Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

1. **The Law of Total Expectation.** Let $X \in L^2$. Show that if $G_2 \subseteq G_1$ then $\mathbb{E} [\mathbb{E} [X | G_1] | G_2] = \mathbb{E} [X | G_2]$.

2. Show that if an irreducible chain is not aperiodic then for every $x \in S$ there is a $k \in \mathbb{N}$ so that $P^m(x, x) = 0$ for all $m$ not divisible by $k$.

3. Show that the simple random walk on a directed graph is irreducible iff the graph is strongly connected.

4. Show that the simple random walk on an undirected connected graph is aperiodic iff the graph is not bipartite.

5. Prove that the simple random walk on $\mathbb{Z}^2$ (given by $P(x, y) = \frac{1}{4} \mathbbm{1}_{\{|x-y|=1\}}$) is recurrent, but that the simple random walk on $\mathbb{Z}^d$ (given by $P(x, y) = \frac{1}{2d} \mathbbm{1}_{\{|x-y|=1\}}$) is transient for all $d \geq 3$.

6. **Coupling.** Fix $1/2 < q < p < 1$, and consider two random walks on $\mathbb{Z}$: $(X^q_0, X^q_1, \ldots)$ with transition matrix

$$P_q(x, y) = \begin{cases} 
q & \text{if } y = x + 1 \\
1 - q & \text{if } y = x - 1 \\
0 & \text{otherwise},
\end{cases}$$

and $(X^p_0, X^p_1, \ldots)$ defined analogously. Show that for every $z \in \mathbb{Z}$ and $n \in \mathbb{N}$ it holds that $\mathbb{P} [X^q_n \leq z] \geq \mathbb{P} [X^p_n \leq z]$.

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