LECTURE NOTES ON GAME THEORY
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1. Extensive form games with perfect information

1.1. Chess. We assume the students are familiar with chess. We will choose the following (non-standard) rules for the ending of chess: the game ends either by the capturing of a king, in which case the capturing side wins and the other loses, or else in a draw, which happens when there a player has no legal moves, or more than 100 turns have elapsed.

As such, this game has the following features:

- There are two players, white and black.
- There are (at most) 100 time periods.
- In each time period one of the players chooses an action. This action is observed by the other player.
- The sequence of actions taken by the players so far determines what actions the active player is allowed to take.
- Every sequence of alternating actions eventually ends with either a draw, or one of the players winning.

We say that white can force a victory if, for any moves that black chooses, white can choose moves that will end in its victory. Zermelo showed in 1913 [21] that in the game of chess, as described above, one of the following three holds:

- White can force a victory.
- Black can force a victory.
- Both white and black can force a draw.

We will prove this later.

We can represent the game of chess mathematically as follows.

- Let $N = \{W, B\}$ be the set of players.
- Let $A$ be the set of actions (or moves) that any of the players can potentially take at any stage of the game. E.g., rook from $A1$ to $B3$.
- A history $h = (a_1, \ldots, a_n)$ is a finite sequence of elements of $A$; we denote the empty sequence by $\emptyset$. We say that $h$ is legal if each move $a_i$ in $h$ is allowed in the game of chess, given the previous moves $(a_1, \ldots, a_{i-1})$, and if the length of $h$ is at most 100. We denote by $H$ the set of legal histories, including $\emptyset$.
- Let $Z \subset H$ be the set of terminal histories; these are histories of game plays that have ended (as described above, by capturing or a draw). Formally, $Z$ is the set of histories in $H$ that are not prefixes of other histories in $H$.
- Let $O = \{W, B, D\}$ be the set of outcomes of the game (corresponding to $W$ wins, $B$ wins and draw). Let $o: Z \rightarrow O$ be a
function that assigns to each terminal history the appropriate outcome.

- Let \( P: H \setminus Z \to N \) be the player function. \( P \) assigns to each non-terminal history the player whose turn it is to play. Specifically, \( P(h) \) will be \( W \) if the length of \( h \) is even, and \( B \) otherwise.

- Let \( \preceq_W \) and \( \preceq_B \) be the white player’s and the black player’s (obvious) preferences over \( O \).

A strategy of a player is a complete description of what that player would do in her turns, given every possible history. Formally, a strategy \( s \) of \( W \) is a mapping \( s: P^{-1}(W) \to A \) that assigns to each every history \( h \) with \( P(h) = W \) an \( a \in A \) such that \((h, a) \in H \). Here we denote by \((h, a)\) the strategy \((a_1, \ldots, a_n, a)\), where \( h = (a_1, \ldots, a_n) \).

Let \((s_W, s_B)\) be a pair of strategies, one for each player. Then we can define the history \( h(s_W, s_B) \in H \) to be the terminal history that is reached when the two players chooses these strategies. We can also define \( o(s_W, s_B) = o(h(s_W, s_B)) \) to be the resulting outcome of the game.

Given these definitions, we can formally state Zermelo’s Theorem.

**Theorem 1.1** (Zermelo, 1913). In the game of chess, as described above, exactly one of the following three holds:

- There exists a strategy \( s_W^* \) for \( W \) such that, for any strategy \( s_B \) of \( B \), \( o(s_W^*, s_B) = W \).
- There exists a strategy \( s_B^* \) for \( B \) such that, for any strategy \( s_W \) of \( W \), \( o(s_W, s_B^*) = B \).
- There exist strategies \( s_W^* \) for \( W \) and \( s_B^* \) for \( B \) such that, for any strategies \( s_W \) and \( s_B \), \( o(s_W^*, s_B) \neq B \) and \( o(s_W, s_B^*) \neq W \). (It follows that \( o(s_W^*, s_B^*) = D \).)

Note that it is not known which of the three is true.

1.2. **Definition of extensive form games with perfect information.** In general, an extensive form game (with perfect information) \( G \) is a tuple \( G = (N, A, H, O, o, P, \{\preceq_i\}_{i \in N}) \) where

1. \( N \) is the set of players.
2. \( A \) is the set of actions.
3. \( H \) is the set of allowed histories. This is a set of sequences of elements of \( A \) such that if \( h \in H \) then every prefix of \( h \) is also in \( H \).
   
   \( Z \) is the set of sequences in \( H \) that are not subsequences of others in \( H \). Note that we can specify \( H \) by specifying \( Z \); \( H \) is the set of subsequences of sequences in \( Z \).
In general, the sequences in $H$ needn’t be finite. In this case we need an additional requirement, namely that if $H$ contains every subsequence of some infinite sequence then it also contains that infinite sequence. The set $Z$ of terminal histories - those that are not prefixes of others - includes all the infinite ones.

(4) $O$ is the set of outcomes.
(5) $o$ is a function from $Z$ to $O$.
(6) $P$ is a function from $H \setminus Z$ to $N$.
(7) For each player $i \in N$, $\preceq_i$ is a preference relation over $O$. (I.e., a complete, transitive and reflexive binary relation).

We denote by $A(h)$ the actions available to player $P(h)$ after history $h$:

$$A(h) = \{a \in A : ha \in H\}.$$ 

A game is called a finite horizon game if there is a bound on the length of histories in $H$. A game is called finite if $H$ is finite.

Strategies are defined as for chess. A strategy profile $s = \{s_i\}_{i \in N}$ constitutes a strategy for each player. We can, as for chess, define $h(s)$ and $o(s)$ as the history and outcome associated with a strategy profile.

1.3. The ultimatum game. In the ultimatum game player 1 makes an offer $a \in \{0, 1, 2, 3, 4\}$ to player 2. Player 2 either accepts or rejects. If player 2 accepts then she receives $a$ dollars and player 1 receives $4 - a$ dollars. If 2 rejects then both get nothing. This is how this game can be written in extensive form:

(1) $N = \{1, 2\}$.
(2) $A = \{0, 1, 2, 3, 4, a, r\}$.
(3) $Z = \{0a, 1a, 2a, 3a, 4a, 0r, 1r, 2r, 3r, 4r\}$.
(4) $O = \{(0, 0), (0, 4), (1, 3), (2, 2), (3, 1), (4, 0)\}$. Each pair corresponds to what players 1 receives and what player 2 receives.
(5) For $b \in \{0, 1, 2, 3, 4\}$, $o(ba) = (4 - b, b)$ and $o(br) = (0, 0)$.
(6) $P(\emptyset) = 1$, $P(0) = P(1) = P(2) = P(3) = P(4) = 2$.
(7) For $a_1, b_1, a_2, b_2 \in \{0, 1, 2, 3, 4\}$, $(a_1, a_2) \preceq_1 (b_1, b_2)$ iff $a_1 \leq b_1$, and $(a_1, a_2) \preceq_2 (b_1, b_2)$ iff $a_2 \leq b_2$.

A strategy for player 1 is just a choice among $\{0, 1, 2, 3, 4\}$. A strategy for player 2 is a map from $\{0, 1, 2, 3, 4\}$ to $\{a, r\}$: player 2’s strategy describes whether or not she accepts or rejects any given offer.

1.4. Equilibria. Given a strategy profile $s = \{s_i\}_{i \in N}$, we denote by $(s_{-i}, s'_i)$ the strategy profile in which $i$’s strategy is changed from $s_i$ to $s'_i$ and the rest remain the same.
A strategy profile $s^*$ is a *Nash equilibrium* if for all $i \in N$ and strategy $s_i$ of player $i$ it holds that

$$o(s^*_i, s_i) \preceq_i o(s^*).$$

When $s$ is the equilibrium $h(s)$ is also known as the *equilibrium* path associated with $s$.

Example: in the ultimatum game, consider the strategy profile in which player 1 offers 3, and player 2 accepts 3 or 4 and rejects 0, 1 or 2. It is easy to check that this is an equilibrium profile.

1.5. *The centipede game.* In the centipede game there are $n$ time periods and 2 players. The players alternate in turns, and at each turn each player can either stop (S) or continue (C), except at the last turn, where they must stop. Now, there is a piggy bank which initially has in it 2 dollars. If a player decides to stop, she is awarded half of what’s in the bank, plus one dollar, and the other player is awarded the remainder. If a player decides to continue, 2 dollars are added to the bank. Hence, in period $m$, a player is awarded $m + 1$ if she decided to stop, and the other player is given $m - 1$.

**Exercise 1.2.** Define the centipede game formally, for $n = 6$. How many strategies does each player have?

**Exercise 1.3.** Show that the strategy profile in which both players play $S$ in every time period is an equilibrium.

**Theorem 1.4.** In every Nash equilibrium, player 1 plays $S$ in the first period.

*Proof.* Assume by contradiction that player 1 plays $C$ in the first period under some equilibrium $s$. Then there is some period $m > 1$ in which $S$ is played for the first time on the equilibrium path. It follows that the player who played $C$ in the previous period is awarded $m - 1$. But she could have been awarded $m$ by playing $S$ in the previous period, and therefore $s$ is not an equilibrium. □

1.6. *Subgames and subgame perfect equilibria.* A *subgame* of a game $G = (N, A, H, O, o, P, \{\preceq_i\}_{i \in N})$ is a game that starts after a given finite history $h \in H$. Formally, the subgame $G(h)$ associated with $h = (h_1, \ldots, h_n) \in H$ is $G(h) = (N, A, H|_h, O|_h, o|_h, P|_h, \{\preceq_i\}_{i \in N})$, where

$H|_h = \{(a_1, a_2, \ldots) : (h_1, \ldots, h_n, a_1, a_2, \ldots) \in H\}$

and

$$o|_h(h') = o(hh') \quad P|_h(h') = P(hh').$$
A strategy $s$ of $G$ can likewise used to define a strategy $s|_{h}$ of $G(h)$.

A subgame perfect equilibrium of $G$ is a Nash equilibrium $s^*$ such that for every subgame $G(h)$ it holds that $s^*|_{h}$ is a Nash equilibrium of $G(h)$.

An important property of finite horizon games is the one deviation property. Before introducing it we make the following definition.

Let $s$ be a strategy profile. We say that $s'_i$ is a profitable deviation from $s$ for player $i$ at history $h$ if $s'_i$ is a strategy for $G$ such that

$$o|_{h}(s_{-i}|_{h}, s'_i|_{h}) \succ_i o|_{h}(s|_{h}).$$

**Theorem 1.5** (The one deviation principle). Let $G = (N, A, H, O, o, P, \{\preceq_i \}_{i \in N})$ be a finite horizon, extensive form game with perfect information. Let $s$ be a strategy profile that is not a subgame perfect equilibrium. There there exists some history $h$ and a profitable deviation $s'_i$ for player $i = P(h)$ at history $h$ such that $s'_i(h') = s_i(h')$ for all $h' \neq h$.

**Proof.** Let $s$ be a strategy profile that is not a subgame perfect equilibrium. Then there is a subgame $G(h)$ and a strategy $s'_i$ for player $i = P(h)$ such that $s'_i$ is a profitable deviation for $i$ in $h$. Let $h^*$ be a maximal length history for which there exists a profitable deviation, and let $s'_i$ be a profitable deviation for $G(h^*)$ that minimizes the number of histories by which it differs from $s_i$. It follows that $s'_i(h) = s_i(h)$ for any history $h$ that does not have $h^*$ as a prefix, since otherwise $s'_i$ would not be minimal.

Now, we claim that $s'_i$ differs from $s_i$ only at $h^*$, which will prove the claim. By the definition of $s'_i$, this need only be demonstrated for histories that have $h^*$ as a prefix. Assume by contradiction that $s'_i(h^*h) \neq s_i(h^*h)$ for some non-empty $h$. Then, by the maximality of $h^*$, $s'_i|_{h^*h}$ is not a profitable deviation for $G(h^*h)$. But this then contradicts the minimality of $s'_i$. \[\square\]

### 1.7 Backward induction, Kuhn’s Theorem and a proof of Zermelo’s Theorem

Let $G = (N, A, H, O, o, P, \{\preceq_i \}_{i \in N})$ be an extensive form game with perfect information. Recall that $A(\emptyset)$ is the set of allowed initial actions for player $i = P(\emptyset)$. For each $b \in A(\emptyset)$, let $\sigma(G(b))$ be some strategy profile for the subgame $G(b)$. Given some $a \in A(\emptyset)$, we denote by $a\sigma$ the strategy profile for $G$ in which player $i = P(\emptyset)$ chooses the initial action $a$, and for each action $b \in A(\emptyset)$ the subgame $G(b)$ is played according to $\sigma(G(b))$. That is, $[a\sigma]_i(\emptyset) = a$ and for every player $j$, $b \in A(\emptyset)$ and $bh \in H \setminus Z$, $[a\sigma]_j(bh) = [\sigma(G(b))]_j(h)$.

**Lemma 1.6** (Backward Induction). Let $G = (N, A, H, O, o, P, \{\preceq_i \}_{i \in N})$ be an extensive form game with perfect information. Assume that
for each $b \in A(\emptyset)$ the subgame $G(b)$ has a subgame perfect equilibrium $\sigma(G(b))$. Assume also that there is an $a^* \in A(\emptyset)$ that $\lhd_i$-maximizes $o(a^* \sigma)$, where $i = P(\emptyset)$. Then $a^* \sigma$ is a subgame perfect equilibrium of $G$.

Proof. Denote $s^* = a^* \sigma$. Assume by contradiction that $s^*$ is not a subgame perfect equilibrium of $G$. Then some player $j$ has a profitable deviation $s_j$ at some history $h$ for which $j = P(h)$. Let $h$ be a minimal length history at which such a deviation exists.

We first claim that $h = \emptyset$. Otherwise, $h = ah'$ for some $a \in A(\emptyset)$, and so $s_j|_a$ would be a profitable deviation for $j$ in $\sigma(G(a))$ and history $h'$; this is impossible since $\sigma(G(a))$ is a subgame perfect equilibrium.

We have thus shown that $i = P(\emptyset)$ has a profitable deviation $s_i$ at $h = \emptyset$. Denote $b = s_i(\emptyset)$.

Since $s_i$ is profitable then

$$o(s^*_{-i}, s_i) \succ_i o(s^*).$$

The left hand side equals $o(b(s^*_{-i}|b, s_i|b))$. As for the right hand side, by the definition of $s^*$ it holds that

$$o(s^*) \succeq_i o(b \sigma) = o(b(s^*_{-i}|b, s^*_i|b)).$$

Putting this together yields

$$o(b(s^*_{-i}|b, s_i|b) \succ_i o(b(s^*_{-i}|b, s^*_i|b)).$$

But $s^*|_b = \sigma(G(b))$, and so we can write this as

$$o(b(\sigma(G(b)), s_i|b) \succ_i o(b(\sigma(G(b)), s^*_i|b)).$$

Hence $s_i|_b$ is a profitable deviation from $\sigma(G(b))$ for player $i$ in the subgame $G(b)$, in contradiction to the fact that $\sigma(G(b))$ is an equilibrium. $\square$

Kuhn [13] proved the following theorem.

**Theorem 1.7** (Kuhn, 1953). *Every finite extensive form game with perfect information has a subgame perfect equilibrium.*

Given a game $G$ with allowed histories $H$, denote by $\ell(G)$ the maximal length of any history in $H$.

**Proof of Theorem 1.7.** We prove the claim by induction on $\ell(G)$. For $\ell(G) = 0$ the claim is immediate, since the trivial strategy profile is an equilibrium, and there are no proper subgames. Assume we have proved the claim for all games $G$ with $\ell(G) < n$.

Let $\ell(G) = n$, and denote $i = P(\emptyset)$. For each $b \in A(\emptyset)$, let $\sigma(G(b))$ be some subgame perfect equilibrium of $G(b)$. These exist by our inductive assumption, as $\ell(G(b)) < n$. 
Let \( a^* \in A(\emptyset) \) be a \( \preceq_i \)-maximizer of \( o(a^*\sigma) \). Then by the Backward Induction Lemma \( a^*\sigma \) is a subgame perfect equilibrium of \( G \), and our proof is concluded. \( \square \)

Given Kuhn’s Theorem, Zermelo’s Theorem admits a simple proof.

**Proof of Theorem 1.1.** Let \( s^* = (s^*_W, s^*_B) \) be a subgame perfect equilibrium of any finite extensive form game with two players (\( W \) and \( B \)) and outcomes \( \{W, B, D\} \) (e.g., chess). Hence the outcome \( o(s^*) \) is either \( W, B \) or \( D \). Consider these three cases. If \( o(s^*) = W \) then for any \( s_B \)
\[
o(s^*_W, s_B) \preceq_B o(s^*) = W.
\]
But \( W \) is the unique \( \preceq_B \) minimizer, and so \( o(s^*_W, s_B) = W \). That is, white can force victory by playing \( s^*_W \). The same argument shows that if \( o(s^*) = B \) then black can force victory. Finally, if \( o(s^*) = D \) then for any \( s_B \)
\[
o(s^*_W, s_B) \preceq_B o(s^*) = D,
\]
so \( o(s^*_W, s_B) \) is either \( D \) or \( W \). By the same argument \( o(s^*_W, s^*_B) \) is either \( D \) or \( B \) for any \( s_W \), and so we have proved the claim. \( \square \)
2. Strategic form games

2.1. Definition. A game in strategic form (or normal form) is a tuple $G = (N, \{S_i\}_{i \in N}, O, o, \{\preceq_i\}_{i \in N})$ where

- $N$ is the set of players.
- $S_i$ is the set of actions or strategies available to player $i$. We denote by $S = \prod_i S_i$ the set of strategy profiles.
- $O$ is the set of outcomes.
- $o : S \rightarrow O$ is a function that assigns an outcome to each strategy profile.
- For each player $i \in N$, $\preceq_i$ is a preference relation over $O$.

Sometimes it is more convenient to define a game by $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$, where for each $i \in N$, the function $u_i : S \rightarrow \mathbb{R}$ is player $i$’s utility (or payoff) for each strategy profile.

We say that $G$ is finite if $N$ is finite and $S$ is finite.

2.2. Nash equilibria. Given a strategy profile $s$, a profitable deviation for player $i$ is a strategy $s_i$ such that $o(s_{-i}, s_i) \succ_i o(s)$.

A strategy profile $s$ is a Nash equilibrium if no player has a profitable deviation. These are also called pure Nash equilibria, for reasons that we will see later. They are often just called equilibria.

2.3. Examples.

- Extensive form game with perfect information. Let $G = (N, A, H, O, o, P, \{\preceq_i\}_{i \in N})$ be an extensive form game with perfect information. Let $G'$ be the strategic form game given by $G' = (N', \{S_i\}_{i \in N}, O', o', \{\preceq'_i\}_{i \in N})$, where
  - $N' = N$.
  - $S_i$ is the set of $G$-strategies of player $i$.
  - $O' = O$.
  - For every $s \in S$, $o'(s) = o(s)$.
  - $\preceq'_i = \preceq_i$.

We have thus done nothing more than having written the same game in a different form. Note, however, that not every game in strategic form can be written as an extensive form game with perfect information.

Exercise 2.1. Show that $s \in S$ is a Nash equilibrium of $G$ iff it is a Nash equilibrium of $G'$.
Note that a disadvantage of the strategic form is that there is no natural way to define subgames or subgame perfect equilibria.

**Matching pennies.** In this game, and in the next few, there will be two players: a row player (R) and a column player (C). We will represent the game as a payoff matrix, showing for each strategy profile \( s = (s_R, s_C) \) the payoffs \( u_R(s) \), \( u_C(s) \) of the row player and the column player.

\[
\begin{array}{c|cc}
 & H & T \\
\hline
H & 1,0 & 0,1 \\
T & 0,1 & 1,0 \\
\end{array}
\]

In this game each player has to choose either heads (H) or tails (T). The row player wants the choices to match, while the row player wants them to mismatch.

**Exercise 2.2.** Show that matching pennies has no pure Nash equilibria.

**Prisoner’s dilemma.**

Two prisoners are faced with a dilemma. A crime was committed in the prison, and they are the only two who could have done it. Each prisoner has to make a choice between testifying against the other (and thus betraying the other) and keeping her mouth shut. In the former case we say that the prisoner defected (i.e., betrayed the other), and in the latter she cooperated (with the other prisoner, not with the police).

If both cooperate (i.e., keep their mouths shut), they will have to serve the remainder of their sentences, which are 2 years each. If both defect (i.e., agree to testify against each other), each will serve 3 years. If one defects and the other cooperates, the defector will be released immediately, and the cooperator will serve 10 years for the crime.

Assuming that a player’s utility is minus the number of years served, the payoff matrix is the following.

\[
\begin{array}{c|cc}
 & D & C \\
\hline
D & -3,-3 & 0,-10 \\
C & -10,0 & -2,-2 \\
\end{array}
\]

**Exercise 2.3.** Show that the unique pure Nash equilibrium is \((D, D)\).
• **Battle of the sexes**: Adam and Steve are a married couple. They have to decide whether to spend the night at the monster truck show (M) or at the opera (O). Adam (the row player) prefers the truck show, while Steve prefers the opera. Both would rather go out then do nothing, which is what would happen if they could not agree. Their payoff matrix is the following.

<table>
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<tr>
<th></th>
<th>M</th>
<th>O</th>
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<tr>
<td><strong>M</strong></td>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td><strong>O</strong></td>
<td>0, 0</td>
<td>1, 2</td>
</tr>
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**Exercise 2.4.** *Find all the equilibria of this game.*

• **Bertrand.** There are *n* companies that are selling the same product. There are 1,000 customers who want to buy one product each, but will not buy if the price is more than $1,000. Each company has to set a price, which we will assume has to be an integral amount of dollars.

  All customers buy from the company that set the lowest price (unless all prices are more than $1,000, in which case they buy nothing). If more than one has set the lowest price then the revenue is split evenly between these companies. Therefore, if *m* companies chose the lowest price *p*, then the payoff to each of these companies is $1000 \cdot p/m$, and the payoff to the rest is zero.

**Exercise 2.5.** *Find all the equilibria of this game.*

Thought experiment: what can the members of the cartel do (in real life) to ensure that all set the price to $1,000?

• **The regulated cartel.** As in the previous game, there are *n* companies and 1,000 customers who will buy at $1,000 or less. There is a production cost *p*\textsubscript{prod} to this product (which is the same for all companies) and the payoff for the company from selling a product is its price minus the production cost. As before, customers are split between the lowest priced companies.

  Regulation dictates that companies must sell at production cost plus $1. However, the regulator does not know the production cost *p*\textsubscript{prod}, which the companies do know.

  Hoping to solve this problem, the regulator has dictated that any company that sets a price *p* that is higher than the lowest set price *p*\textsubscript{min} has to pay a fine of *p* − *p*\textsubscript{min}.

\[\text{The name of this game seems archaic in this day and age, but we will keep it, as it is standard in the literature}\]
Exercise 2.6. Find the equilibria of this game.

- **Cournot.** Consider a finite set of companies $N$ that are selling the same product. Each has to decide on a quantity $q_i$ to manufacture. The function $D(p)$ describes the demand as a function of price, and we set denote its inverse by $P = D^{-1}$. Let $c(q)$ be the cost of manufacturing, as a function of quantity. Assuming prices are set so that all the manufactured goods are sold, company $i$’s payoff is

$$u_i(q_1, \ldots, q_n) = q_i \cdot P\left(\sum_i q_i\right) - c(q_i).$$

Exercise 2.7. Let $P(q) = A - B \cdot q$, and let $c(q) = C \cdot q$ (so that the cost per unit is independent of the amount manufactured), and assume $A > C$. Show that there is a Nash equilibrium in which all companies set the same price $q$, and calculate this $q$.

2.4. Dominated strategies. A strategy $s_i$ of player $i$ in $G = (N, \{S_i\}_{i \in N}, O, \prec_i, \{\preceq_i\}_{i \in N})$ is strictly dominated (or just dominated) if there exists another strategy $t_i$ such that, for all choices of strategy $s_{-i}$ of the other players it holds that

$$o(s_{-i}, t_i) \succ_i o(s_{-i}, s_i).$$

That is, regardless of what the other players do, $t_i$ is a better choice for $i$ than $s_i$.

We say that $s_i$ is weakly dominated if there exists another strategy $t_i$ such that for all $s_{-i}$

$$o(s_{-i}, t_i) \succeq_i o(s_{-i}, s_i),$$

and furthermore for some $s_{-i}$

$$o(s_{-i}, t_i) \succ_i o(s_{-i}, s_i).$$

Exercise 2.8. Does matching pennies have strictly dominated strategies? Weakly dominated strategies? How about the prisoner’s dilemma? How about the regulated cartel?

2.5. Repeated elimination of dominated strategies. It seems unreasonable that a reasonable person would choose a strictly dominated strategy, because she has an obviously better choice. Surprisingly, taking this reasoning to its logical conclusion leads to predictions that sharply contradict observed human behavior.

Consider the regulated cartel game $G$, and assume that $p_{\text{prod}} = 0$ and that the set of possible strategies is \{$1, \ldots, 1000\}. It is easy to see that $1,000 is a dominated strategy; if all other companies choose
$1,000 then $999 is a better strategy. If the lowest price $p_{min}$ is lower than $1,000$, then still $999 is a better strategy, since then the fine is smaller. It is likewise easy to check that $1,000 is the only dominated strategy.

Since no reasonable player would choose $1,000, it is natural to define a new game $G'$ which is identical to $G$, except that the strategy space of every player no longer includes $1,000. Indeed, assuming that we are interested in equilibria, the following theorem guarantees that analyzing $G'$ is equivalent to analyzing $G$.

**Theorem 2.9.** Let $G = (N, \{S_i\}_{i \in N}, O, o, \{\preceq_i\}_{i \in N})$, let $d_j \in S_j$ be a dominated strategy of player $j$, and let $G' = (N, \{S'_i\}_{i \in N}, O, o', \{\preceq_i'\}_{i \in N})$, where

$$S'_i = \begin{cases} S_i & \text{for } i \neq j \\ S_j \setminus \{d_j\} & \text{for } i = j \end{cases}$$

and $o'$ is the restriction of $o$ to $S'$. Then every Nash equilibrium $s \in S$ of $G$ is in $S'$. Furthermore $s \in S'$ is a Nash equilibrium of $G$ if and only if it is a Nash equilibrium of $G'$.

The proof of this Theorem is straightforward and is left as an exercise.

Now, as before, it is easy to see that $999 is a dominated strategy for $G'$, and to therefore remove it from the set of strategies and arrive at a new game, $G''$. Indeed, if we repeat this process, we will arrive at a game in which the single strategy is $1. However, in experiments this is very rarely the chosen strategy\footnote{Except in populations of game theory students}.

The following theorem of Gilboa, Kalai and Zemel \cite{gilboa1990van} shows that the order of elimination of dominated strategies does not matter, as long as one continues eliminating until there are no more dominated strategies. Note that this is not true for weakly dominated strategies \cite{gilboa1990van}.

**Theorem 2.10** (Gilboa, Kalai and Zemel, 1990). Fix a finite game $G$, and let $G_1$ be a game that

- is the result of repeated elimination of dominated strategies from $G$, and
- has no dominated strategies.

Let $G_2$ be a game with the same properties. Then $G_1 = G_2$. 
2.6. **Dominant strategies.** A strategy \( s_i \) of player \( i \) in \( G = (N, \{S_i\}_{i \in N}, O, o, \{\preceq_i\}_{i \in N}) \) is **strictly dominant** if for every other strategy \( t_i \) it holds that

\[
o(s_{-i}, s_i) \succ_i o(s_{-i}, t_i).
\]

That is, regardless of what the other players do, \( t_i \) is a better choice for \( i \) than \( s_i \).

**Exercise 2.11.** Which of the games above have a strictly dominant strategy?

The proof of the following theorem is straightforward.

**Theorem 2.12.** If player \( i \) has a strictly dominant strategy \( d_i \) then, for every pure Nash equilibrium \( s^* \) it holds that \( s^*_i = d_i \).

2.7. **Mixed equilibria and Nash’s Theorem.** Given a finite set \( X \), denote by \( \Delta X \) set of probability distributions over \( X \).

Let \( G = (N, \{S_i\}, \{u_i\}) \) be a finite game. A **mixed strategy** \( \sigma_i \) is an element of \( \Delta S_i \). Given a **mixed strategy profile** \( \sigma = (\sigma_1, \ldots, \sigma_n) \), we overload notation and let \( \sigma \) be the element of \( \Delta S \) given by the product \( \prod_i \sigma_i \). That is, \( \sigma \) is the distribution over \( \prod_i S_i \) in which we pick independently from each \( S_i \), with distribution \( \sigma_i \).

Define a game \( \hat{G}(N, \{\hat{S}_i\}, \{\hat{u}_i\}) \) as follows:

- \( \hat{S}_i = \Delta S_i \).
- \( \hat{u}_i(\sigma) = \mathbb{E}_{s \sim \sigma} [u_i(s)] \).

That is, \( \hat{G} \) is a game whose strategies are the mixed strategies of \( G \), and whose utilities are the expected utilities of \( G \), taken with respect to the given mixed strategies. A pure Nash equilibrium of \( \hat{G} \) is called a **mixed Nash equilibrium** of \( G \). That is, a mixed strategy profile \( \sigma \in \prod_i \Delta S_i \) is a mixed Nash equilibrium if no player can improve her expected utility by deviating to another mixed strategy. We will often just say “Nash equilibrium” when referring to mixed equilibria. We will use \( u_i \) to also mean \( \hat{u}_i \); that is, we will extend \( u_i \) from a function \( S \to \mathbb{R} \) to a function \( \Delta S \to \mathbb{R} \).

Note that as a function \( \Delta S \to \mathbb{R} \), \( u_i \) is linear. That is,

\[
u_i(\alpha \sigma + (1 - \alpha) \tau) = \alpha u_i(\sigma) + (1 - \alpha) u_i(\tau).
\]

Nash’s celebrated theorem [17] states that every finite game has a mixed equilibrium.

**Theorem 2.13** (Nash, 1950). *Every finite game \( G \) has a mixed Nash equilibrium.*

To prove Nash’s Theorem we will need Brouwer’s Fixed Point Theorem.
Theorem 2.14 (Brouwer’s Fixed Point Theorem). Let $X$ be a compact convex subset of $\mathbb{R}^d$. Let $T : X \to X$ be continuous. Then $T$ has a fixed point. I.e., there exists an $x \in X$ such that $T(x) = x$.

Corollary: if you are in a room and hold a map of the room horizontally, then there is a point in the map that is exactly above the point it represents.

Exercise 2.15. Prove Brouwer’s Theorem for the case that $X$ is a convex compact subset of $\mathbb{R}$.

Consider a “lazy player” who, given that all players are currently playing a mixed strategy profile $\sigma$, has utility for playing some mixed strategy $\sigma'_i$ which is given by

$$g^i(\sigma'_i) = u_i(\sigma_{-i}, \sigma'_i) - \lambda \|\sigma'_i - \sigma_i\|^2$$

for some $\lambda > 0$. That is, her utility has an extra addend which is lower the further away the new strategy is from her current strategy. Analyzing what happens when all players are lazy is the key to the following proof of Nash’s Theorem.

Proof of Theorem 2.13. Let $G = (N, \{S_i\}, \{u_i\})$ be a finite game. Let $f : \Delta S \to \Delta S$ be given by

$$[f(\sigma)]_i = \arg\max_{\sigma'_i \in \Delta S_i} g^i(\sigma'_i).$$

It is straightforward to show that $f$ is continuous. A more subtle point is that this argmax may not be unique. However, for low enough values of $\lambda$ it is unique, and we thus choose $\lambda$ accordingly.

Since we can think of $\Delta S$ as a convex subset of $\mathbb{R}^{|S|}$, $f$ has a fixed point $\sigma^*$, by Brouwer’s Theorem. We will show that $\sigma^*$ is a mixed Nash equilibrium.

Assume by contradiction that there exists a player $i$ and a $\sigma_i \in \Delta S_i$ such that

$$\delta := u_i(\sigma^*_{-i}, \sigma_i) - u_i(\sigma^*) > 0.$$

Given any $\varepsilon > 0$, let $\tau^\varepsilon_i = (1 - \varepsilon) \cdot \sigma^*_{-i} + \varepsilon \cdot \sigma_i$. Then

$$g^i(\tau^\varepsilon_i) = u_i(\sigma^*_{-i}, \tau^\varepsilon_i) - \lambda \|\tau^\varepsilon_i - \sigma^*_{i}\|^2.$$

(2.1)

Now, by the linearity $u_i$,

$$u_i(\sigma^*_{-i}, \tau^\varepsilon_i) = u_i(\sigma^*_{-i}, (1 - \varepsilon) \cdot \sigma^*_{-i} + \varepsilon \cdot \sigma_i)$$

$$= (1 - \varepsilon) \cdot u_i(\sigma^*) + \varepsilon \cdot u_i(\sigma^*_{-i}, \sigma_i)$$

$$= u_i(\sigma^*) + \varepsilon \cdot \delta.$$
By the definition of $\tau^\varepsilon_i$
\[ \|\tau^\varepsilon_i - \sigma^*_i\|_2^2 = \|\varepsilon \cdot (\sigma_i - \sigma^*_i)\|_2^2 = \varepsilon^2 \|\sigma_i - \sigma^*_i\|_2^2. \]
Plugging these expressions back into (2.1) we get
\[ g^*_i(\tau^\varepsilon_i) = u_i(\sigma^*) + \varepsilon \cdot \delta - \varepsilon^2 \lambda \|\sigma_i - \sigma^*_i\|_2^2, \]
which, for $\varepsilon$ small enough, is greater than $u_i(\sigma^*)$. Hence
\[ \sigma^* \neq \arg \max_{\sigma^i \in \Delta S_i} g^*_i(\sigma^i), \]
and $\sigma^*$ is not a fixed point of $f$ - contradiction. \qed

2.8. **Best responses.** Let $G = (N, \{S_i\}, \{u_i\})$ be a finite game, and let $\sigma$ be a mixed strategy profile in $G$. We say that $s_i \in S_i$ is a **best response** to $\sigma_{-i}$ if, for all $t_i \in S_i$, \[ u_i(\sigma_{-i}, s_i) \geq u_i(\sigma_{-i}, t_i). \]
This notion can be naturally extended to mixed strategies.

The following proposition is helpful for understanding and calculating mixed equilibria.

**Proposition 2.16.** Let $G = (N, \{S_i\}, \{u_i\})$ be a finite game, and let $\sigma^*$ be a mixed Nash equilibrium of $G$. Then any $s_i \in S_i$ (i.e., any $s_i$ to which $\sigma^*_i$ assigns positive probability) is a best response to $\sigma^*_{-i}$.

**Proof.** Denote by $\delta_s \in \Delta S_i$ the point mass distribution which assigns probability one to $s_i$.

Suppose $s_i \in S_i$ is not a best response to $\sigma^*_{-i}$. We will prove the claim by showing that $\sigma^*_i(s_i) = 0$.

Let $t_i \in S_i$ be some best response to $\sigma^*_{-i}$. Then \[ C := u_i(\sigma_{-i}, t_i) - u_i(\sigma_{-i}, s_i) > 0. \]

Let $\sigma_i = \sigma^*_i + \sigma^*_i(s_i)(\delta_{t_i} - \delta_{s_i})$ be the distribution in which we transfer the probability of $s_i$ to $t_i$. That is, let
\[
\sigma_i(s) = \begin{cases} 
0 & \text{if } s = s_i \\
\sigma^*_i(t_i) + \sigma^*_i(s_i) & \text{if } s = t_i \\
\sigma^*_i(s) & \text{otherwise}
\end{cases}
\]
Then
\[
u_i(\sigma^*_i - \sigma_i) = \nu_i(\sigma^*_i, \sigma^*_i + \sigma^*_i(s_i)(\delta_{t_i} - \delta_{s_i}))
\]
\[
= \nu_i(\sigma^*) + \sigma^*_i(s_i) \cdot (\nu_i(\sigma^*_{-i}, \delta_{t_i}) - \nu_i(\sigma^*_i, \delta_{s_i}))
\]
\[
= \nu_i(\sigma^*) + \sigma^*_i(s_i) \cdot C.
\]
But since $\sigma^*$ is an equilibrium we know that

$$u_i(\sigma^*_{-i}, \sigma_i) \leq u_i(\sigma^*).$$

Hence $\sigma^*_i(s_i) = 0$, and $s_i$ is not in the support of $\sigma^*_i$. \qed

It follows that if $\sigma^*$ is an equilibrium then $u_i(\sigma^*_{-i}, s_i)$ is the same for every $s_i$ in the support of $\sigma^*_i$. That is, $i$ is indifferent between all the pure strategies in the support of her mixed strategy.

2.9. Correlated equilibria.

2.9.1. Motivating example. Aumann \cite{Aumann1974} introduced the notion of a correlated equilibrium. Consider the following game, which is usually called “Chicken”. Consider two drivers who arrive at the same time to an intersection. Each one would like to drive on (strategy D) rather than yield (strategy Y), but if both drive then they run the risk of damaging the cars. If both yield time is wasted, but no egos are hurt.

The payoff matrix is the following.

$$
\begin{array}{c|cc}
  & Y & D \\
\hline
  Y & 3, 3 & 0, 5 \\
  D & 5, 0 & -4, -4 \\
\end{array}
$$

This game has three Nash equilibria: two pure ($\{Y, D\}$ and $\{D, Y\}$) and one mixed, in which each player drives with probability $1/3$.

**Exercise 2.17.** Show that this is indeed a mixed equilibrium.

The players’ expected utilities in these equilibria are $(0, 5), (5, 0)$ and $(2, 2)$.

A natural way to resolve this conflict is by the installation of a traffic light which would instruct each player whether to yield or drive. For example, the light could choose uniformly at random from $\{Y, D\}$ and $\{D, Y\}$. It is easy to convince oneself that a player has no incentive to disobey the traffic light, assuming that the other player is obeying it. The players’ utilities in this case become $(2, 5, 2)$.

One could imagine a traffic light that chooses from $\{(Y, Y), (Y, D), (D, Y)\}$, where the first option is chosen with probability $p$ and the second and third are each chosen with probability $(1 - p)/2$. Now, given that a player is instructed to drive, she knows that the other player has been instructed to yield, and so, if we again assume that the other player is obedient, she has no reason to yield.

Given that a player has been instructed to yield, she knows that the other player has been told to yield with conditional probability $p_Y = p/(p + (1 - p)/2)$ and to drive with conditional probability $p_D = ((1 - p)/2)/(p + (1 - p)/2)$. Therefore, her utility for yielding is $3p_Y$, where $p_Y$ is the probability that she is instructed to yield.
while her utility for driving is $5p_Y - 4p_D$. It thus follows that she is not better off disobeying, as long as $3p_Y \geq 5p_Y - 4p_D$. A simple calculation shows that this is satisfied as long as $p \leq 1/2$.

Now, each player’s expected utility is $3p + 5(1 - p)/2$. Therefore, if we choose $p = 1/2$, the players’ expected utilities are $(2.75, 2.75)$. In this equilibrium the sum of the players’ expected utilities is larger than in any Nash equilibrium.

2.9.2. Definition. We now generalize and formalize this idea. Let $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ be a finite game. A distribution $\mu \in \Delta S$ is a correlated equilibrium if for every player $i$ and every $s_i, t_i \in S_i$ it holds that

$$\sum_{s \neq i} \mu(s \neq i, s_i)u_i(s \neq i, s_i) \geq \sum_{s \neq i} \mu(s \neq i, s_i)u_i(s \neq i, t_i). \tag{2.2}$$

Player $i$’s expected utility under a correlated equilibrium $\mu$ is simply $\mathbb{E}_{s \sim \mu} [u_i(s)]$.

Note that for given $i$ and $s_i$, the condition in (2.2) is closed (i.e., if each of a converging sequence $\{\mu_n\}$ of distributions satisfies it then so does $\lim_n \mu_n$). Note also that if $\mu_1$ and $\mu_2$ satisfy (2.2) then so does any convex combination of $\mu_1$ and $\mu_2$. These observations immediately imply the following claim.

Claim 2.18. The set of correlated equilibria is a compact, convex subset of $\Delta S$.

An advantage of correlated equilibria is that they are easier to calculate than Nash equilibria, since they are simply the set of solutions to a linear program. It is even easy to find a correlated equilibrium that maximizes (for example) the sum of the players’ expected utilities, or indeed any linear combination of their utilities. Finding Nash equilibria can, on the other hand, be a difficult problem.

2.10. Zeros-sum games. A two player game $G = (\{1, 2\}, \{S_i\}, \{u_i\})$ is called zero-sum if $u_1 + u_2 = 0$. For such games we drop the subscript on the utility functions and use $u := u_1$.

A general normal form game may have cooperative components (see, e.g., the battle of the sexes game), in the sense that moving from one strategy profile to another can benefit both players. However, zero-sum games are competitive: whatever one player gains the other loses. Hence a player may want to prepare herself for the worst possible outcome, namely one in which, given her own strategy, her opponent will choose the strategy yielding her the minimal utility. Hence an interesting
quantity for a strategy \( \sigma_1 \) for player 1 is the guaranteed utility \( u_g \) when playing \( \sigma_1 \):

\[
u_g(\sigma_1) = \min_{\sigma_2 \in \Delta S_2} u(\sigma_1, \sigma_2).
\]

Continuing with this line of reasoning, player 1 will choose a strategy that maximizes her guaranteed utility. She would thus choose an action in

\[
\arg\max_{\sigma_1 \in \Delta S_1} u_g(\sigma_1) = \arg\max_{\sigma_1 \in \Delta S_1} \min_{\sigma_2 \in \Delta S_2} u(\sigma_1, \sigma_2).
\]

Any such strategy is called a maxmin strategy for player 1. It gives her the best possible guaranteed utility, which is

\[
\max_{\sigma_1 \in \Delta S_1} u_g(\sigma_1) = \max_{\sigma_1 \in \Delta S_1} \min_{\sigma_2 \in \Delta S_2} u(\sigma_1, \sigma_2).
\]

The next theorem shows that maxmin strategies and mixed Nash equilibria are closely related in zero-sum games.

**Theorem 2.19** (Borel, 1921, von Neumann, 1928). Let \( G \) be a finite zero-sum game.

1. In every mixed Nash equilibrium \( \sigma^* \) each strategy is a maxmin strategy for that player.
2. There is a number \( v \in \mathbb{R} \) such that

\[
\max_{\sigma_1 \in \Delta S_1} \min_{\sigma_2 \in \Delta S_2} u(\sigma_1, \sigma_2) = v.
\]

and

\[
\max_{\sigma_2 \in \Delta S_2} \min_{\sigma_1 \in \Delta S_1} -u(\sigma_1, \sigma_2) = -v.
\]

The quantity \( v \) is called the value of \( G \). It follows that \( u(\sigma^*) = v \) for any equilibrium \( \sigma^* \).

**Proof of Theorem 2.19** In this proof we will disencumber notation by shortening \( \max_{\sigma_1 \in \Delta S_1} \) to simply \( \max_{\sigma_1} \), etc.

1. Let \( \sigma^* \) be a mixed Nash equilibrium of \( G \). Without loss of generality, it suffices to show that \( \sigma_1^* \) is a maxmin strategy for player 1.

   Since \( \sigma_1^* \) is a best response of player 1,

   \[
   \max_{\sigma_1} u(\sigma_1, \sigma_2^*) = u(\sigma_1^*, \sigma_2^*).
   \]

   Since \( \min_{\sigma_2} u(\sigma_1, \sigma_2) \leq u(\sigma_1, \sigma_2^*) \), we can write

   \[
   \max_{\sigma_1} \min_{\sigma_2} u(\sigma_1, \sigma_2) \leq u(\sigma_1^*, \sigma_2^*).
   \]
Now, assume by contradiction that $\sigma^*_1$ is not a maxmin strategy, so that $\max_{\sigma_1} \min_{\sigma_2} u_1(\sigma_1, \sigma_2) > u_2(\sigma^*_1)$. I.e.,

$$\max_{\sigma_1} \min_{\sigma_2} u(\sigma_1, \sigma_2) > \min_{\sigma_2} u(\sigma^*_1, \sigma_2).$$

Since $\sigma^*_2$ is a best response of player 2, it maximizes her utility, which, since this is a zero-sum game, minimizes player 1’s utility. Hence we have that $\min_{\sigma_2} u(\sigma^*_1, \sigma_2) = u(\sigma^*_1, \sigma^*_2)$, and we deduce that

$$\max_{\sigma_1} \min_{\sigma_2} u(\sigma_1, \sigma_2) > u(\sigma^*_1, \sigma^*_2).$$

But this contradicts (2.3). We have thus shown that $\sigma^*_1$ is a maxmin strategy for player 1.

(2) Let $\sigma^*$ be an arbitrary mixed equilibrium of $G$; at least one exists by Nash’s Theorem. We first show that

$$\max_{\sigma_1} \min_{\sigma_2} u(\sigma_1, \sigma_2) = u(\sigma^*).$$

Note that

$$u(\sigma^*) = \min_{\sigma_2} u(\sigma^*_1, \sigma_2),$$

since, as argued above, $\sigma_2$ is a best response for player 2, which by the zero-sum nature of the game minimizes player 1’s utility. But certainly

$$\min_{\sigma_2} u(\sigma^*_1, \sigma_2) \leq \max_{\sigma_1} \min_{\sigma_2} u(\sigma^*_1, \sigma_2),$$

and so we have shown that

$$\max_{\sigma_1} \min_{\sigma_2} u(\sigma^*_1, \sigma_2) \geq u(\sigma^*).$$

By (2.3) the other inequality also holds, and thus

$$\max_{\sigma_1} \min_{\sigma_2} u(\sigma^*_1, \sigma_2) = u(\sigma^*).$$

The same argument applied with the roles of the players reversed shows that

$$\max_{\sigma_1} \min_{\sigma_2} -u(\sigma^*_1, \sigma_2) = -u(\sigma^*),$$

and thus our proof is complete, with $v = u(\sigma^*)$.

\[\square\]

In this proof we used Nash’s Theorem to generate a mixed Nash equilibrium of this game. In fact, the existence of Nash equilibria in zero-sum games can be proved using convex analysis methods, and this was done by Borel [4] and von Neumann [20] long before Nash’s
Theorem. Moreover, in zero-sum games equilibria can be found by solving a linear program, which, as observed in the previous section, is easier than the general problem of finding mixed Nash equilibria.
3. Belief and Knowledge

3.1. Beliefs. Let \((\Omega, \Sigma)\) be a measurable space; that is, a set equipped with a sigma-algebra. A belief of a player over \((\Omega, \Sigma)\) is a probability measure on \((\Omega, \Sigma)\). It is often sufficient to consider a finite \(\Omega\), in which case the natural choice for \(\Sigma\) is the set \(2^\Omega\) of all subsets of \(\Omega\).

3.1.1. A motivating example. Beliefs are useful for modeling uncertainty. For example, consider a monopoly seller who would like to sell a single object to a single buyer. The value of the product to the buyer is \(v\). The seller sets a price \(p\); hence her strategy space is \(\mathbb{R}\). The buyer learns \(p\) and then decides to either buy or not to buy. Thus her strategies can each be characterized by a set \(B \subset \mathbb{R}\) of the prices in which she decides to buy.

If the buyer buys then her utility is \(v - p\), and the seller’s utility is \(p\). Otherwise both have zero utility. Note that this is an (infinite) extensive form game with perfect information.

Exercise 3.1. Prove that if the seller knows \(v\) then a subgame perfect equilibrium is for the seller to choose \(p = v\) and for the buyer to buy if and only if \(p \leq v\).

We would like to model the case in which the seller does not know \(v\) exactly. One way to do this is to assume that the seller has some belief \(\mu\) over \(v\). That is, she believes that \(v\) is chosen at random from \(\mu\). Given this belief, she now has to choose a price \(p\). As before, the seller learns \(p\) and has to decide whether or not to buy. Additionally, she also learns \(v\). Thus a strategy of the buyer is, for each price \(p\), a (measurable) set \(B(p) \subset \mathbb{R}\), where she buys at value \(v\) and price \(p\) iff \(v \in B(p)\). The buyer’s utility is now the expected price that the seller paid, given that she bought, and where expectations are taken with respect to \(\mu\). The buyer’s utility is, as before, \(v - p\) if she buys and zero otherwise.

Assume that the seller’s belief can be represented by a probability distribution function \(f: [0, \infty) \rightarrow \mathbb{R}\), so that the probability that \(v \in [a, b]\) is \(\int_a^b f(v) dv\). Then the seller’s utility for strategy profile \((p, B)\) is

\[
    u_S(p, B) = \int_0^\infty \mathbb{1}_{\{B(p)\}}(v) \cdot p \cdot f(v) dv.
\]

Assume that the buyer’s strategy is to buy whenever \(p \leq v\) or whenever \(p < v\); the following calculation will be the same in both cases,
since player 1’s belief is non-atomic. Then the seller’s utility is

$$u_S(p) = \int_p^\infty p \cdot f(v) dv,$$

which we can write as $$p \cdot (1 - F(p))$$, where $$F(x) = \int_0^x f(x) dx$$ is the cumulative distribution function associated with $$f$$. To find the seller’s best response we need to find a maximum of $$u_S$$. Its derivative with respect to $$p$$ is

$$\frac{du_S}{dp} = 1 - F(p) - p \cdot f(p).$$

Hence in any maximum it will be the case that

$$p = \frac{1 - F(p)}{f(p)}.$$

It is easy to show that if this equation has a single solution then this solution is a maximum. In this case $$p$$ is called the monopoly price.

**Exercise 3.2.** Let

$$f(v) = \begin{cases} 0 & \text{if } v < 1 \\ \frac{1}{v^2} & \text{if } v \geq 1 \end{cases}.$$

Calculate $$u_S(p)$$. Under the seller’s belief, what is the buyer’s expected utility as a function of $$p$$?

**Exercise 3.3.** Let $$f(v) = \frac{1}{v_0} \cdot e^{-v/v_0}$$ for some $$v_0 > 0$$. Calculate the monopoly price. Under the seller’s belief, what is the buyer’s expected utility?

3.1.2. **Belief spaces.** When we have more than one player we sometimes want each player to have access to different information. Accordingly, a belief space is a tuple $$(N, (\Omega, \Sigma), \{\mu_i\}_{i \in N}, \{T_i\}_{i \in N}, \{t_i\}_{i \in N})$$ where

- $$N$$ is the set of players.
- $$(\Omega, \Sigma)$$ is measurable space of the states of the world.
- $$\mu_i$$ is player $$i$$’s belief over $$(\Omega, \Sigma)$$.
- $$T_i$$ the space of possible types of player $$i$$. In general it will also be equipped with a sigma-algebra, which we suppress here.
- $$t_i: \Omega \rightarrow T_i$$ is player $$i$$’s private signal or type.

We naturally require that $$t_i$$ and $$u_i$$ be measurable. To facilitate notation, especially in the finite case, we define $$P_i: \Omega \rightarrow \Sigma$$ by

$$P_i(\omega) = t_i^{-1}(t_i(\omega)).$$

That is, $$P_i(\omega)$$ is the set of states of the world for which player $$i$$ has the same type as she does in $$\omega$$. It is easy to show that $$\omega \in P_i(\omega)$$.
The set \( \{ P_i(\omega) \}_{\omega \in \Omega} \) is easily seen to be a partition of \( \Omega \), and is usually called \( i \)'s information partition.

We denote by \( \mu_i(E|t_i) \) the probability, under \( \mu_i \), of the event \( E \in \Sigma \), conditioned on \( t_i \). When \( \Omega \) is finite, this is given by

\[
(3.1) \quad \mu_i(E|t_i(\omega)) = \frac{\mu_i(E \cap P_i(\Omega))}{\mu_i(P_i(\omega))}.
\]

If the different \( \mu_i \)'s are equal then we say that the players have a common prior. This will be a common assumption.

As an example, consider two weather forecasters. Each has a computer program to which she feeds the current meteorological data, and which runs a random simulation. The program’s output is either rain (R) or shine (S). Hence we let \( \Omega = \{R,S\}^3 \); the first coordinate is the first forecaster’s program’s output, and the second coordinate is the second’s output. The last coordinate is the “truth” - the actual weather tomorrow.

The forecasters believe that about half the days have rain and a half do not, that their programs predict the future with some probability \( p > 1/2 \), and that the programs fail or succeed independently from each other. This is reflected in the common prior \( \mu \), as detailed in Table 3.1.2.

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( \mu(\omega) )</th>
<th>( t_1(\omega) )</th>
<th>( t_2(\omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>RRR</td>
<td>( \frac{1}{2}p^2 )</td>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>RRS</td>
<td>( \frac{1}{2}(1-p)^2 )</td>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>RSR</td>
<td>( \frac{1}{2}p(1-p) )</td>
<td>R</td>
<td>S</td>
</tr>
<tr>
<td>RSS</td>
<td>( \frac{1}{2}p(1-p) )</td>
<td>R</td>
<td>S</td>
</tr>
<tr>
<td>SRR</td>
<td>( \frac{1}{2}p(1-p) )</td>
<td>S</td>
<td>R</td>
</tr>
<tr>
<td>SRS</td>
<td>( \frac{1}{2}p(1-p) )</td>
<td>S</td>
<td>R</td>
</tr>
<tr>
<td>SSR</td>
<td>( \frac{1}{2}(1-p)^2 )</td>
<td>S</td>
<td>S</td>
</tr>
<tr>
<td>SSS</td>
<td>( \frac{1}{2}p^2 )</td>
<td>S</td>
<td>S</td>
</tr>
</tbody>
</table>

Table 1. The forecasters’ common prior.

Each of the players gets to observe her program’s forecast only. Hence \( t_1(\omega_1,\omega_2,\omega_3) = \omega_1 \) and \( t_2(\omega_1,\omega_2,\omega_3) = \omega_2 \) with \( T_1 = T_2 = \{R,S\} \).
Exercise 3.4. What is the probability that player 1 assigns to the event \( t_2 = R \), conditioned on \( t_1 = R \)? Conditioned on \( t_1 = S \)? What probability does she assign to \( \omega_3 = R \) in each of these cases?

3.2. Knowledge. Let \( (N, (\Omega, \Sigma), \{\mu_i\}_{i \in N}, \{T_i\}_{i \in N}, \{t_i\}_{i \in N}) \) be a belief space. Let \( A \in \Sigma \) be an event. We say that player \( i \) believes \( A \) at some \( \omega \in \Omega \) if \( \mu_i(A|t_i(\omega)) = 1 \). If \( \Omega \) is finite and \( \Sigma = 2^\Omega \) then we define that \( i \) knows \( A \) at \( \omega \) if \( t_i^{-1}(t_i(\omega)) \subseteq A \). Recalling our notation
\[
P_i(\omega) = t_i^{-1}(t_i(\omega)),
\]
we can, in the finite case, define that \( i \) knows \( A \) if \( P_i(\omega) \subseteq A \).

Claim 3.5. Let \( \Omega \) be finite. Then if \( i \) knows \( A \) at \( \omega \) then \( i \) believes \( A \) in \( \omega \). If furthermore \( \mu_i(\theta) > 0 \) for all \( \theta \in \Omega \), then the opposite implication holds too.

Proof. Assume first that \( P_i(\omega) \subseteq A \). Then
\[
\mu_i(A|t_i(\omega)) \geq \mu_i(P_i(\omega)|t_i(\omega)).
\]
By (3.1), when \( \Omega \) is finite the right hand side is equal to 1, and we have shown the first part of the claim.

Assume now that \( \mu_i(A|t_i(\omega)) = 1 \) and that \( \mu_i(\theta) > 0 \) for all \( \theta \in \Omega \). Then
\[
1 = \mu_i(A|t_i(\omega)) = \frac{\mu_i(A \cap P_i(\omega))}{\mu_i(P_i(\omega))}.
\]
Hence
\[
\mu_i(A \cap P_i(\omega)) = \mu_i(P_i(\omega)),
\]
which implies that
\[
\mu_i(P_i(\omega) \setminus (A \cap P_i(\omega))) = 0.
\]
But since \( \mu_i \) is supported everywhere we have that the argument to \( \mu_i \) in the above display is the empty set, and so \( P_i(\omega) \subseteq A \).

In the remainder of this section we will focus on the case of finite \( \Omega \). In this case the measures \( \{\mu_i\} \) will not play a role, and so we will specify belief spaces without them.

Given an event \( A \in \Sigma \), let \( K_i A \) be the set of states of the world in which \( i \) knows \( A \):
\[
K_i A = \{\omega : P_i(\omega) \subseteq A\}.
\]

Theorem 3.6 (Kripke’s S5 system). (1) \( K_i \Omega = \Omega \). A player knows that some state of the world has occurred.

(2) \( K_i A \cap K_i B = K_i (A \cap B) \). A player knows \( A \) and a player knows \( B \) if and only if she knows \( A \) and \( B \).
(3) Axiom of knowledge. \( K_iA \subseteq A \). If a player knows \( A \) then \( A \) has indeed occurred.

(4) Axiom of positive introspection. \( K_iK_iA = K_iA \). If a player knows \( A \) then she knows that she knows \( A \).

(5) Axiom of negative introspection. \( (K_iA)^c = K_i(K_iA)^c \). If a player does not know \( A \) then she knows that she does not know \( A \).

Proof. (1) This follows immediately from the definition.

(2) 
\[
K_iA \cap K_iB = \{ \omega : P_i(\omega) \subseteq A \} \cap \{ \omega : P_i(\omega) \subseteq B \} = \{ \omega : P_i(\omega) \subseteq A, P_i(\omega) \subseteq B \} = \{ \omega : P_i(\omega) \subseteq A \cap B \} = K_i(A \cap B).
\]

(3) By definition, if \( \omega \in K_iA \) then \( P_i(\omega) \subseteq A \). Since \( \omega \in P_i(\omega) \), it follows that \( \omega \in A \). Hence \( K_iA \subseteq A \).

(4) By the previous part we have that \( K_iK_iA \subseteq K_iA \).

To see the other direction, let \( \omega \in K_iA \), so that \( P_i(\omega) \subseteq A \). Choose any \( \theta' \in P_i(\omega) \). Hence \( P_i(\theta') = P_i(\omega) \), and it follows that \( \theta \in K_iA \). Since \( \theta \) was an arbitrary element of \( P_i(\omega) \), we have shown that \( P_i(\omega) \subseteq K_iA \). Hence, by definition, \( \omega \in K_iK_iA \).

(5) The proof is similar to that of the previous part.

Interestingly, if a map \( L : 2^\Omega \to 2^\Omega \) satisfies the Kripke S5 system axioms, then it is the knowledge operator for some type: there exists a type space \( T \) and a function \( t : \Omega \to 2^\Omega \) such that \( L \) is equal to the associated knowledge operator given by

\[
KA = \{ \omega : P(\omega) \subseteq A \},
\]

where \( P(\omega) = t^{-1}(t(\omega)) \).

3.2.1. The hats riddle. Consider \( n \) players, each of which is wearing a hat that is either red (r) or blue (b). The players each observe the others’ hats, but do not observe their own.

An outside observer announces in the presence of all the players that “At least one of you has a red hat.” They now play the following (non-strategic) game: a clock is set to ring every minute. At each ring, anyone who knows the color of their hat announces it, in which case the game ends. Otherwise the game continues. We will now analyze what transpires, after formally defining the belief space.
Since the players all know that at least one hat is red, we can take the space of states of the world to be \( \Omega = \{ r, b \}^n \setminus \{(b, \ldots, b)\} \), equipped with the maximal sigma-algebra \( \Sigma = 2^\Omega \).

Initially, each player observes the others’ colors, and so \( t_i(\omega_1, \ldots, \omega_n) = \omega_{-i} \in \{ r, b \}^{n-1} \). Accordingly,

\[
P_i(\omega) = \{(\omega_{-i}, r), (\omega_{-i}, b)\},
\]

unless \( \omega_{-i} = (b, \ldots, b) \), in which case

\[
P_i(\omega) = \{(\omega_{-i}, r)\} = \{\omega\},
\]

since \( (b, \ldots, b) \notin \Omega \).

Denote by \( \omega^k \) the state of the world in which players 1 through \( k \) have a red hat, and the rest have a blue hat.

Assume that the actual state of the world is \( \omega^k \) for some \( k \geq 1 \) (this is without loss of generality, by renaming the players), and consider the situation from the point of view of player \( k \).

If \( k = 1 \) then initially player 1 already knows her color, since \( \omega_{-1} = (b, \ldots, b) \), and so \( P_1(\omega) = \{\omega\} \). No other player knows her color, and hence at time period \( t = 1 \) player 1 alone will announce it, and the game will end.

If \( k > 1 \), player 1’s information partition is \( P_k(\omega^k) = \{\omega^{k-1}, \omega^k\} \). Hence she knows that there are either \( k - 1 \) or \( k \) red hats, depending on whether or not her hat is red. Denote by \( R^{\geq k} \) the event that there are at least \( k \) red hats. Then \( \omega^k \in K_k R^{\geq k-1} \) (and of course by symmetry \( \omega^k \in K_i R^{\geq k-1} \) for all \( i \leq k \)). That is, player \( k \) knows that there are at least \( k - 1 \) red hats. Note that \( \omega^k \notin K_k R^{\geq k} \); that is, player \( k \) does not know that there are at least \( k \) red hats.

We now make the following crucial observation.

**Claim 3.7.** \( \omega^k \notin K_k K_{k-1} R^{\geq k-1} \).

That is, player \( k \) does not know that player \( k - 1 \) also knows that there are at least \( k - 1 \) red hats.

**Proof.** Note that \( \omega^{k-1} \in P_k(\omega^k) \), but that \( \omega^{k-1} \notin K_{k-1} R^{\geq k-1} \), since player \( k - 1 \) does not know that there are at least \( k - 1 \) red hats when the state of the world is \( \omega^{k-1} \). Hence \( P_k(\omega^k) \notin K_{k-1} R^{\geq k-1} \), and so indeed \( \omega^k \notin K_k K_{k-1} R^{\geq k-1} \).

For example, if \( k = 100 \), then player \( k \) knows that there are at least 99 red hats, but she does not know that player \( k - 1 \) knows that.

The next claim generalized the previous one.

**Claim 3.8.** For all \( 1 < k \leq n \) and \( i < k \) it holds that

\[
\omega^k \notin K_k K_{k-1} \cdots K_{k-i} R^{\geq k-i}.
\]
The proof of this claim is essentially a repetition of the argument above.

Proof. We prove by induction on $i$. For $i = 0$ and any $1 < k \leq n$ the claim can be restated to say that

$$\omega^k \not\in K_k R^{\geq k}.$$ 

This is immediate, by (3.2): intuitively, player 1 observes only $k - 1$ red hats and so does not know that there are at least $k$ red hats.

For general $k$ and $i$, assume that we have proved the claim for $i' = i - 1$ and $k' = k - 1$. Then by the inductive assumption

$$\omega^{k-1} \not\in K_{k-1} \cdots K_{k-i} R^{\geq k-i}.$$ 

But $\omega^{k-1} \in P_k(\omega^k)$, and so

$$P_k(\omega^k) \not\subseteq K_{k-1} \cdots K_{k-i} R^{\geq k-i}.$$ 

Thus

$$\omega^k \not\in K_{k-1} \cdots K_{k-i} R^{\geq k-i}.$$ 

It follows that in particular, for $k > 1$,

$$\omega^k \not\in K_k K_{k-1} \cdots K_2 R^{\geq 2}.$$ 

That is, player $k$ does not know that player $k - 1$ knows that player $k - 2$ knows... that player 2 knows that there are at least 2 red hats. This holds even when $k$ is (say) 100, and so everyone knows that there are either 99 or 100 red hats!

We will explain why, when the state is $\omega^k$, then at time $k$ the first $k$ players will announce their hat color.

The case $k = 1$ is explained above: player 1 will announce her color in the first time period. If player 1 does not announce her color at time 1 then all players observe this, and thus they now know that $k \geq 2$. Moreover, and most importantly, they know that everyone knows this, and know that everyone knows that everyone knows this, etc. Hence Claim 3.8 no longer holds for the new belief space.

If $k = 2$ then players 1 and 2 now announce that they have red hats, since they each only see one other red hat, and since all know that there are 2 red hats.

If $k = 3$ then players 1 and 2 do not announce in the second time period, and so all learn that $k \geq 3$. By applying the same argument inductively it is possible to show that in all time period $t < k$ all the players know that all the players know... that $k \geq t + 1$. Thus, indeed, when the state is $\omega^k$, then at time $k$ the first $k$ players will know that...
there are at least \( k \) red hats, induce that they have a red hat, and announce their color.

### 3.2.2. Knowledge in terms of sigma-algebras

A useful way to formulate knowledge is in terms of sigma-algebras. Note this is not the usual choice in this literature.

Let \((N, (\Omega, \Sigma), \{T_i\}_{i \in N}, \{t_i\}_{i \in N})\) be a finite belief space. Denote by \( \Sigma_i \subseteq \Sigma \) the sigma-algebra generated by \( i \)'s information partition \( \{P_i(\omega)\}_{\omega \in \Omega} \). That is, \( \Sigma_i \) is the collection of all sets that are unions of sets of the form \( P_i(\omega) \), for some \( \omega \in \Omega \).

**Proposition 3.9.** Let \( A \in \Sigma \). Then \( K_i A \in \Sigma_i \), and in particular

\[
K_i A = \cup \{ S \in \Sigma_i : S \subseteq A \}.
\]

Since any finite sigma-algebra is closed to unions, it follows that \( K_i A \) is the largest element of \( \Sigma_i \) that is contained in \( A \). The proof of this claim follows immediately from the definition of \( K_i \). It is important to note that we can take this to be a definition of \( K_i \). The advantage of this definition is that it is entirely in terms of our newly defined sigma-algebras \( \{\Sigma_i\} \). Also, Theorem 3.6 is (more) immediate, using this definition.

We will occasionally define belief spaces using these sigma-algebras only. That is, a belief space will be given by a tuple \((N, (\Omega, \Sigma), \{\Sigma_i\}_{i \in N})\).

**Exercise 3.10.** Given a finite belief space \( B = (N, (\Omega, \Sigma), \{\Sigma_i\}_{i \in N}) \) find a belief space \( B' = (N, (\Omega, \Sigma), \{T_i\}_{i \in N}, \{t_i\}_{i \in N}) \) such that each \( \Sigma_i \) is the sigma-algebra generated by player \( i \)'s information partition in \( B' \).

### 3.2.3. Common knowledge

Let \((N, (\Omega, \Sigma), \{\mu_i\}_{i \in N}, \{T_i\}_{i \in N}, \{t_i\}_{i \in N})\) be a finite belief space. An event \( A \in \Sigma \) is said to be common knowledge at \( \omega \in \Omega \) if for any sequence \( i_1, i_2, \ldots, i_k \in N \) it holds that

\[
\omega \in K_{i_1} K_{i_2} \cdots K_{i_k} A.
\]

We will give two characterizations of common knowledge events. In order to introduce these we will need a few definitions.

Let \( \Sigma, \Pi \) be two sub-sigma-algebras of some sigma-algebra. We say that \( \Sigma \) is a *refinement* of \( \Pi \) if \( \Pi \subseteq \Sigma \). In this case we say that \( \Pi \) is a *coarsening* of \( \Sigma \).

The *meet* of two sigma-algebras is \( \Sigma_1, \Sigma_2 \subseteq \Sigma \) is the finest sub-sigma-algebra of \( \Sigma \) that is a coarsening of each \( \Sigma_i \). Their *join* is the coarsest sub-sigma-algebra of \( \Sigma \) that is a refinement of each \( \Sigma_i \).

---

\(^4\)We do not specify the measures \( \{\mu_i\} \), which will again not play a role here.
Exercise 3.11. Show that the meet of $\Sigma_1$ and $\Sigma_2$ is their intersection $\Sigma_1 \cap \Sigma_2$. Show that their join is the sigma-algebra generated by $\{S_1 \cap S_2 : S_1 \in \Sigma_1, S_2 \in \Sigma_2\}$.

Given a belief space $(N, (\Omega, \Sigma), \{\Sigma_i\}_{i \in N})$, let $\Sigma_C = \cap_i \Sigma_i$ be the meet of the player's sigma-algebras.

Claim 3.12. If $C \in \Sigma_C$ then $K_i C = C$ for all $i \in N$.

Proof. This is an immediate consequence of the sigma-algebraic definition of $K_i$, and the fact that any $C \in \Sigma_C$ is also in $\Sigma_i$. $\square$

We will now define an undirected graph $G$, whose vertices are the elements of $\Omega$, and where there is an edge between $\omega, \omega' \in \Omega$ if there exists a player $i$ such that $P_i(\omega) = P_i(\omega')$. Let $C(\omega)$ be the connected component of $\omega$; that is the maximal set of vertices in the graph for which there exists a path from $\omega$. Here, a path from $\omega$ to $\omega'$ is a sequence of edges $(\omega_1, \omega_2), (\omega_2, \omega_3), (\omega_3, \omega_4), \ldots, (\omega_{k-1}, \omega_k)$ such that $\omega_1 = \omega$ and $\omega_k = \omega'$.

Theorem 3.13. Given a finite belief space $(N, (\Omega, \Sigma), \{\Sigma_i\}_{i \in N})$, a state of the world $\omega \in \Omega$ and an event $A \in \Sigma$, the following are equivalent.

1. $A$ is common knowledge at $\omega$.
2. $C(\omega) \subseteq A$.
3. There exists a $C \in \Sigma_C$ such that $\omega \in C$ and $C \subseteq A$.

Proof. We first show that (1) implies (2). Choose an arbitrary $\omega' \in C(\omega)$; we will show that $\omega' \in A$. Since $\omega' \in C(\omega)$, there exists a path $(\omega_1, \omega_2), (\omega_2, \omega_3), \ldots, (\omega_{k-1}, \omega_k)$ such that $\omega_1 = \omega$ and $\omega_k = \omega'$. There exists therefore a sequence of players $i_1, \ldots, i_{k-1}$ such that $P_{i_{j}}(\omega_{j}) = P_{i_{j}}(\omega_{j+1})$.

Now, by the common knowledge assumption

$\omega = \omega_1 \in K_{i_1}K_{i_2} \cdots K_{i_{k-1}}A$.

Hence

$P_{i_1}(\omega_1) \in K_{i_2} \cdots K_{i_{k-1}}A$

and in particular

$\omega_2 \in K_{i_2} \cdots K_{i_{k-1}}A$,

since $\omega_2 \in P_{i_1}(\omega_1)$. Applying this argument inductively yields $\omega' = \omega_k \in A$, and thus we have shown that $C(\omega) \subseteq A$.

To show that (2) implies (3) we show that $C(\omega) \in \Sigma_C$. To this end it suffices to show that $C(\omega) \in \Sigma_i$ for all $i \in N$. Let $\omega' \in C(\omega)$. Then, by the definition of the graph $G$, $P_i(\omega') \in C(\omega)$. Hence $C(\omega)$ is a union of sets of the form $P_i(\omega')$, and thus it is an element of $\Sigma_i$. 


Finally, we show that (3) implies (1). Choose $C \in \Sigma_C$ such that $\omega \in C$ and $C \subseteq A$. Let $i_1, i_2, \ldots, i_k$ be an arbitrary sequence of players. We first prove that
$$\omega \in K_{i_1} K_{i_2} \cdots K_{i_k} C.$$ 

By Claim 3.12
$$K_{i_1} K_{i_2} \cdots K_{i_k} C = C,$$
and since $\omega \in C$ we have shown that
$$\omega \in K_{i_1} K_{i_2} \cdots K_{i_k} C.$$

Finally, since $C \subseteq A$,
$$K_{i_1} K_{i_2} \cdots K_{i_k} C \subseteq K_{i_1} K_{i_2} \cdots K_{i_k} A$$
and thus we also have that
$$\omega \in K_{i_1} K_{i_2} \cdots K_{i_k} A.$$  

$\square$

3.3. Agreeing to disagree. Let $(N, (\Omega, \Sigma), \{\mu_i\}_{i \in N}, \{\Sigma_i\}_{i \in N})$ be a finite belief space with common priors, so that $\mu_i = \mu$ for all $i$. Given a random variable $X : \Omega \rightarrow \mathbb{R}$, we denote by $\mathbb{E}[X]$ the expectation of $X$ according to $\mu$. The expectation of $X$ conditioned on player $i$’s information at $\omega$ is
$$\mathbb{E}[X|\Sigma_i] (\omega) = \mathbb{E}[X|P_i(\omega)] = \frac{\sum_{\omega' \in P_i(\omega)} \mu(\omega') X(\omega')}{\sum_{\omega' \in P_i(\omega)} \mu(\omega')}.$$ 

Aumann [1] proved the following theorem.

Theorem 3.14 (Aumann’s Agreement Theorem, 1976). Let $X : \Omega \rightarrow \mathbb{R}$ be a random variable. Suppose that at $\omega_0$ it is common knowledge that $\mathbb{E}[X|\Sigma_i] (\omega_0) = q_i$ for $i = 1, \ldots, n$ and some $(q_1, q_2, \ldots, q_n) \in \mathbb{R}^n$. Then $q_1 = q_2 = \cdots = q_n$.

Note that we implicitly assume that the conditional expectations $\mathbb{E}[X|\Sigma_i] (\omega_0)$ are well defined. Before proving the Theorem we will recall the law of total expectation. Let $S$ be an element of a sigma-algebra $\Pi \subseteq 2^\Omega$, and let $X : \Omega \rightarrow \mathbb{R}$ be a random variable. Then
$$\mathbb{E}[X|S] = \mathbb{E}[\mathbb{E}[X|\Pi]|S].$$

Exercise 3.15. Prove the law of total expectation for a finite probability space. Hint: write $C$ as a disjoint union of elements of the partition that generates $\Pi$: $C = \bigcup_j P_j$, with $P_j \in \Pi$ begin elements with no proper, non-empty subsets in $\Pi$. 

Proof of Theorem 3.14. By the common knowledge hypothesis there is for each player $i$ a $C_i \in \Sigma$ with $\omega_0 \in C_i$ and such that $\mathbb{E}[X|\Sigma_i](\omega) = q_i$ for all $\omega \in C_i$. Let $C = \cap_i C_i$, and note that $\omega_0 \in C_i$. We prove the claim by showing that $q_i = \mathbb{E}[X|C]$ for all $i$.

Fix $i$. Since $C \in \Sigma_i$, by the law of total expectation

$$\mathbb{E}[X|C] = \mathbb{E}[\mathbb{E}[X|\Sigma_i]|C].$$

But $\mathbb{E}[X|\Sigma_i](\omega) = q_i$ for all $\omega \in C$, and thus

$$\mathbb{E}[X|C] = \mathbb{E}[q_i|C] = q_i.$$

In his paper, Aumann stated this theorem for the case that $X$ is the indicator of some event:

Corollary 3.16. If two players have common priors over a finite space, and it is common knowledge that their posteriors for some event are $q_1$ and $q_2$, then $q_1 = q_2$.

3.3.1. No trade. Milgrom and Stokey [15] apply Aumann’s theorem to show that rational agents with common priors can never agree to trade. Here we give a theorem that is less general than their original.

Consider two economic agents. The first one has an indivisible good that she might be interested to sell to the second. This good can be sold tomorrow at an auction for an unknown price that ranges between $0$ and $1,000$, in integer increments. Let $\Omega = \Theta \times P$, and let the common prior be some $\mu$. Here $P = \{0, 1, \ldots, 1000\}$ represents the auction price of the good, and $\Theta$ is some finite set that describes many possible events that may influence the price. Accordingly, $\mu$ is not a product measure, so that conditioning on different $\theta \in \Theta$ yields different conditional distributions on $P$. We assume that the players’ types are a function of the first coordinate only: $t_i(\theta, k) = t_i(\theta)$. Let $p: \Omega \to P$ be the auction price $p(\theta, k) = k$.

We denote by $U_1$ player 1’s conditional expected utility from trading for a price $q$:

$$U_1^q = \mathbb{E}[q - p|t_1] = q - \mathbb{E}[p|t_1].$$

Analogously,

$$U_2^q = \mathbb{E}[p - q|t_2] = \mathbb{E}[p|t_2] - q.$$

For example, let $\Theta = \Theta_1 \times \Theta_2$ with $\Theta_1 = \Theta_2 = P$, let $t_1(\theta_1, \theta_2, k) = \theta_1$, $t_2(\theta_1, \theta_2, k) = \theta_2$, and $\mu$ be the uniform distribution over $\{(\theta_1, \theta_2, k) :$
Then
\[ U_1^q = q - \frac{1}{2} (500 + t_1) \]
and
\[ U_2^q = \frac{1}{2} (500 + t_2) - q. \]
Thus in any state in which \( t_1 < t_2 \) we have, for \( q = \frac{1}{2}(t_1 + t_2) \), that
\[ U_1^q > 0 \text{ and } U_2^q > 0: \]
both players expect a positive return for trading the good for \( q \) dollars. However, note that the players do not know that the other player also has positive expectation. What if they knew that the other person is willing to trade?

**Theorem 3.17** (Milgrom and Stokey, 1982). If at some \( \omega \in \Omega \) it is common knowledge that \( U_1^q(\omega) \geq 0 \) and \( U_2^q(\omega) \geq 0 \) then \( U_1^q(\omega) = U_2^q(\omega) = 0 \).

**Exercise 3.18.** Prove Theorem 3.17.

**Exercise 3.19.** Construct an example in which \( U_1^q(\omega) > 0 \) and \( U_2^q(\omega) > 0 \), player 1 knows that \( U_2^q(\omega) > 0 \) and player 2 knows that \( U_1^q(\omega) > 0 \).

3.4. Reaching common knowledge. We suggest that the reader becomes familiar with the content of Appendix A before reading this section.

Geanakoplos and Polemarchakis [8] show that repeatedly communicating posteriors leads agents to convergence to a common posterior, which is then common knowledge. We state this theorem in somewhat greater generality than in the original paper, requiring a more abstract mathematical formulation.

**Theorem 3.20** (Geanakoplos and Polemarchakis, 1982). Let \((\Omega, \Sigma, \mathbb{P})\) be a probability space, and fix \( A \in \Sigma \). Let \( X_1 \) and \( X_2 \) be two random variables on this space. Denote \( P_1^0 = \mathbb{P}[A|X_1] \) and \( P_2^0 = \mathbb{P}[A|X_2] \). For \( t \in \{1, 2, \ldots \} \) let
\[ P_1^t = \mathbb{P}[A|X_1, P_1^0, P_2^1, \ldots, P_2^{t-1}] \]
and
\[ P_2^t = \mathbb{P}[A|X_2, P_2^0, P_1^1, \ldots, P_1^{t-1}] \].
Then \( \lim_{t \to \infty} P_1^t \) and \( \lim_{t \to \infty} P_2^t \) almost surely exist and are equal.

**Proof.** Note that by Lévy’s zero-one law (Theorem A.7)
\[ P_1^\infty := \lim_{t \to \infty} P_1^t = \mathbb{P}[A|X_1, P_2^0, P_2^1, \ldots] \]
and an analogous statement holds for \( \lim_{t} P_{2}^{t} \). Let \( \Sigma_{1} \) be the sigma-algebra generated by \( \{X_{1}, P_{2}^{0}, P_{2}^{1}, \ldots\} \), so that
\[
P_{1}^{\infty} = \mathbb{P}[A|\Sigma_{1}].
\]
Since \( P_{2}^{\infty} \) is \( \Sigma_{1} \)-measurable, it follows that
\[
P_{1}^{\infty} = \mathbb{P}[A|\Sigma_{1}] = \mathbb{P}[\mathbb{P}[A|P_{2}^{\infty}]|\Sigma_{1}] = \mathbb{P}[P_{2}^{\infty}|\Sigma_{1}] = P_{2}^{\infty}.
\]
\[\blacksquare\]

3.5. **Bayesian games.** The buyer-seller game described in Section 3.1.1 is an example of a Bayesian game. In these games there is uncertainty over the payoffs to the players in given pure strategy profiles.

A Bayesian game is a tuple \( G = (N, \{A_{i}\}_{i \in N}, (\Omega, \Sigma), \{\mu_{i}\}_{i \in N}, \{T_{i}\}_{i \in N}, \{t_{i}\}_{i \in N}, \{u_{i}\}_{i \in N}) \) where

- \( N \) is the set of players.
- \( A_{i} \) is the set of actions of player \( i \).
- \( (\Omega, \Sigma) \) is a set with a sigma-algebra. This set is called the set of states of the world.
- \( \mu_{i} \) is player \( i \)'s belief over \( (\Omega, \Sigma) \).
- \( t_{i} : \Omega \to T_{i} \) is player \( i \)'s private signal or type, with \( T_{i} \) the space of types of player 1.
- \( u_{i} : A \times \Omega \to \mathbb{R} \) is player \( i \)'s utility function.

We naturally require that \( t_{i} \) and \( u_{i} \) be measurable. We denote by \( \mu_{i}(E|t_{i}) \) the probability of the event \( E \in \Sigma \), conditioned on \( t_{i} \).

The set of pure strategies of player \( i \) is the set of measurable functions from \( T_{i} \) to \( A_{i} \). That is, a strategy of a player is a choice of action, given her private signal realization. Given a strategy profile \( (s_{1}, \ldots, s_{n}) \), player \( i \)'s expected utility is
\[
\mathbb{E}_{\mu_{i},s}[u_{i}] = \int_{\Omega} u_{i}(s_{1}(t_{1}(\omega)), \ldots, s_{n}(t_{n}(\omega)), \omega) d\mu_{i}(\omega).
\]

A Bayes-Nash equilibrium is a strategy profile in which no player can improve her expected utility by changing her strategy. That is, for any player \( i \) and strategy \( s_{i}' \) it holds that
\[
\mathbb{E}_{\mu_{i},s}[u_{i}] \geq \mathbb{E}_{\mu_{i},(s_{-i},s_{i}')}[u_{i}]. \tag{3.3}
\]

An alternative definition of a Bayes-Nash equilibrium is a strategy profile in which, for each player \( i \) and each type \( \tau_{i} \in T_{i} \) it holds that
\[
\mathbb{E}_{\mu_{i},s}[u_{i}|t_{i} = \tau_{i}] = \int_{\Omega} u_{i}(s_{1}(t_{1}(\omega)), \ldots, s_{i}(\tau_{i}), \ldots, s_{n}(t_{n}(\omega)), \omega) d\mu_{i}(\omega|t_{i} = \tau_{i}) \tag{3.4}
\]
cannot be improved:
\[ E_{\mu_i,s} [u_i | t_i = \tau_i] \geq E_{\mu_i,(s_{-i},s'_i)} [u_i | t_i = \tau_i] \]
for all \( s'_i \) and \( \tau_i \in T_i \).

This is not an equivalent definition, but the second is stronger than the first.

**Proposition 3.21.** (3.4) implies (3.3).

**Proof.** Let \( s \) satisfy (3.4). Then by the law of total expectation
\[
E_{\mu_i,s} [u_i] = E_{\mu_i,s} [E_{\mu_i,s} [u_i | t_i]] \\
\geq E_{\mu_i,s} [E_{\mu_i,(s_{-i},s'_i)} [u_i | t_i]] \\
= E_{\mu_i,(s_{-i},s'_i)} [u_i].
\]

Conversely, if \( \Omega \) is finite and there is a common prior \( \mu = \mu_1 = \cdots = \mu_n \) then, if \( s \) satisfies (3.3) then there is an \( s' \) that satisfies (3.4), and such that the probability (under \( \mu \)) that \( s_i(t_i) \neq s'_i(t_i) \) is zero.

**Exercise 3.22.** Find a finite Bayesian game that has a (pure) strategy profile that satisfies (3.3) but does not have one that satisfies (3.4).
4. Auctions

Auctions have been used throughout history to buy and sell goods. They are still today very important in many markets, including on-line markets that run massive computerized auctions.

4.1. Classical auctions. In this section we will consider \( n \) players, each of which have a fixed valuation \( v_i \) for some item that is being auctioned. We assume that each \( v_i \) is a non-negative integer. Furthermore, to avoid having to deal with tie-breaking, we assume that each \( v_i = i + 1 \mod n \). We also assume without loss of generality that \( v_1 > v_2 > \cdots > v_n \).

If it is agreed that player \( i \) buys the item for some price \( p \) then that player’s utility is \( v_i - p \). If a player does not buy then she pays nothing and her utility is zero.

We will consider a number of possible auctions.

4.1.1. First price, sealed bid auction. In this auction each player submits a bid \( b_i \), which has to be a non-negative integer, congruent to \( i \mod n \). Note that this means that a player cannot ever bid her valuation (which is congruent to \( i + 1 \mod n \)), but can bid one less than her valuation.

For example, consider the case that \( n = 2 \). Then possible valuations are \( v_1 = 10 \) and \( v_2 = 5 \), and \( b_1 \) must be odd while \( b_2 \) must be even.

The bids \( b = (b_1, \ldots, b_n) \) are submitted simultaneously. The player who submitted the highest bid \( b_{\text{max}}(b) = \max_i b_i \) buys the item, paying \( b_{\text{max}} \).

Hence player \( i \)'s utility for strategy profile \( b \) is given by

\[
 u_i(b) = \begin{cases} 
 v_i - b_i & \text{if } b_i = b_{\text{max}}(b) \\ 
 0 & \text{otherwise} 
\end{cases}
\]

We now analyze this game. We first note that \( b_i = v_i - 1 \) guarantees utility at least 0. Next, we note that any \( b_i > v_i \) is weakly dominated by \( b_i = v_i - 1 \), since it guarantees utility at most 0, but can result in negative utility if \( b_i = b_{\text{max}} \). Furthermore, it is impossible that in a pure equilibrium the winner of the auction bid more than \( v_i \), since then she could increase her utility by lowering her bid to \( v_i - 1 \).

Assume that \( b^* \) is an equilibrium.

**Claim 4.1.** Player 1 wins the auction: \( b_1^* = b_{\text{max}}^* \).

\(^5\)Think of this as one cent less.
Proof. Assume by contradiction that player $i > 1$ wins the auction. As we noted above, $b_i^* \leq v_i - 1$. Hence $b_{\text{max}}^* = b_i^* \leq v_i - 1 < v_1 - 1$. Hence player 1 could improve her utility to 1 by bidding $v_1 - 1$ and winning the auction. □

We have thus shown that in any equilibrium the first player wins. It thus remains to show that one exists.

Claim 4.2. Let $b_1^*$ be the smallest allowed bid that is larger than $v_2 - 1$. Let $b_2 = v_2 - 1$. For $i > 2$ (if there are more than 2 players) let $b_i^*$ be any allowed bid that is less than $v_2$. Then $b^*$ is an equilibrium.

Exercise 4.3. Prove Claim 4.2.

We note a few facts about this equilibrium.

- The item was allocated to the player who values it the most.
- The player who won did not base her bid on her own valuation, but on the other players’ valuations, and in particular on the second highest one.

Note that other equilibria exist. For example, if $n = 2$ and $v_1 = 10$ and $v_2 = 5$ then $b_1 = 9$ and $b_2 = 8$ is again an equilibrium. Player 2 gets zero payoff, but can only decrease her utility by raising her price and winning the auction. Player 1 gets positive utility (1), but cannot improve it by lowering her bid.

4.1.2. Second price, sealed bid auction. In this auction each player again submits a bid $b_i$, which this time has to be a non-negative integer, congruent to $i + 1 \mod n$; that is, it can be equal to $v_i$. Again, the player who submitted the highest bid $b_{\text{max}}$ wins. However, in this case she does not pay her bid, but rather the second highest bid $b_{\text{nd}}$. Hence

$$u_i(b) = \begin{cases} v_i - b_{\text{nd}} & \text{if } b_i = b_{\text{max}}(b) \\ 0 & \text{otherwise} \end{cases}$$

As in the first price auction, any $b_i > v_i$ is weakly dominated by $b_i = v_i$; when bidding more than $v_i$ the player can at most make 0, but in some cases may have negative utility, which is not possible when bidding $v_i$.

Moreover, in this auction any $b_i < v_i$ is also weakly dominated by $b_i = v_i$. To see this, let $b'$ be the highest bid of the rest of the players. If $b' > v_i$ then in either bid the player loses the auction, and so both strategies yield zero. If $b' < v_i$ then bidding $b_i < v_i$ may either cause

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6Congruent to $1 \mod n$. 
the loss of the auction and utility zero (if \( b_i < b' \)) or otherwise gaining \( v_i - b' \). But bidding \( b_i = v_i \) guarantees utility \( v_i - b' \).

Hence \( b_i = v_i \) is a weakly dominant strategy, and so this is an equilibrium. Auctions in which bidding your valuation is weakly dominant are called truthful.

Note that in this equilibrium the item is allocated to the player who values it the most, as in the first price auction. However, the player based her bid on her own valuation, independently of the other.

4.1.3. **English auction.** This auction is an extensive form game with complete information. The players take turns, starting with player 1, then player 2 and so on up to player \( n \), and then player 1 again etc. Each player can, at her turn, either leave the auction or stay in. Once a player has left she must choose to leave in all the subsequent turns.

The auction ends when all players but one have left the auction. If this happens at round \( t \) then the player left wins the auction and pays \( t - 1 \).

**Claim 4.4.** There is a subgame perfect equilibrium of this game in which each player \( i \) stays until period \( t = v_i \) and leaves once \( t > v_i \).

**Exercise 4.5.** Prove Claim 4.4.

**Exercise 4.6.** What is the relation between this English auction and the second price auction?

4.1.4. **Social welfare.** Imagine that the person running the auction is also a player in the game. Her utility is simply the payment she receives; she has no utility for the auctioned object. Then the social welfare, which we will for now define to be the sum of all the players’ utilities, is equal to the utility of the winner — her value minus her payment — plus the utility of the losers (which is zero), plus the utility of the auctioneer, which is equal to the payment. This sum is the value of the object to the winner. Hence social welfare is maximized when the winner is a person who values the object most.

4.2. **Bayesian auctions.** In this section we will consider auctions in which the players do not know the others’ valuations exactly. Specifically, the auctions will be Bayesian games with common priors.

We will again have \( n \) players. Each player’s type will be her valuation \( v_i \), and the players will have some common prior \( \mathbb{P} \) over \( (v_1, \ldots, v_n) \). Formally, the belief space will be \( ((\mathbb{R}^+)^n, \Sigma, \mathbb{P}) \), where \( \Sigma \) is the Borel sigma-algebra, and \( \mathbb{P} \) is some probability distribution. Player \( i \)’s type \( t_i \) is given by \( t_i(v_1, \ldots, v_n) = v_i \).
As before, if a player does not win the auction she has utility zero. Otherwise, assuming she pays a price $p$, she has utility $v - p$. Note that the players’ utilities indeed depend on their types in these Bayesian games.

4.2.1. **Second price, sealed bid auction.** As before, the players will submit bids $b_i$. In this case we do not restrict the bids, and can allow them to take any value in $\mathbb{R}$. Formally, a pure strategy of a player in this game is a measurable function $b_i: \mathbb{R}^+ \rightarrow \mathbb{R}$, assigning a bid to each possible type or valuation.

As before, the player with the highest bid $b_{\text{max}}$ wins and pays the second highest bid $b_{\text{nd}}$. Note that despite the fact that two valuations can be never be the same, it still may be the case that two players choose the same bid. For example, the strategy profile could be such that all players always bid 1. Accordingly, we assume that there is some tie-breaking mechanism (e.g., choose at random from all those with the highest bid), but it will not play a role in our analysis.

**Proposition 4.7.** For any joint distribution $\mathbb{P}$, it is weakly dominant for each player to choose $b^*_i(v_i) = v_i$.

The proof of this is identical to the one in the non-Bayesian case.

As an example, consider the case that the valuations are picked i.i.d. from some non-atomic distribution with cumulative distribution function $F$. Then $b^*_i(v_i) = v_i$ is the unique Bayes-Nash equilibrium.

Assume that there are two players. Player 1 wins the auction if she has the highest valuation. Conditioning on her valuations $v_1$, her probability of winning is therefore $F(v_1)$. If she wins then she expects to pay $\mathbb{E}_F[v_2|v_2 < v_1]$.

4.2.2. **First price, sealed bid auction.** In this auction, as in the classical one, each player will submit a bid $b_i$, and the player with the highest bid $b_{\text{max}}$ will win and pay $b_{\text{max}}$. We assume here that the valuations are picked i.i.d. from some non-atomic distribution with cumulative distribution function $F$ with derivative $f$. To simplify our calculations we will assume that there are only two players; the general case is almost identical.

**Claim 4.8.** $b^*_i(v_i) = v_i$ is not an equilibrium of this auction.

We will try to construct a Bayes-Nash equilibrium with the following properties:

- Symmetry: there is a function $b: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $b_i(v_i) = b(v_i)$ for all players.
- Monotony and differentiability: $b$ is monotone increasing.
Thus, to construct such an equilibrium we assume all players play $b$, and try to calculate $b$ assuming it is some player’s (say player 1’s) best response.

Assume then that players 2 plays $b_2(v_2) = b(v_2)$. Fix $v_1$, and assume that player 1 bids $b_1$. Denote by $G$ the cumulative distribution function of $b(v_2)$, and let $g$ be its derivative. Note that we can write $G$ and $g$ in terms of $F$ and $f$:

\[ F(v) = G(b(v)) \]

and

\[ f(v) = g(b(v)) \cdot b'(v). \]

The probability that $b_1$ is the highest bid is $G(b_1)$. It follows that player 1’s expected utility (conditioned on $v_1$) is

\[ u_1(v_1, b_1) = G(b_1) \cdot (v_1 - b_1). \]

Therefore, to maximize expected utility, $b_1$ must satisfy

\[ 0 = \frac{du_1(v_1, b_1)}{db_1} = g(b_1) \cdot (v_1 - b_1) - G(b_1), \]

or

\[ G(b_1) = g(b_1) \cdot (v_1 - b_1). \]

Note that $b_1 = v_1$ is a solution only if $G(b_1) = 0$.

Since we are looking for a symmetric equilibrium, we can now plug in $b_1 = b(v)$ to arrive at the condition

\[ G(b(v)) = g(b(v)) \cdot (v - b(v)). \]

Translating back to $F$ and $f$ using (4.1) and (4.2) yields

\[ F(v) = \frac{f(v)}{b'(v)} \cdot (v - b(v)). \]

Rearranging, we can write

\[ F(v)b'(v) + f(v) \cdot b(v) = f(v) \cdot v \]

or

\[ \frac{d}{dv} [F(v)b(v)] = f(v) \cdot v. \]

Now, clearly $b(0) = 0$ is weakly dominant, and so we will assume that this is indeed the case. We can therefore solve the above expression to arrive at

\[ b(v) = \frac{1}{F(v)} \int_0^v f(u) \cdot u \, du. \]
Note that this is equal to $E_F[v_2|v_2 < v_1]$, the expectation of $v_2$, conditioned on $v_2$ being less than $v$. It remains to be shown that this strategy profile is indeed a maximum (we only checked the first order condition).

A player’s expected utility, conditioned on having valuation $v$, is simply $F(v)$ (the probability that she wins) times $v - b(v)$. Interestingly, in the second price auction the expected utility is identical: the probability of winning is still $F(v)$, and the expected utility is $v - b(v)$, since, conditioned on winning, the expected payment is the expected valuation of the other player.

4.3. **Truthful mechanisms and the revelation principle.** The revelation principle is important in mechanism design. The basic idea is due to Gibbard [9], with generalizations by others [5,12,16].

In this section we will call Bayesian games of incomplete information mechanisms. We will say that a mechanism is truthful if $A_i = T_i$ and $\tilde{s}(t_i) = t_i$ is an equilibrium. Note that sometimes this term is used to describe mechanisms in which the same $\tilde{s}$ is weakly dominant.

**Theorem 4.9.** Let $G = (N, \{A_i\}_{i \in N}, (\Omega, \Sigma), \{\mu_i\}_{i \in N}, \{T_i\}_{i \in N}, \{t_i\}_{i \in N}, \{u_i\}_{i \in N})$ be a mechanism with an equilibrium $s^*$. Then there exists a truthful mechanism $G' = (N, \{A'_i\}_{i \in N}, (\Omega, \Sigma), \{\mu_i\}_{i \in N}, \{T_i\}_{i \in N}, \{t_i\}_{i \in N}, \{u'_i\}_{i \in N})$ such that

$$
E_{\mu_i, \tilde{s}}[u'_i|t_i] = E_{\mu_i, s^*}[u_i|t_i].
$$

That is, for every game and equilibrium one can design a truthful game in which playing truthfully yields the same conditionally expected utilities as in the original game. The idea of the proof is simple: in the new mechanism, the players reveal their types, the mechanism calculates their equilibrium actions, and then implements the original mechanism on those actions.

**Proof of Theorem 4.9.** Let

$$u'_i(\tau_1, \ldots, \tau_n, \omega) = u_i(s^*_1(\tau_1), \ldots, s^*_n(\tau_n), \omega).$$

Then

$$
E_{\mu_i, \tilde{s}}[u'_i|t_i = \tau_i] = \int_{\Omega} u'_i(\tilde{s}_1(t_1(\omega)), \ldots, \tilde{s}_n(t_n(\omega)), \omega) d\mu_i(\omega|t_i = \tau_i)
$$

$$= \int_{\Omega} u'_i(t_1(\omega), \ldots, t_n(\omega), \omega) d\mu_i(\omega|t_i)
$$

$$= \int_{\Omega} u_i(s^*_1(t_1(\omega)), \ldots, s^*_n(t_n(\omega)), \omega) d\mu_i(\omega|t_i)
$$

$$= E_{\mu_i, s^*}[u_i|t_i].$$
To see that this mechanism is truthful, note that for any player $i$ with type $\tau_i \in T_i$ and action $\tau'_i \in A'_i = T_i$ it holds that the utility for playing $\tau'_i$ (instead of $\tau_i$) is

$$\int_{\Omega} u'_i(\tilde{s}_1(t_1(\omega)), \ldots, \tau'_i, \ldots, \tilde{s}_n(t_n(\omega)), \omega) d\mu_i(\omega | t_i = \tau_i)$$

$$= \int_{\Omega} u'_i(t_1(\omega), \ldots, \tau'_i, \ldots, t_n(\omega), \omega) d\mu_i(\omega | t_i = \tau_i)$$

$$= \int_{\Omega} u_i(s^*_1(t_1(\omega)), \ldots, s^*_i(\tau'_i), \ldots, s^*_n(t_n(\omega)), \omega) d\mu_i(\omega | t_i = \tau_i) \cdot \omega.$$

But since $s^*$ is an equilibrium this is

$$\leq \int_{\Omega} u_i(s^*_1(t_1(\omega)), \ldots, s^*_i(\tau_i), \ldots, s^*_n(t_n(\omega)), \omega) d\mu_i(\omega | t_i = \tau_i)$$

$$= \int_{\Omega} u'_i(\tilde{s}_1(t_1(\omega)), \ldots, \tilde{s}_n(t_n(\omega)), \omega) d\mu_i(\omega | t_i = \tau_i),$$

which is the utility for playing according to $\tilde{s}_i$. □

It follows that when designing auctions we can assume without loss of generality that the players reveal their types to the auctioneer.
5. Repeated games

5.1. Extensive form games with perfect information and simultaneous moves. Recall (Section 1.2) that an extensive form game with perfect information is given as a tuple $G = (N, A, H, O, o, P, \{\preceq_i\}_{i \in N})$, where $N$ is the set of players, $A$ is the set of actions, $H$ is the set of allowed histories (with $Z$ the terminal histories), $O$ is the set of outcomes, $o$ is a function $Z \rightarrow O$, $P$ is a function $H \setminus Z \rightarrow N$, and for each player $i \in N$, $\preceq_i$ is a preference relation over $O$.

In this section we slightly extend this definition by allowing two or more players to make simultaneous moves. The definitions of $N$ and $A$ remain the same. $P$ will now be a function from $H \setminus Z$ to $2^N$, the power set of the set of players, so that after history $h \in H$ the set of players who play simultaneously is $P(h)$.

Recall that a history was defined to be a finite sequence in $A$; it will now be a sequence of tuples $a_t$ in $A$, with each tuple in the sequences potentially of different size. Specifically, a history $h$ is a sequence $(a_1, a_2, \ldots)$, such that, for each $t$, $a_t \in A^M$ where $M = P(a_1, \ldots, a_{t-1})$. The set of histories will have to satisfy an additional criterion: if $h = (a_1, \ldots, a_n) \in H$ then there must be for each player $i \in P(h)$ a set $A_i(h) \subset A$ such that for any $a \in \prod_i A_i$ the history $(a_1, \ldots, a_n, a)$ is also in $H$. That is, each player has a set of allowed moves, and any combination of allowed moves by the players is allowed.

A strategy in such a game is a function that assigns to each history $h \in H$ and player $i \in P(h)$ an action in $A_i(h)$.

5.1.1. Equilibria. The definitions of a Nash equilibrium, a subgame, and a subgame perfect equilibrium carry unchanged to this new setting.

Note that Kuhn’s Theorem does not apply here: for example, any strategic form game is a one-period extensive form game with perfect information and simultaneous moves. But not every such games has a pure equilibrium. On the other hand, the one deviation principle (Theorem 1.5) still applies; the proof is identical.

5.2. Definition of repeated games. Let $G_0 = (N, \{A_i\}, \{u_i\})$ be a strategic form game, and, as usual, let $A = \prod_i A_i$. We will only consider games in which $A$ is compact and each $u_i$ is continuous.

Let $T$ be the number of periods (or repetitions) of the repeated game. $T$ can be either finite or infinite. A $T$-repeated game of $G_0$ is an extensive form game $G = (N, A', H, O, o, P, \{\preceq_i\}_{i \in N})$, where

- $P(h) = N$ for all $h \in H \setminus Z$.
- The set of histories $H$ has terminal histories $Z = \prod_{t=1}^{T} A$; recall that $Z$ uniquely determines $H$ as the set of all prefixes of $Z$. 

• $A'$ is likewise determined by $H$.
• $O = Z$ and $o$ is the identity.
• $\preceq_i$ is some preference relation on $Z$ that satisfies the following condition: If $u_i(a^t) \leq u_i(b^t)$ for all $t$ and some $(a^1, a^2, \ldots), (b^1, b^2, \ldots) \in Z$ then
  \[(a^1, a^2, \ldots) \preceq_i (b^1, b^2, \ldots).\]

We call $G_0$ the base game of $G$.

If we assign utilities $\{v_i\}_{i \in N}$ to elements of $Z$, then we call $(v_1(a), \ldots, v_n(a))$ the payoff profile associated with $a$. Given a strategy profile $s$ of $G$, we likewise define the payoff profile associated with $s$ to be the payoff profile associated with the path of play generated by $s$.

When $T$ is finite then a natural choice is to let the players’ utilities in the repeated game be the sum of their base game utilities in each of the periods. When $T = \infty$ we will consider two types of preference relations: discounting and limit of means, or limiting mean. In fact, in both we will define utilities that induce the equivalence relations.

5.2.1. Discounting. We fix some $\delta \in (0, 1)$ and let $v_i : O \to \mathbb{R}$ be given by the discounted sum

$$v_i(a^1, a^2, \ldots) = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_i(a^t).$$

These will be the player’s utilities in the repeated game $G$, and $\preceq_i$ will simply be given by these utilities. Note that we chose the scaling of $v_i$ to make it a weighted average of the utilities.

Discounting has the advantage that it is stationary: every subgame of $G$ is isomorphic to $G$.

5.2.2. Limit of means, limiting mean. In the usual definition of “limit of means”, $(a^1, a^2, \ldots) \succ_i (b^1, b^2, \ldots)$ by player $i$ if

$$\liminf_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} u_i(a^t) - \frac{1}{\tau} \sum_{t=1}^{\tau} u_i(b^t) > 0.$$

Note that, under this definition, a pair $(a^1, a^2, \ldots)$ and $(b^1, b^2, \ldots)$ that differ only in finitely many periods are equivalent.

We will take a slightly different approach to limits of means, using limiting means. Let $\ell^\infty(N) \subset \mathbb{R}^N$ denote the bounded real sequences.

A mean $m$ is a map $\ell^\infty(N) \to \mathbb{R}$ with the following properties

• Linearity:
  $$m(x + y) = m(x) + m(y)$$
and
\[ m(\alpha \cdot x) = \alpha \cdot m(x). \]

- Positivity: if \( x_n \geq 0 \) for all \( n \) then \( m(x) \leq 0. \)
- Unity: \( m(1, 1, \ldots) = 1. \)

For example, if \( \sum_{n=1}^{\infty} \alpha_n = 1 \) then
\[ m(x) = \sum_{n=1}^{\infty} \alpha_n x_n \]
is a mean. However, there are also more interesting examples:

**Theorem 5.1.** There exists a mean \( m \) such that \( m(x) \) is always the limit of some subsequence of \( \{ \frac{1}{n} \sum_{i=1}^{n} x_i \} \) \( n \in \mathbb{N} \).

In particular, if a pair \( x, y \in \ell^\infty(\mathbb{N}) \) differ in only finitely many coordinates then \( m(x) = m(y) \). Note also that such a mean \( m \) is shift-invariant: \( m(x_1, x_2, x_3, \ldots) = m(x_2, x_3, \ldots) \).

It follows that \( m(x_1, x_2, \ldots) > m(y_1, y_2, \ldots) \) whenever
\[ \lim inf_n \frac{1}{n} \sum_{i=1}^{n} x_i - \frac{1}{n} \sum_{i=1}^{n} y_i > 0. \]

Therefore, if we choose such an mean \( m \) and let
\[ v_i(a^1, a^2, \ldots) = m(u_i(a^1), u_i(a^2), \ldots) \]
then \( v_i(a) > v_i(b) \) whenever \( a \succ_i b. \) That is, the preference relation induced by \( v_i \) has all the strict preferences that \( \succeq_i \) has, and perhaps also additional ones.

Limiting means utilities, like discounting utilities, are stationary.

### 5.3. Folk theorems.

#### 5.3.1. Example.** What payoffs profiles are achievable in Nash equilibria of infinite repeated games? It turns out that the answer is: more or less all of them. To get some intuition as to how this is done, consider the following example. Let \( G_0 \) be the following prisoner’s dilemma game:\n
<table>
<thead>
<tr>
<th></th>
<th>D</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>0, 0</td>
<td>1, 0</td>
</tr>
<tr>
<td>C</td>
<td>0, 1</td>
<td>1/2, 1/2</td>
</tr>
</tbody>
</table>

Consider the following symmetric strategy profile, called “grim trigger”: start with \( C \), and keep on playing \( C \) until the other person plays \( D \). Then play \( D \) henceforth. It is easy to see that this is an equilibrium under both limiting means and discounting, for \( \delta \) close enough to one.

---

\(^7\)For a “real life” example: [https://www.youtube.com/watch?v=y6GhbT-zEfc](https://www.youtube.com/watch?v=y6GhbT-zEfc)
5.3.2. **Enforceable and feasible payoffs.** We fix an infinitely repeated game $G$ with base game $G_0$. Define the minmax payoff of player $i$ in the base game $G_0$ as the lowest payoff that the rest of the players can force on $i$:

$$u_{mm}^i = \min_{a_{-i}} \max_{a_i} u_i(a_{-1}, a_i).$$

Equivalently, this is the payoff that player one can guarantee for herself, regardless of the other player’s actions.

We say that a payoff profile $w \in \mathbb{R}^n$ is **enforceable** if $w_i \geq u_{mm}^i$ for all $i \in N$. It is **strictly enforceable** if $w_i > u_{mm}^i$ for all $i \in N$.

**Claim 5.2.** Let $s^*$ be a Nash equilibrium of $G$, under either discounting or limiting means. Then the payoff profile associated with $s$ is enforceable.

**Proof.** Since player $i$ can guarantee a stage utility of $u_{mm}^i$, she can always choose a strategy such that her stage utilities will each be at least $u_{mm}^i$. Hence under both discounting and limiting means her payoff will be at least $u_{mm}^i$ in $G$. □

We say that a payoff profile $w$ is **feasible** if it is a convex combination of utilities achievable in $G_0$. That is, if for all $i \in N$

$$w_i = \sum_{a \in A} \alpha_a \cdot u_i(a)$$

for some $\{\alpha_a\}$ that sum to one. Clearly, every payoff profile in $G$ is feasible.

**Exercise 5.3.** Consider the game

$\begin{array}{ccc}
D & C & F \\
D & 0,0 & 1,0 & 0,1 \\
C & 1,1 & 2,2 & -2,3 \\
F & 0,1 & 2,-3 & -2,-2 \\
\end{array}$

Draw a diagram of the feasible and enforceable profiles.

5.3.3. **Nash folk theorems.** Aumann and Shapley [3] proved the Nash folk theorem for limit of means. We state it for limiting means.

**Theorem 5.4.** For every feasible, enforceable payoff profile $w$ there exists a Nash equilibrium of $G$ with limiting means utilities whose associated payoff profile is $w$.

The construction of these equilibria involves **punishing**: players all play some equilibrium, and if anyone deviates the rest punish them.
Proof of Theorem 5.4. Let \( w_i = \sum_{a \in A} \alpha_a \cdot u_i(a) \). Let \((a^1, a^2, \ldots)\) be a sequence in \( A \) such that \( m(\{a^1 = a\}, \{a^2 = a\}, \ldots) = \alpha_a \). Such a sequence exists since (for example) if we choose each \( a^t \) independently at random to equal \( a \) with probability \( \alpha_a \) then with probability one the sequence has this property. If the coefficients \( \alpha_a \) are rational then one can simply take a periodic sequence with period equal to the lowest common denominator.

Let \( s^* \) be the following strategy profile. For each player \( i \) let \( s^*_i \) be the strategy in which she chooses \( a^t_i \), unless in some previous period \( \tau \) some player \( j \) did not choose \( a^\tau_j \), in which case she chooses a strategy \( b_i \), where

\[
b_{-j} \in \arg\min_{a_{-j}} \max_{a_j} u_j(a_{-j}, a_j).\]

Hence the stage utilities of a player \( i \) who deviates will be, from the point of deviation on, at most \( u^{mm}_i \). Therefore her utility will be at most \( u^{mm}_i \), since utilities do not depend on any finite set of stage utilities. Since \( w \) is enforceable, it follows that \( w_i \geq u^{mm}_i \), and so no deviation is profitable, and \( s^* \) is an equilibrium. \( \square \)

A similar proof technique can be used to show the following theorem, which is due to Fudenberg and Maskin [7], with an earlier, weaker version by Friedman [6].

Theorem 5.5. For every feasible, strictly enforceable payoff profile \( w \) and \( \varepsilon > 0 \) there is a \( \delta_0 > 0 \) such that for all \( \delta > \delta_0 \) there exists a Nash equilibrium of \( G \) with \( \delta \)-discounting utilities whose associated payoff profile \( w' \) satisfies \( |w_i' - w_i| < \varepsilon \) for all \( i \in N \).

The discount factor needs to be large enough to make eternal punishment pose more of a loss than can be gained by a single deviation. It also needs to be large enough to allow for the discounted averages to approximate a given convex combination of the base game utilities.

5.3.4. Perfect folk theorems. Consider the following base game (taken from Osborne and Rubinstein [18]):

\[
\begin{array}{cc}
D & C \\
D & 0.1 & 0.1 \\
C & 1.5 & 2.3
\end{array}
\]

Here, an equilibrium built in Theorems 5.4 and 5.5 that achieves payoff profile \((2, 3)\) has the players playing \((C, C)\) on the equilibrium path, and punishing by playing \( D \) forever after a deviation. Note, however, that for the row player, action \( D \) is strictly dominated by \( C \). Hence this equilibrium is not a subgame perfect equilibrium: regardless of
what the column player does, the row player can increase her subgame utility by at least 1 by always playing $C$ rather than $D$. It is therefore interesting to ask if there are subgame perfect equilibria that can achieve the same set of payoff profiles.

The following theorem is due to Aumann and Shapley [3], as well as Rubinstein [19].

**Theorem 5.6.** For every feasible, strictly enforceable payoff profile $w$ there exists a subgame perfect Nash equilibrium of $G$ with limiting means utilities whose associated payoff profile is $w$.

The idea behind these equilibria is still of punishing, but just for some time rather than for all infinity.

**Proof of Theorem 5.6.** As in the proof of Theorem 5.4, let $w_i = \sum_{a \in A} \alpha_a \cdot u_i(a)$, and let $(a^1, a^2, \ldots)$ be a sequence in $A$ such that $m(\mathbb{1}_{\{a^1=a\}}, \mathbb{1}_{\{a^2=a\}}, \ldots) = \alpha_a$. Likewise, for each player $i$ let $s^*_i$ be the strategy in which she chooses $a^t_i$, unless in some previous period $\tau$ some player $j$ deviated and did not choose $a^\tau_j$. In the latter case, we find for each such $\tau$ and $j$ a $\tau'$ large enough so that, in time periods $(\tau + 1, \ldots, \tau')$ then the average of player $j$’s payoffs in periods $(\tau, \tau + 1, \ldots, \tau')$ is lower than $w_j$. Such a $\tau'$ exists since the payoffs in periods $(\tau + 1, \ldots, \tau')$ will all be at most $u^\text{mm}_j$, and since $w_j > u^\text{mm}_j$. We let all players but $j$ play $b_{-j} \in \arg\min_{a_{-i}} \max_{a_i} u_i(a_{-1}, a_i)$, in time periods $(\tau + 1, \ldots, \tau')$ then the average of player $j$’s payoffs in periods $(\tau, \tau + 1, \ldots, \tau')$ is lower than $w_j$. Such a $\tau'$ exists since the payoffs in periods $(\tau + 1, \ldots, \tau')$ will all be at most $u^\text{mm}_j$, and since $w_j > u^\text{mm}_j$. We let all players but $j$ play $b_{-j}$ in time periods $\tau + 1, \ldots, \tau'$. We do not consider these punishments as punishable themselves, and after period $\tau'$ all players return to playing $a^t_j$ (until the next deviation).

To see that $s^*$ is a subgame perfect equilibrium, we consider two cases. First, consider a subgame in which no one is currently being punished. In such a subgame anyone who deviates will be punished and lose more than they gain for each deviation. Hence a deviant $j$’s long run average utility will tend to at most $w_j$, and there is no incentive to deviate.

Second, consider a subgame in which someone is currently being punished. In such a subgame the punishers have no incentive to deviate, since the punishment lasts only finitely many periods, and thus does not affect their utilities; deviating from punishing will not have any consequences (i.e., will not be punished) but will also not increase utilities, and therefore there is again no incentive to deviate.

We next turn to proving perfect folk theorems for discounted utilities. An early, simple result is due to Friedman [6].
Theorem 5.7. Let $G_0$ have a pure Nash equilibrium $s^*$ with payoff profile $z$. Let $w$ be a payoff profile of some strategy profile $a \in A$ of $G_0$ such that $w_i > z_i$ for all $i \in N$. Then there is a $\delta_0 > 0$ such that for all $\delta > \delta_0$ there exists a subgame perfect equilibrium of $G$ under discounting, with payoff profile $w$.

The idea behind this result is simple: the players all play $a$ unless someone deviates. Once anyone has deviated, they all switch to playing $s^*$ henceforth. Given that $\delta$ is close enough to 1, the deviant’s utility from playing $s^*$ henceforth will trump any gains from the deviation. Since $s^*$ is an equilibrium, there is no reason for the punishers to deviate from the punishment.

A harder result is due to Fudenberg and Maskin [7] who, for the two player case, extend the Nash folk theorem 5.5 to a perfect Nash folk theorem.

Theorem 5.8. Assume that $|N| = 2$. For every feasible, strictly enforceable payoff profile $w$ and $\varepsilon > 0$ there is a $\delta_0 > 0$ such that for all $\delta > \delta_0$ there exists a perfect Nash equilibrium of $G$ with $\delta$-discounting utilities whose associated payoff profile $w'$ satisfies $|w'_i - w_i| < \varepsilon$ for all $i \in N$.

Before proving this theorem will we state the following useful lemma. The proof is straightforward.

Lemma 5.9. Let $G$ be a repeated game, with $\delta$-discounted utilities. Let $G$ have a subgame $G'$ that starts after the players play some action profile $a \in A$. Let $(a^1, a^2, \ldots)$ be a sequence of action profiles for $G'$, and define the $G'$ utilities by

$$v'_i(a^1, a^2, \ldots) = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \cdot u_i(a^t).$$

Then the $G$ utilities for playing $(a, a^1, a^2, \ldots)$ are given by

$$v_i(a, a^1, a^2, \ldots) = (1 - \delta)u_i(a) + \delta v'_i(a^1, a^2, \ldots).$$

We prove Theorem 5.8 for the particular case that $(w_1, w_2) = (u_1(a), u_2(a))$ for some feasible and strictly enforceable $a \in A$; in this case we can take $\varepsilon = 0$. The proof of the general case uses the same idea, but requires the usual technique of choosing a sequence of changing action profiles.

We assume that $u_{i,mm} = 0$. This is without loss of generality, since otherwise we can define a game $G'_0$ in which the utilities are $u'_i = u_i - u_{i,mm}$. The analysis of the $G'_0$-repeated game will be identical, up to an additive constant for each player’s utility.
Fix
\[ b_1 \in \text{argmin} \max_{a_1} u_2(a_1, a_2) \]
and
\[ b_2 \in \text{argmin} \max_{a_2} u_1(a_1, a_2) \]
Note that \( u_1(b_1, b_2) \leq 0 \) and likewise \( u_2(b_1, b_2) \leq 0 \), since we assume that \( u_{1mm} = u_{2mm} = 0 \).

As an example, consider the following base game:

<table>
<thead>
<tr>
<th></th>
<th>D</th>
<th>C</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>0, 0</td>
<td>1, 0</td>
<td>0, 1</td>
</tr>
<tr>
<td>F</td>
<td>1, 0</td>
<td>2, 2</td>
<td>−2, −2</td>
</tr>
</tbody>
</table>

With \( a = (C, C) \). It is easy to see that \( u_{1mm} = u_{2mm} = 0 \) and that necessarily \( b = (F, F) \).

Consider the following strategy profile \( s^* \) for the repeated game. Recall that \((w_1, w_2) = (u_1(a), u_2(a))\) for some feasible and strictly enforceable \( a \in A \). In \( s^* \), the game has two "modes": on-path mode and punishment mode.

- In on-path mode each player \( i \) plays \( a_i \).
- In punishment mode each player \( i \) plays \( b_i \).

Both players start in the on-path mode. If any player deviates, the game enters punishment mode for some fixed number of \( \ell \) rounds. This also applies when the game is already in punishment mode: if a player deviates when in punishment mode (i.e., stops punishing and does not play \( b_i \)), the game re-enters punishment mode for \( \ell \) rounds.

Player \( i \)'s utility on path is given by
\[ v_i(s^*) = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_i(a) = w_i. \]

Denote by \( \bar{u}_i = \max_{a \in A} u_i(a) \) the maximal utility achievable by player \( i \) in \( G_0 \). Denote the punishment utility penalty by \( p_i = -u_i(b_1, b_2) \geq 0 \).

In the example above, \( w_1 = w_2 = 2 \) and \( u_1(b_1, b_2) = u_2(b_1, b_2) = -2 \).

To show that \( s^* \) is a subgame perfect equilibrium, we consider the \( \ell + 1 \) types of possible subgames: those that start in punishment mode (and consider separately the number of punishment periods left), and those that start in on-path mode. We begin with the former.

Let the subgame \( G^k \) start in punishment mode, with \( k \) punishment periods left. Then on the (subgame) equilibrium path, player \( i \)'s utility
is given by

\[ v_i^k = (1 - \delta) \sum_{t=1}^{k} \delta^{t-1} \cdot u_i(b_1, b_2) + (1 - \delta) \sum_{t=k+1}^{\infty} \delta^{t-1} \cdot w_i \]

\[ = -(1 - \delta^k) p_i + \delta^k w_i. \]

In particular

\[ v_i^\ell = \delta^\ell w_i - (1 - \delta^\ell) p_i. \]

Let \( k \) be the highest number (\( \leq \ell \)) for which it is profitable to deviate on the first round of \( G^k \). If no such \( k \) exists then we are done with the punishing mode subgames. Otherwise, if a player decides to deviate, she will again and again find it profitable to deviate when arriving at the subgame \( G^k \), and so the game will always be in punishment mode. Hence her utility will be at most 0. It follows that it is profitable to already deviate at the first opportunity, which is \( G^\ell \), and thus ensure a utility of 0. Hence, under our assumption that it is profitable to deviate, a utility of zero is preferable to \( v_i^\ell \), or, equivalently, \( v_i^\ell < 0 \). Therefore, if we choose \( \delta \) and \( \ell \) in such a way that \( v_i^\ell \geq 0 \), player \( i \) will have no incentive to deviate in punishment mode. Since \( w_i > 0 \), this is always possible, by choosing \( \delta^\ell \) close enough to 1.

Now, consider any subgame that starts in on-path mode. Player \( i \)'s utility from playing \( s^* \) is \( w_i \). Her utility from deviating in the first round is at most

\[ (1 - \delta) \bar{u}_i + \delta v_i^\ell, \]

since after the deviation she will enter punishment mode, and there we have ensured that \( v_i^\ell \) is the best utility she can achieve. Hence if we can choose \( \delta \) and \( \ell \) in such a way that

\[ w_i - (1 - \delta) \bar{u}_i - \delta v_i^\ell \geq 0 \]  \hspace{1cm} (5.1)

then she will have no reason to deviate in the first round of any subgame that starts in on-path mode, and will therefore never deviate.

Consider the sequences \( \{\delta_m\} \) and \( \{\ell_m\} \) given by

\[ \delta_m = e^{-1/m^2} \]

and

\[ \ell_m = m, \]

so that

\[ \delta_m^{\ell_m} = e^{-1/m}. \]
Since $\delta_m^\ell$ tends to 1, it follows that $v_i^\ell$ is positive for large enough $m$, and so player $i$ has no incentive to deviate on the first rounds of subgames that start in punishing mode. So see that (5.1) is also satisfied — and so player $i$ has no incentive to deviate in on-path mode — we calculate:

$$w_i - (1 - \delta_m)\bar{u}_i - \delta_m v_i^\ell = w_i - (1 - \delta_m)\bar{u}_i - \delta_m(\delta_m^\ell w_i - (1 - \delta_m^\ell)p)$$

$$= w_i - (1 - e^{-1/m^2})\bar{u}_i - e^{-1/m^2}(e^{-1/m}w_i - (1 - e^{-1/m^2})p)$$

$$\geq w_i - (1 - e^{-1/m^2})\bar{u}_i - e^{-1/m^2-1/m}w_i$$

$$= (1 - e^{-1/m^2-1/m})w_i - (1 - e^{-1/m^2})\bar{u}_i.$$ 

Now, $1 - e^{-x} = x + O(x^2)$, and so this expression is equal to

$$\frac{1}{m}w_i - \frac{1}{m^2}u_i + O\left(\frac{1}{m^2}\right),$$

and in particular is positive for large enough $m$. Hence, for large enough $m$, it will be positive for both $i = 1$ and $i = 2$.

We have thus shown that for an appropriate choice of $\delta$ and $\ell$ it holds that both players have no incentive to deviate in any subgame, and so $s^*$ is a subgame perfect equilibrium. This completes the proof of Theorem 5.8.
Appendix A. Conditional expectations as orthogonal projections

Let $(\Omega, \Sigma, \mathbb{P})$ be a finite probability space. Let $X : \Omega \to \mathbb{R}$ denote a real random variable, and let $\mathcal{V} = \mathbb{R}^\Omega$ be the set of all real random variables. We say that $X_1, X_2 \in \mathcal{V}$ are equivalent if $\mathbb{P}[X_1 = X_2] = 1$; that is, if the subset of $\Omega$ on which they agree has full measure. Let $\mathcal{V}$ be the set of equivalence classes of $\mathcal{V}$; this space is usually denoted $L^2(\Omega, \Sigma, \mathbb{P})$. Note that if each singleton $\{\omega\} \subset \Omega$ has positive probability then we can just take $\mathcal{V} = \mathcal{V}$, and we encourage the uninitiated reader to take this point of view.

As usual, we can define the sum of two random variables in the obvious way, and likewise define the product of a random variable by a real number. These operations extend naturally to operations on $\mathcal{V}$.

Under the above operations it is straightforward to show that $\mathcal{V}$ is a vector space. We can furthermore equip $\mathcal{V}$ with the $L^2$ norm given by $\|X\|^2 = \mathbb{E}[X^2]$.

Exercise A.1. Prove that $\mathcal{V}$, as described above, is indeed a normed vector space.

We next claim that the expectation of a random variable $X$ is the single number that best approximates $X$, in the sense of expected distance squared.

Claim A.2. 

$$\mathbb{E}[X] = \arg\min_{z \in \mathbb{R}} \mathbb{E}[(X - z)^2].$$

Proof. Note first that

$$\mathbb{E}[(X - z)^2] = \mathbb{E}[X^2 - 2z \cdot X + z^2] = \mathbb{E}[X^2] - 2z \cdot \mathbb{E}[X] + z^2.$$ 

Hence

$$\frac{d\mathbb{E}[(X - z)^2]}{dz} = \frac{d(\mathbb{E}[X^2] - 2z\mathbb{E}[X] + z^2)}{dz} = -2\mathbb{E}[X] + 2z,$$

---

8 All the results in this appendix apply to general (separable) probability spaces, and indeed we will state our claims in language that is sometimes too general for the finite case. The proofs in the more general case are slightly more technical, but use the same ideas.

9 In the general (separable) case we will more precisely require that they are equal on some full measure set.
and the unique extremal point is achieved at \( z = \mathbb{E}[X] \). This is a maximum since
\[
\frac{d^2 \mathbb{E}[(X - z)^2]}{dz^2} = 2.
\]
\( \square \)

More generally, if \( \Pi \subseteq \Sigma \) then \( \mathbb{E}[X|\Pi] \) is the best \( \Pi \)-measurable random variable that approximates \( X \). Denote by \( V_\Pi \) the set of \( \Pi \)-measurable random variables.

**Claim A.3.**
\[
\mathbb{E}[X|\Pi] = \arg\min_{Z \in V_\Pi} \mathbb{E}[(X - Z)^2].
\]

**Exercise A.4.** Prove Claim A.3

We next note that \( V_\Pi \) is a vector subspace of \( V \); this follows immediately from the definitions. We also note that \( \mathbb{E}[(X - Z)^2] \), the expression that appears in Claim A.3, is simply \( \|X - Z\|^2 \). Hence \( \mathbb{E}[X|\Pi] \) is the vector in \( V_\Pi \) that has minimum distance to \( X \), and so \( \mathbb{E}[:|\Pi] : V \to V_\Pi \) is simply the orthogonal projection operator to \( V_\Pi \). This fact has many interesting consequences, since orthogonal projection operators have a very simple and rigid structure.

**Claim A.5.** Let \( E : V \to V \) be an orthogonal projection operator. Denote \( K = \ker E \) and the range of \( E \) be \( M = E(V) \). Then \( V = K \oplus M \). That is, there is an orthogonal basis of \( K \), \( (X_1, \ldots, X_k) \), and an orthogonal basis of \( M \), \( (Y_1, \ldots, Y_m) \), such that \( (X_1, \ldots, X_k, Y_1, \ldots, Y_m) \) is an orthogonal basis of \( V \).

Furthermore,
\[
E(\alpha_1 X_1 + \cdots + \alpha_k X_k + \beta_1 Y_1 + \cdots \beta_m Y_m) = \beta_1 Y_1 + \cdots \beta_m Y_m.
\]
It follows that \( E^2 = E \), and that if \( EX \neq X \) then \( \|EX\| < \|X\| \).

**Exercise A.6.** Prove Claim A.5

Using this Claim we can prove Lévy’s zero-one law. Before stating it we will introduce the term *filtration*: a filtration is a sequence of sub-sigma-algebras of \( \Sigma \), \( \{ \Pi_n \}_{n \in \mathbb{N}} \), such that \( \Pi_n \subseteq \Pi_{n+1} \) for all \( n \in \mathbb{N} \).

**Theorem A.7** (Lévy’s Zero-One Law). Let \( \{ \Pi_n \} \) be a filtration, and let \( X \) be a random variable. Let \( \Pi_\infty \) be the sigma-algebra generated by \( \bigcup_n \Pi_n \). Then \( \lim_n \mathbb{E}[X|\Pi_n] \) exists for almost every \( \omega \in \Omega \), and is equal to \( \mathbb{E}[X|\Pi_\infty] \).
Note that this theorem is trivial for finite probability spaces, where there are only finitely many sub-sigma-algebras, and so every filtration must stabilize. The more general case can be proved using the orthogonal projection approach.
References


