Collaboration on homework is encouraged, but individually written solutions are required. Please name all collaborators and sources of information on each assignment. Any such named source may be used.

(1) The envelope paradox. Consider the following Bayesian game: A coin is tossed until it comes out heads. The number of tosses $k$ is recorded. There are two pieces of paper; on one is written the number $10^k$, and on the other $10^{k+1}$. One of the two notes is chosen at random and given to Silvia. The other is given to Vadim. They each look at their own note. If both want to trade then they are allowed to. After trading (or not) each is given an amount of money equal to the number written on his or her paper.

Formally, the states of the world are $\Omega = \{1, 2, 3, \ldots\} \times \{0, 1\}$, where the first coordinate is the number of tosses and the second corresponds to the random allocation of notes. There is a common prior, which, for $(k, b) \in \Omega$ is

$$
\mu(k, b) = \left(\frac{1}{2}\right)^k \cdot \frac{1}{2} = 2^{-(k+1)}.
$$

Silvia’s type is $t_S(k, b) = 10^{k+b}$ and Vadim’s type is $t_V(k, b) = 10^{k+1-b}$.

The action space of each player is $\{T, N\}$, corresponding on whether they want to trade or not. Their utilities are

$$
u_S(a_S, a_V, k, b) = \begin{cases} t_V & \text{if } a_V = a_S = T, \\ t_S & \text{otherwise} \end{cases},$$

and likewise

$$
u_V(a_S, a_V, k, b) = \begin{cases} t_S & \text{if } a_V = a_S = T, \\ t_V & \text{otherwise} \end{cases}.$$

(a) For each possible value of Silvia’s type, calculate her conditional utility for trading and for not trading. Do the same for Vadim.

(b) Describe the unique Bayes-Nash equilibrium of the game in which the players use weakly dominant strategies.

(c) Use the revelation principle to construct an equivalent mechanism in which players play their types.

(d) What is the common knowledge sigma-algebra $\Sigma_C$?

(e) Explain the apparent conflict with the no trade theorem.

(f) If you had a chance to play this game, would you ask to trade? Would you look at your note before asking?
(2) **Reserve prices.** Alex and John would both like to buy an item owned by Xiaomin. Alex and John’s valuations are chosen independently from the uniform distribution on $[0, 1]$.

(a) What is Xiaomin’s expected revenue from a second price auction?

(b) Xiaomin now introduces a *reserve price* $b_r \in [0, 1]$: if the maximum bid was under $b_r$ then the auction is cancelled, no one gets the item and no one pays. Otherwise, the winner pays the maximum of $b_r$ and the loser’s bid. What is her expected revenue, as a function of $b_r$?

(3) **Bundling.** Ali walks into a store with the intention of buying a loaf of bread and a stick of butter. His valuations for the two items are chosen independently from the uniform distribution on $[0, 1]$. Betty, the store owner, has to set the prices. We assume that Ali will buy for any price that is lower than his valuation.

(a) Assume first that Betty sets a price for each item. What is her optimal expected revenue?

(b) Betty now decides to *bundle*: she sets a price for buying both items together, and does not offer each one of them separately. That is, she offers Ali to either buy both, or else get neither. What is her optimal expected revenue?