MA144A, Homework 2
Due by Noon on Tuesday, October 18th

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) Consider the Bernoulli measure on \{0, 1\}^\mathbb{N} which is the unique extension (as we have shown in class / homework) of \(\mu_0 : \mathcal{A}_{\text{clopen}} \to [0, 1]\) with

\[\mu_0(A_x) = 2^{-|x|}.\]

(a) For \(n \in \mathbb{N}\) let the real random variable \(X_n : \Omega \to \mathbb{R}\) be given by \(X_n(\omega) = \omega_n\). Show that \(X_n\) is indeed a random variable, and prove that the random variables \((X_1, X_2, \ldots)\) are independent.

(b) Let the real random variable \(Y : \Omega \to \mathbb{R}\) be given by

\[Y(\omega) = \sum_{n=1}^{\infty} 2^{-n}\omega_n.\]

Prove that \(Y\) is indeed a random variable, and that its law is the uniform (Lebesgue) measure on \([0, 1)\). (Hint: you can use the fact that the algebra of diadic intervals \([m2^{-n}, (m + 1)2^{-n})\), \(m < 2^n\) generates the Borel sigma-algebra on \([0, 1)\).)

(c) Construct independent random variables \((Y_1, Y_2, \ldots)\), each with the uniform distribution on \([0, 1)\).

(2) Consider a casino in which there is an infinite sequence of slot machines. On machine \(n\) a gambler gains a dollar with probability \(1 - 2^{-n}\), and loses \(2^n\) dollars with probability \(2^{-n}\). Consider a gambler who starts out with 0 dollars in her account, and proceeds to gamble on each machine in turn. That is, her balance is the sequence of random variables \(\{X_n\}\) with \(X_0 = 0\) and \(X_{n+1} = X_n + Y_{n+1}\), where \(\{Y_n\}\) is a sequence of independent random variables with \(\mathbb{P}[Y_n = 1] = 1 - 2^{-n}\) and \(\mathbb{P}[Y_n = -2^n] = 2^{-n}\).

Omer Tamuz. Email: tamuz@caltech.edu.
(a) Show that $E[X_n] < 0$ for all $n > 0$.

(b) Show that $P[\lim X_n = \infty] = 1$.

(3) Prove the Dominated Convergence Theorem using the Monotone Convergence Theorem (as they appear in the lecture notes).