PS/Ec 172, Homework 1
Due Wednesday, April 6th

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) *Backward induction.* Let $G = (N, A, H, O, o, P, \{\preceq_i\}_{i \in N})$ be an extensive form game with perfect information. We showed in class that there exists a subgame perfect equilibrium for G that is of the form $a^*\sigma$, where, for each $b \in A(\emptyset)$, $\sigma(G(b))$ is a subgame perfect equilibrium of $G(b)$, and where a^* is a $\preceq_{P(\emptyset)}$-maximizer of $o(a^*\sigma)$.
   (a) *20 points.* Show that *every* subgame perfect equilibrium of G is of this form.

(2) *Alternating ultimata.* Anne and Jared are walking to lunch when they spot a \$7 note in a tree. They both quickly realize that the only way they can reach it is by having one of them climb on the shoulders of the other. It thus remains for them to agree on how they will divide the money between them once they retrieve it.
   Anne first makes an offer to Jared. Her offer has to be one of $\{\$0, \$1, \$2, \$3, \$4, \$5, \$6, \$7\}$, corresponding to the size of Jared's share.
   If Jared accepts they fetch the money and split it accordingly. If Jared rejects then he makes an offer to Anne. If she accepts they fetch the money and split it accordingly. Otherwise she makes an offer again, etc. At most T offers can be made before they have to go to class and the game must end. If T offers are rejected then the money is left in the tree.
   (a) *20 points.* Consider the case that $T = 31415$. Construct a Nash equilibrium in which they both miss lunch and receive no money. What are their possible utilities in subgame perfect equilibria? Hint: use backward induction.
   (b) *20 points.* Repeat for the case that $T = 3141592$.

(3) *Deviations in infinite games.* In the lecture notes we prove (Theorem 1.5) a *one deviation principle*: a strategy profile in a finite horizon game is a Nash equilibrium iff it is impossible to profit by deviating at just *one* history. Read the formal theorem statement and its proof. In this problem we will explain why we prove this claim only for finite horizon games.
   Consider the following game in which there are two players and infinitely many time periods. In the odd time periods player 1 has to decide whether to stop or continue. In the even time periods player 2 has to make the same decision. If any player decides to stop at any period then the utility is 0 for both players. If both players always continue then the utility is 1 for both.
   Consider the strategy profile s in which player 1 always stops, and player 2 always continues.

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(a) *20 points.* Explain why s is not an equilibrium.

(b) *20 points.* Explain why no subgame has a profitable deviation from s that differs from s in only one move. (I.e., the one deviation principle does not apply to this game).

(4) *Bonus question.* Recall that a set S is *countable* if there exists a bijection (one-to-one correspondence) $f\colon \mathbb{N} \to S$ from the natural numbers to S. Recall also that the interval $[0,1]$ is not countable (Cantor, 1874). We will prove this using a game. This proof is due to Grossman and Turett (1998).

Consider the following game. Fix a subset $S \subseteq [0,1]$, and let $a_0 = 0$ and $b_0 = 1$. The players Al and Betty take alternating turns, starting with Al. In Al's n^{th} turn he has to choose some a_n which is strictly larger than a_{n-1}, but strictly smaller than b_{n-1}. At Betty's n^{th} turn she has to choose a b_n that is strictly smaller than b_{n-1} but strictly larger than a_n. Thus the sequence $\{a_n\}$ is strictly increasing and the sequence $\{b_n\}$ is strictly decreasing, and furthermore $a_n < b_m$ for every $n, m \in \mathbb{N}$.

Since a_n is a bounded increasing sequence, it has a limit $a = \lim_n a_n$. Al wins the game if $a \in S$, and Betty wins the game otherwise.

(a) *10 points.* Let S be countable, so we can write it as $S = \{s_1, s_2, \ldots\}$. Prove that the following is a winning strategy for Betty: in her n^{th} turn she chooses $b_n = s_n$ if she can (i.e., if $a_n < s_n < b_{n-1}$). Otherwise she chooses any other allowed number.

(b) *10 points.* Explain why this implies that $[0,1]$ is uncountable.