Planes and distances in $\mathbb{R}^3$

Based on lecture notes by James McKernan

How do we represent a plane $\Pi$ in $\mathbb{R}^3$? In fact the best way to specify a plane is to give a normal vector $\vec{n}$ to the plane and a point $P_0$ on the plane. Then if we are given any point $P$ on the plane, the vector $\overrightarrow{P_0P}$ is a vector in the plane, so that it must be orthogonal to the normal vector $\vec{n}$. Algebraically, we have

$$\overrightarrow{P_0P} \cdot \vec{n} = 0.$$ 

Let’s write this out as an explicit equation.

**Blackboard 1.** Suppose that the point $P_0 = (x_0, y_0, z_0)$, $P = (x, y, z)$ and $\vec{n} = (A, B, C)$. Then we have

$$(x - x_0, y - y_0, z - z_0) \cdot (A, B, C) = 0.$$ 

Expanding, we get

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

We can also rewrite this as

$$Ax + By + Cz = D.$$ 

Here

$$D = Ax_0 + By_0 + Cz_0 = (A, B, C) \cdot (x_0, y_0, z_0) = \vec{n} \cdot \overrightarrow{OP_0}.$$ 

This is perhaps the most common way to write down the equation of a plane.

**Example 2.** What is the equation of a plane passing through $(1, -1, 2)$, with normal vector $\vec{n} = (2, 1, -1)$? We have

$$(x - 1, y + 1, z - 2) \cdot (2, 1, -1) = 0.$$ 

So

$$2(x - 1) + y + 1 - (z - 2) = 0,$$

so that in other words,

$$2x + y - z = -1.$$ 

**Example 3.**

$$3x - 4y + 2z = 6,$$

is the equation of a plane. A vector normal to the plane is $(3, -4, 2)$.

A line is determined by two points; a plane is determined by three points, provided those points are not collinear (that is, provided they don’t lie on the same line). So given three points $P_0$, $P_1$ and $P_2$, what is the equation of the plane $\Pi$ containing $P_0$, $P_1$ and $P_2$? Well, we would like to find a vector $\vec{n}$ orthogonal to any vector in the plane. Note that $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_2}$ are two vectors in the plane, which by assumption are not parallel. The cross product is a vector which is orthogonal to both vectors.

**Blackboard 4.** Let $P_0, P_1$, and $P_2$ be three vectors in $\mathbb{R}^3$. Let

$$\vec{n} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}.$$ 

Then the equation of the plane containing the three points is

$$\overrightarrow{P_0P} \cdot (\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}) = 0.$$
We can rewrite this a little, using $\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0}$. Expanding and rearranging gives
\[
\overrightarrow{OP} \cdot (\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}) = \overrightarrow{OP_0} \cdot (\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}).
\]
Note that both sides involve the triple scalar product.

**Example 5.** What is the equation of the plane $\Pi$ through the three points, $P_0 = (1,1,1)$, $P_1 = (2,-1,0)$ and $P_2 = (0,-1,-1)$?

$\overrightarrow{P_0P_1} = (1,-2,-1)$ and $\overrightarrow{P_0P_2} = (-1,-2,-2)$.

Now a vector orthogonal to both of these vectors is given by the cross product:
\[
\vec{n} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \begin{vmatrix} i & j & k \\ 1 & -2 & -1 \\ -1 & -2 & -2 \end{vmatrix} = i(-2-2)-j(1+2)+k(-2+1) = 2\hat{i} + 3\hat{j} - 4\hat{k}.
\]

Note that $\vec{n} \cdot \overrightarrow{P_0P_1} = 2 - 6 + 4 = 0$, as expected. It follows that the equation of $\Pi$ is
\[
2(x - 1) + 3(y - 1) - 4(z - 1) = 0,
\]
so that
\[
2x + 3y - 4z = 1.
\]
For example, if we plug in $P_2 = (0,-1,-1)$, then
\[
2 \cdot 0 + 3 \cdot -1 + 4 = 1,
\]
as expected.

**Example 6.** What is the parametric equation for the line $l$ given as the intersection of the two planes $2x - y + z = 1$ and $x + y - z = 2$?

Well we need two points on the intersection of these two planes. If we set $z = 0$, then we get the intersection of two lines in the $xy$-plane,
\[
\begin{align*}
2x - y &= 1 \\
x + y &= 2.
\end{align*}
\]
Adding these two equations we get $3x = 3$, so that $x = 1$. It follows that $y = 1$, so that $P_0 = (1,1,0)$ is a point on the line.

Now suppose that $y = 0$. Then we get
\[
\begin{align*}
2x + z &= 1 \\
x - z &= 2.
\end{align*}
\]
As before this says $x = 1$ and so $z = -1$. So $P_1 = (1,0,-1)$ is a point on $l$.

\[
\overrightarrow{P_0P} = t\overrightarrow{P_0P_1},
\]
for some parameter $t$. Expanding
\[
(x - 1, y - 1, z) = t(0, -1, -1),
\]
We can also calculate distances between planes and points, lines and points, and lines and lines.

**Example 7.** What is the distance between the plane \( x - 2y + 3z = 4 \) and the point \( P = (1, 2, 3) \)?

Call the closest point \( R \). Then \( \overrightarrow{PR} \) is orthogonal to every vector in the plane, that is, \( \overrightarrow{PR} \) is normal to the plane. Note that \( \vec{n} = (1, -2, 3) \) is normal to the plane, so that \( \overrightarrow{PR} \) is parallel to \( \vec{n} \).

Pick any point \( Q \) belonging to the plane. Then the triangle \( PQR \) has a right angle at \( R \), so that \( \overrightarrow{PR} = \pm \text{proj}_{\vec{n}} \overrightarrow{PQ} \).

When \( x = z = 0 \), then \( y = -2 \), so that \( Q = (0, -2, 0) \) is a point on the plane.

\[
\overrightarrow{PQ} = (-1, -4, -3).
\]

Now
\[
\|\vec{n}\|^2 = \vec{n} \cdot \vec{n} = 1^2 + 2^2 + 3^2 = 14 \quad \text{and} \quad \vec{n} \cdot \overrightarrow{PQ} = -2.
\]

So
\[
\text{proj}_{\vec{n}} \overrightarrow{PQ} = \frac{1}{14}(-1, 2, -3).
\]

So the distance is
\[
\frac{1}{14} \sqrt{14}.
\]

Here is another way to proceed. The line through \( P \), pointing in the direction \( \vec{n} \), will intersect the plane at the point \( R \). Now this line is given parametrically as

\[
(x - 1, y - 2, z - 3) = t(1, -2, 3),
\]

so that
\[
(x, y, z) = (t + 1, 2 - 2t, 3 + 3t).
\]

The point \( R \) corresponds to

\[
(t + 1) - 2(2 - 2t) + 3(3 + 3t) = 4,
\]

so that
\[
14t = -2 \quad \text{that is} \quad t = -\frac{1}{7}.
\]

So the point \( R \) is
\[
\frac{1}{7}(6, 16, 18).
\]

It follows that
\[
\overrightarrow{PR} = \frac{1}{7}(-1, 2, -3),
\]

the same answer as before (phew!).
Example 8. What is the distance between the two lines

\((x, y, z) = (t - 2, 3t + 1, 2 - t)\) and \((x, y, z) = (2t - 1, 2 - 3t, t + 1)\)?

If the two closest points are \(R\) and \(R'\) then \(\overrightarrow{RR'}\) is orthogonal to the direction of both lines. Now the direction of the first line is \((1, 3, -1)\) and the direction of the second line is \((2, -3, 1)\). A vector orthogonal to both is given by the cross product:

\[
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 3 & -1 \\
2 & -3 & 1 \\
\end{vmatrix} = -3\mathbf{j} - 9\mathbf{k}.
\]

To simplify some of the algebra, let’s take

\(\mathbf{n} = \mathbf{j} + 3\mathbf{k},\)

which is parallel to the vector above, so that it is still orthogonal to both lines.

It follows that \(\overrightarrow{RR'}\) is parallel to \(\mathbf{n}\). Pick any two points \(P\) and \(P'\) on the two lines. Note that the length of the vector

\[
\text{proj}_{\mathbf{n}} \overrightarrow{P'P},
\]

is the distance between the two lines.

Now if we plug in \(t = 0\) to both lines we get

\(P' = (-2, 1, 2)\) and \(P = (-1, 2, 1)\).

So

\(\overrightarrow{P'P} = (1, 1, -1)\).

Then

\[\|\mathbf{n}\|^2 = 1^2 + 3^2 = 10\quad \text{and} \quad \mathbf{n} \cdot \overrightarrow{P'P} = -2.\]

It follows that

\[
\text{proj}_{\mathbf{n}} \overrightarrow{P'P} = \frac{-2}{10} (0, 1, 3) = \frac{-1}{5} (0, 1, 3).
\]

and so the distance between the two lines is

\[\frac{1}{5} \sqrt{10}.\]