

SS 201B, HOMEWORK 5  
DUE TUESDAY, FEBRUARY 20<sup>ST</sup>

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

- (1) Consider the following example that appears in Geanakoplos and Polemarchakis (1982); they write that “This example was provided by Robert Aumann in exchange for a Kosher dinner.”

Let  $\Omega = \{1, 2, \dots, 16\}$  and  $\Sigma = 2^\Omega$ . There are two players, with  $\mu_1 = \mu_2 = \mu$  being the uniform distribution  $\mu(\omega) = 1/16$ . Player 1’s initial information partition is

$$\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9, 10, 11, 12\}, \{13, 14, 15, 16\}.$$

Player 2’s partition is

$$\{1, 2, 3, 4, 5\}, \{6, 7, 8, 9, 10\}, \{11, 12, 13, 14, 15\}, \{16\}.$$

Let  $A = \{1, 6, 11, 16\}$ .

There are discrete time periods  $t = \{1, 2, 3, 4, \dots\}$ . In each time period, the players simultaneously announce to each other their posterior probabilities for  $A$ . Thus, if  $\omega = 1$ , then in the first period player 1 announces  $1/4$ , and player 2 announces  $1/5$ .

Assuming  $\omega = 1$ , calculate their posteriors at each time period and show that eventually their posteriors become common knowledge.

- (2) *Reserve prices.* Jeff and Anastasia would both like to buy an item owned by Daniel. Jeff and Anastasia’s valuations are chosen independently from the uniform distribution on  $[0, 1]$ , and each is known only to himself.

(a) What is Daniel’s expected revenue from a second price auction?

(b) Daniel now introduces a *reserve price*  $b_r \in [0, 1]$ : if the maximum bid is under  $b_r$  then the auction is canceled, no one gets the item and no one pays. Otherwise, the winner pays the maximum of  $b_r$  and the loser’s bid. What is Daniel’s expected revenue, as a function of  $b_r$ ?

(c) What is the maximal expected revenue Daniel can get by choosing  $b_r$  optimally?

- (3) *Bundling.* Sumit walks into a store with the intention of buying a loaf of bread and a stick of butter. His valuations for the two items are chosen independently from the uniform distribution on  $[0, 1]$ . Joanna, the store owner, has to set the prices. We assume that Sumit will buy for any price that is lower than his valuation.

(a) Assume first that Joanna sets a price  $b_l$  for the loaf and  $b_s$  for the stick. What is her expected revenue, as a function of  $b_l$  and  $b_s$ ?

(b) What is the maximal expected revenue she can get?

- (c) Joanna now decides to *bundle*: she sets a price  $b_b$  for buying both items together, and does not offer each one of them separately. That is, she offers Sumit to either buy both for  $b_b$ , or else get neither. What is her expected revenue, as a function of  $b_b$ ?
- (d) What is the maximal expected revenue she can get now?
- (e) *Bonus question.* Assume now that Joanna sets three different prices:  $b_l$  for the loaf,  $b_s$  for the stick, and  $b_b$  for both, so that Sumit can choose if to buy just the loaf (for  $b_l$ ), just the stick (for  $b_s$ ), or both (for  $b_b$ ). Assume that he will choose to buy whichever items maximize his utility, which is his value for the bought items minus the price paid. What is the maximal expected revenue she can get now?
- (4) *Bonus: a riddle with both prisoners and hats (Gabay-O'Connor game).* There are  $n$  prisoners standing in a line. The first can observe all the rest. The second can observe all except the first, etc. Each is given either a red or a blue hat which he cannot see. Now, starting with the first prisoner, each in turn has to guess the color of his hat, a guess which the rest can hear.
- (a) The prisoners are allowed to decide on a strategy ahead of time. Find one in which they all guess the color correctly, except maybe the first prisoner.
- (b) Do the same, but for an infinite line of prisoners.
- (c) For an infinite line of deaf prisoners, find a strategy in which at most finitely many of them guess incorrectly.
- (d) For an infinite line of deaf prisoners, assume that each is assigned a hat independently and uniformly at random. Show that regardless of the strategy the prisoners agree on, each has a probability of  $1/2$  of guessing his hat color correctly. Explain why this means that with probability one infinitely many prisoners will guess incorrectly. Resolve the apparent conflict with your answer from the previous question.