MA144A, Homework 1
Due by Noon on Thursday, October 12th

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) Let $I$ be a set, and let $\{F_i\}_{i \in I}$ be a collection of sigma-algebras of subsets of $\Omega$. Show that $\bigcap_{i \in I} F_i$ is a sigma-algebra.

(2) Let $\mathcal{C}$ be a collection of subsets of $\Omega$. Show that there exists a unique minimal (under inclusion) sigma-algebra $\mathcal{F} \supseteq \mathcal{C}$.

(3) Consider $\mathcal{A}_{\text{clopen}}$, as defined in Example 2.5 in the lecture notes. Prove that there exists an additive $\mu_0: \mathcal{A}_{\text{clopen}} \to [0, 1]$ with

$$\mu_0(A_x) = 2^{-|x|}.$$  

*Bonus:* Prove that there exists a countably additive such $\mu_0$, so that whenever $(A_1, A_2, \ldots)$ are disjoint elements of $\mathcal{A}_{\text{clopen}}$ with $\bigcup_n A_n \in \mathcal{A}_{\text{clopen}}$ then $\mu_0(\bigcup_n A_n) = \sum_n \mu_0(A_n)$.

(4) Let $\Omega = \mathbb{Z}$ and $\mathcal{F}$ be the power set of $\mathbb{Z}$. A finitely additive probability measure $\mu: \mathcal{F} \to [0, 1]$ is *shift-invariant* if for all $A \in \mathcal{F}$ it holds that $\mu(A) = \mu(A + 1)$, where $A + 1 = \{n + 1 : n \in A\}$.

(a) Prove that if $\mu$ is shift-invariant then it is not sigma-additive.

(b) *Bonus.* Prove that there exists such a shift-invariant $\mu$.

(5) Let $(\Omega, \mathcal{F})$ and $(\Theta, \mathcal{G})$ be measurable spaces. Prove that $f: \Omega \to \Theta$ is measurable iff the collection

$$\sigma(f) = \{f^{-1}(A) : A \in \mathcal{G}\}$$

is a sub-sigma-algebra of $\mathcal{F}$.

(6) *Bonus.* Prove or disprove: in the setting of riddle 2 in the lecture notes, for any (measurable) choice of functions $f_n: \{0, 1\}^\mathbb{N} \to \{0, 1\}$, it holds that $\mathbb{P}[Y_n = X_n \text{ for all } n > 1] < 1$.

Omer Tamuz. Email: tamuz@caltech.edu.