for any d and n, $g^M(d,\delta) < y_P^*(\delta,d,n)$ for all $\delta \in [0,1]$. Define $y_j^M(g;d,\delta)$ recursively with $y_0^M(g;d,\delta) = g$ and $y_j^M(g;d,\delta) = y^M(y_{j-1}^M(g;d,\delta);d,\delta)$. For any g, $y_j^M(g;d,\delta) \to g^M(d,\delta)$ as $j \to \infty$. It follows that $\lim_{\delta\to 1}\left[(1-\delta)\, v^M(g;d,\delta)\right] = \left(W - dg^M(d,1)\right)/n + u(g^M(d,1))$. Let $y^P(g;d,\delta)$ be the efficient investment function characterized in Section 3 with steady state $g^P(d,\delta) = y_P^*(\delta, d, n)$, and let $v^P(g; d, \delta)$ be the associated expected utility for a player. Similarly, it is easy to see that $\lim_{\delta\to 1}\left[(1-\delta)\, v^P(g; d, \delta)\right] = \left(W - dg^P(d,1)\right)/n + u(g^P(d,1))$, where $y^P(g; d, \delta)$ be the efficient investment function characterized in Section 3 with steady state $g^P(d,\delta) = y_P^*(\delta, d, n)$). It follows that $\lim_{\delta\to 1}\left[(1-\delta)\, v^P(g; d, \delta)\right] > \lim_{\delta\to 1}\left[(1-\delta)\, v^M(g; d, \delta)\right]$.

Associated to an aggregate investment function $y^l(g; d, \delta)$, $l = \{M, P\}$, we have the individual contribution function: $i^l(g; d, \delta) = \left[y^l(g; d, \delta) - (1-d)g\right]/n$. To construct the equilibrium, consider the following trigger strategies. If $g_\tau = y_\tau^P(g_0; d, \delta)$ for all $\tau \leq t$, then $i^t(g_t; d, \delta) = i^P(g; d, \delta)$, where $i_j^t(g_t)$ is the investment at time t of an agent. If $\exists \tau \leq t$ such that $g_\tau \neq y_\tau^P(g_0; d, \delta)$, then $i^t(g_t) = i^M(g; d, \delta)$. Note that, by construction, deviations are not profitable after a τ such that $g_\tau \neq y_\tau^P(g_0; d, \delta)$. For the remaining histories note that the average utility of a deviating agent must converge to $(1-\delta)\, v^M(g; d, \delta) < (1-\delta)\, v^P(g; d, \delta)$, so there must be a $\delta_1 < 1$, such that for $\delta > \delta_1$ no deviation is profitable.

The result that we also have a $\delta_2 < 1$, such that for $\delta > \delta_2$ the efficient path is a SPE path in a reversible investment economy can be proven analogously. From Battaglini et al. [2012], we know that there is a Markov equilibrium $\widetilde{y}^M(g; d, \delta)$, $\widetilde{v}^M(g; d, \delta)$ with steady state $\widetilde{g}^M(d, \delta) \leq [u']^{-1}(1 - \delta(1-d)/n)$, and so strictly lower than the steady state $g^P(d,1)$ of the planner's solution for all $\delta \in [0,1]$. Proceeding exactly as above we can see that $\lim_{\delta\to 1}\left[(1-\delta)\, v^P(g; d, \delta)\right] > \lim_{\delta\to 1}\left[(1-\delta)\, \widetilde{v}^M(g; d, \delta)\right]$. Associated to an aggregate investment function $\widetilde{y}^M(g; d, \delta)$ we define as above the individual contribution function: $\widetilde{i}^M(g; d, \delta) = \left[\widetilde{y}^M(g; d, \delta) - (1-d)g\right]/n$. To construct the equilibrium, consider the following trigger strategies. If $g_\tau = y_\tau^P(g_0; d, \delta)$ for all $\tau \leq t$, then $i^t(g_t; d, \delta) = i^P(g; d, \delta)$, where $i^t(g_t)$ is the investment at time t of an agent. If $\exists \tau \leq t$ such that $g_\tau \neq y_\tau^P(g_0; d, \delta)$, then $i^t(g_t) = \widetilde{i}^M(g; d, \delta)$. Note that, by construction, deviations are not profitable after a τ such that $g_\tau \neq y_\tau^P(g_0; d, \delta)$. For the remaining histories note that the average utility of a deviating agent must converge to $(1-\delta)\, v^M(g; d, \delta) < (1-\delta)\, v^P(g; d, \delta)$, so there must be a $\delta_2 < 1$, such that for $\delta > \delta_2$ no deviation is profitable. Given this, the statement of the proposition follows immediately by defining $\overline{\delta} = \max(\delta_1, \delta_2)$. ■