MA144A, Homework 7
Due by Noon on Monday, November 27th

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

1) Use the Ergodic Theorem to prove the strong law of large numbers: let \((X_1, X_2, \ldots)\) be i.i.d. random variables in \(L^1\). Then
\[
P \left[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} X_k = \mathbb{E}[X_1] \right] = 1.
\]

2) Prove the following claim.
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, with \(T: \Omega \to \Omega\) an ergodic measure preserving transformation. If \(Z: \Omega \to \mathbb{R}\) is a \(T\)-invariant random variable (i.e., \(Z(\omega) = Z(T(\omega))\) for all \(\omega \in \Omega\)) then there is some \(z \in \mathbb{R}\) such that \(\mathbb{P}[Z = z] = 1\).

3) Prove that irrational rotations are ergodic transformations of \([0, 1)\), equipped with the Lebesgue measure. Hint: use the fact that a random variable \(X\) on \([0, 1)\) is uniquely determined by the values of its characteristic function \(\varphi_X(k) = \mathbb{E}[e^{2\pi i k X}]\) for integer \(k\).

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