PS/EC 172, HOMEWORK 4
DUE TUESDAY, FEBRUARY 13TH

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) Construct an example of a knowledge space with two players, a finite set of states of the world, an event A and a state of the world ω such that $\omega \in K_1A$, $\omega \in K_2A$, $\omega \in K_1K_2A$, $\omega \in K_2K_1A$, but $\omega \notin K_1K_2K_1A$.

(2) Consider a finite knowledge space. Prove that the collection of connected components $\{C(\omega)\}_{\omega\in\Omega}$ of the graph G_C is the partition that generates the common knowledge algebra Σ_C. (Hint: you can use anything that is proved in the lecture notes.)

(3) In the class of models described in the "No trade" section of the lecture notes, find an example of a belief space in which, for some $\omega \in \Omega$, $Q_1(\omega) < q < Q_2(\omega)$ and at ω both players know that $Q_1 < q < Q_2$.

(4) *The envelope paradox.* A coin is tossed until it comes out heads. Let k denotes the (random) number of tosses. There are two pieces of paper; on one is written the number 10^k, and on the other 10^{k+1}. One of the two notes is chosen at random and given to Robin. The other is given to Peter. They each look at their own note. If both want to trade then they are allowed to. After trading (or not) each is given an amount of money equal to the number written on his or her paper.

Formally, the states of the world are $\Omega = \{1, 2, 3, \ldots\} \times \{0, 1\}$, where the first coordinate is the number of tosses and the second corresponds to the random allocation of notes. There is a common prior, which, for $(k, b) \in \Omega$ is

$$\mu(k, b) = \left(\frac{1}{2}\right)^k \cdot \frac{1}{2} = 2^{-(k+1)}.$$

Robin's type is $t_R(k, b) = 10^{k+b}$ and Peter's type is $t_P(k, b) = 10^{k+1-b}$. Robin's utility for not trading is t_R. Her utility for trading is t_P. Peter's utility for not trading is t_P, and his utility for trading is t_R.

(a) What is the common knowledge algebra Σ_C?

(b) For each possible value of Robin's type, calculate her conditional expected utility for trading and for not trading. Do the same for Peter.

(c) Let A be the event that both want to trade. Is it common knowledge?

(d) Explain the apparent conflict with the no trade theorem.

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(e) If you had a chance to play this game, and you got to decide if a trade would take place, would you decide to trade? If so, would you look at your note before asking?

(5) *Bonus question.* A prisoner escapes to $\mathbb{Z}^2$ on Sunday. Every day he must move either one up (i.e., add $(0,1)$ to his location) or one to the right (add $(1,0)$), except on Saturdays, when he must rest. The detective can, once a day, check one element of $\mathbb{Z}^2$ and see if the prisoner is there. If she finds him then she wins. He wins if she never finds him.

Formally, the prisoner's strategy is an element

$$(z, f) \in \mathbb{Z}^2 \times \{(1,0),(0,1),(0,0)\}^{\mathbb{N}}$$

such that $f(n) = (0,0)$ whenever $n \equiv 0 \mod 7$, and $f(n) \in \{(1,0),(0,1)\}$ otherwise. The detective's strategy is a sequence $\{z_n\}_{n\in\mathbb{N}}$ with $z_n \in \mathbb{Z}^2$.

The prisoner's current location when using strategy (z,f) is

$$\ell_n = z + \sum_{k=1}^{n-1} f(k).$$

The detective wins if $\ell_n = z_n$ for some n. The prisoner wins otherwise.

(a) Show that the detective has a winning strategy.

(b) Show that if we remove the requirement that the prisoner rests on Saturdays then the detective does not have a winning strategy.