PS/Ec 172, Homework 5
Due Wednesday, May 4^{th}

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) *50 points.* Consider the following example that appears in Geanakoplos and Polemarchakis (1982); they write that "This example was provided by Robert Aumann in exchange for a Kosher dinner."

Let $\Omega = \{1, 2, \ldots, 16\}$ and $\Sigma = 2^\Omega$. There are two players, with $\mu_1 = \mu_2 = \mu$ being the uniform distribution $\mu(\omega) = 1/16$. Player 1's initial information partition is

$$\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9, 10, 11, 12\}, \{13, 14, 15, 16\}.$$

Player 2's partition is

$$\{1, 2, 3, 4, 5\}, \{6, 7, 8, 9, 10\}, \{11, 12, 13, 14, 15\}, \{16\}.$$

Let $A = \{1, 6, 11, 16\}$.

There are discrete time periods $t = \{1, 2, 3, 4, \ldots\}$. In each time period, the players simultaneously announce to each other their posterior probabilities for A. Thus, if $\omega = 1$, then in the first period player 1 announces $1/4$, and player 2 announces $1/5$.

Assuming $\omega = 1$, calculate their posteriors at each time period and show that eventually their posteriors become common knowledge.

(2) *The envelope paradox.* A coin is tossed until it comes out heads. Let the number of tosses be k. There are two pieces of paper; on one is written the number 10^k, and on the other 10^{k+1}. One of the two notes is chosen at random and given to Oriel. The other is given to Wade. They each look at their own note. If both want to trade then they are allowed to. After trading (or not) each is given an amount of money equal to the number written on his or her paper.

Formally, the states of the world are $\Omega = \{1, 2, 3, \ldots\} \times \{0, 1\}$, where the first coordinate is the number of tosses and the second corresponds to the random allocation of notes. There is a common prior, which, for $(k, b) \in \Omega$ is

$$\mu(k, b) = \left(\frac{1}{2}\right)^k \cdot \frac{1}{2} = 2^{-(k+1)}.$$

Oriel's type is $t_O(k, b) = 10^{k+b}$ and Wade's type is $t_W(k, b) = 10^{k+1-b}$. Oriel's utility for not trading is t_O. Her utility for trading is t_W. Wade's utility for not trading is t_W, and his utility for trading is t_O.

(a) *10 points.* What is the common knowledge sigma-algebra Σ_C?

(b) *10 points.* For each possible value of Oriel's type, calculate her conditional expected utility for trading and for not trading. Do the same for Wade.

Omer Tamuz. Email: tamuz@caltech.edu.

(c) *10 points.* Let A be the event that both want to trade. Is it common knowledge?

(d) *10 points.* Explain the apparent conflict with the no trade theorem.

(e) *10 points.* If you had a chance to play this game, and you got to decide if a trade would take place, would you decide to trade? If so, would you look at your note before asking?

(3) *Bonus: a riddle with* **both** *prisoners* **and** *hats.* There are n prisoners standing in a line. The first can observe all the rest. The second can observe all except the first, etc. Each is given either a red or a blue hat which he cannot see. Now, starting with the first prisoner, each in turn has to guess the color of his hat, a guess which the rest can hear.

(a) *3 points.* The prisoners are allowed to decide on a strategy ahead of time. Find one in which they all guess the color correctly, except maybe the first prisoner.

(b) *5 points.* Do the same, but for an infinite line of prisoners.

(c) *7 points.* For an infinite line of deaf prisoners, find a strategy in which at most finitely many of them guess incorrectly.

(d) *10 points.* For an infinite line of deaf prisoners, assume that each is assigned a hat independently and uniformly at random. Show that regardless of the strategy the prisoners agree on, each has a probability of 1/2 of guessing his hat color correctly. Explain why this means that with probability one infinitely many prisoners will guess incorrectly. Resolve the apparent conflict with your answer from the previous question.