

SS 201A, HOMEWORK 5  
DUE FRIDAY, DECEMBER 6<sup>TH</sup>

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

- (1) *Reserve prices.* Jacob and Zhuofang would both like to buy an item owned by Fan. Jacob and Zhuofang's valuations are chosen independently from the uniform distribution on  $[0, 1]$ , and each is known only to her/himself.
  - (a) What is Fan's expected revenue from a second price auction?
  - (b) Fan now introduces a *reserve price*  $b_r \in [0, 1]$ : if the maximum bid is under  $b_r$  then the auction is canceled, no one gets the item and no one pays. Otherwise, the winner pays the maximum of  $b_r$  and the loser's bid. What is Fan's expected revenue, as a function of  $b_r$ ?
  - (c) What is the maximal expected revenue Fan can get by choosing  $b_r$  optimally?
- (2) *Bundling.* Alena walks into a store with the intention of buying a loaf of bread and a stick of butter. Her valuations for the two items are chosen independently from the uniform distribution on  $[0, 1]$ . Kate, the store owner, has to set the prices. We assume that Alena will buy for any price that is lower than her valuation.
  - (a) Assume first that Kate sets a price  $b_l$  for the loaf and  $b_s$  for the stick. What is her expected revenue, as a function of  $b_l$  and  $b_s$ ?
  - (b) What is the maximal expected revenue she can get?
  - (c) Kate now decides to *bundle*: she sets a price  $b_b$  for buying both items together, and does not offer each one of them separately. That is, she offers Alena to either buy both for  $b_b$ , or else get neither. What is her expected revenue, as a function of  $b_b$ ?
  - (d) What is the maximal expected revenue she can get now?
  - (e) *Bonus question.* Assume now that Kate sets three different prices:  $b_l$  for the loaf,  $b_s$  for the stick, and  $b_b$  for both, so that Alena can choose if to buy just the loaf (for  $b_l$ ), just the stick (for  $b_s$ ), or both (for  $b_b$ ). Assume that Alena will choose to buy whichever items maximize her utility, which is his value for the bought items minus the price paid. What is the maximal expected revenue Kate can get now?
- (3) *Bonus: a riddle with both prisoners and hats (Gabay-O'Connor game).* There are  $n$  prisoners standing in a line. The first can observe all the rest. The second can observe all except the first, etc. Each is given either a red or a blue hat which he cannot see. Now, starting with the first prisoner,

each in turn has to guess the color of his hat, a guess which the rest can hear.

- (a) The prisoners are allowed to decide on a strategy ahead of time. Find one in which they all guess the color correctly, except maybe the first prisoner.
- (b) Do the same, but for an infinite line of prisoners.
- (c) For an infinite line of deaf prisoners, find a strategy in which at most finitely many of them guess incorrectly.
- (d) For an infinite line of deaf prisoners, assume that each is assigned a hat independently and uniformly at random. Show that regardless of the strategy the prisoners agree on, each has a probability of  $1/2$  of guessing his hat color correctly. Explain why this means that with probability one infinitely many prisoners will guess incorrectly. Resolve the apparent conflict with your answer from the previous question.