Line integrals

Based on lecture notes by James McKernan

Let $I$ be an open interval and let
\[ \vec{r}: I \rightarrow \mathbb{R}^n, \]
be a parametrised differentiable curve. If $[a, b] \subset I$ then let $C = \vec{r}([a, b])$ be the image of $[a, b]$ and let $f: C \rightarrow \mathbb{R}$ be a function.

**Definition 1.** The line integral of $f$ along $C$ is
\[ \oint_C f \, ds = \int_a^b f(\vec{r}(u)) \| \vec{r}'(u) \| \, du. \]

Let $u: J \rightarrow I$ be a diffeomorphism between two open intervals. Suppose that $u$ is $C^1$. We think of $u$ as a coordinate transformation $u = u(t)$; we want to transform from the variable $u$ to the variable $t$.

**Definition 2.** We say that $u$ is **orientation-preserving** if $u'(t) > 0$ for every $t \in J$.

We say that $u$ is **orientation-reversing** if $u'(t) < 0$ for every $t \in J$.

Notice that $u$ is always either orientation-preserving or orientation-reversing (this is a consequence of the intermediate value theorem, applied to the continuous function $u'(t)$).

Define a function $\vec{y}: J \rightarrow \mathbb{R}^n$, by composition,
\[ \vec{y}(t) = \vec{r}(u(t)), \]
so that $\vec{y} = \vec{r} \circ u$.

Now suppose that $u([c, d]) = [a, b]$. Then $C = \vec{y}([c, d])$, so that $\vec{y}$ gives another parametrisation of $C$.

**Lemma 3.**
\[ \int_a^b f(\vec{r}(u)) \| \vec{r}'(u) \| \, du = \int_c^d f(\vec{y}(t)) \| \vec{y}'(t) \| \, dt. \]

**Proof.** We deal with the case that $u$ is orientation-preserving. The case that $u$ is orientation-reversing is similar.

As $u$ is orientation-preserving, we have $u(c) = a$ and $u(d) = b$ and so,
\[ \int_c^d f(\vec{y}(t)) \| \vec{y}'(t) \| \, dt = \int_c^d f(\vec{r}(u(t))) \| u'(t) \| \vec{r}'(u(t)) \| \, dt \]
\[ = \int_c^d f(\vec{r}(u(t))) \| \vec{r}'(u(t)) \| u'(t) \, dt \]
\[ = \int_a^b f(\vec{r}(u)) \| \vec{r}'(u) \| \, du. \]

Now suppose that we have a vector field on $C$,
\[ \vec{F}: C \rightarrow \mathbb{R}^n. \]
Definition 4. The line integral of $\vec{F}$ along $C$ is

$$\oint_C \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{r}(u)) \cdot \vec{r}'(u) \, du.$$ 

Note that now the orientation is very important:

Lemma 5.

$$\int_a^b \vec{F}(\vec{r}(u)) \cdot \vec{r}'(u) \, du = \begin{cases} \int_c^d \vec{F}(\vec{y}(t)) \cdot \vec{y}'(t) \, dt & u'(t) > 0 \\ -\int_c^d \vec{F}(\vec{y}(t)) \cdot \vec{y}'(t) \, dt & u'(t) < 0 \end{cases} \quad \blacksquare$$

Proof. We deal with the case that $u$ is orientation-reversing. The case that $u$ is orientation-preserving is similar and easier.

As $u$ is orientation-reversing, we have $u(c) = b$ and $u(d) = a$ and so,

$$\int_c^d \vec{F}(\vec{y}(t)) \cdot \vec{y}'(t) \, dt = \int_c^d \vec{F}(\vec{r}(u(t))) \cdot \vec{r}'(u(t))u'(t) \, dt$$

$$= \int_b^a \vec{F}(\vec{r}(u)) \cdot \vec{r}'(u) \, du$$

$$= -\int_a^b \vec{F}(\vec{r}(u)) \cdot \vec{r}'(u) \, du. \quad \blacksquare$$

Example 6. If $C$ is a piece of wire and $f(\vec{r})$ is the mass density at $\vec{r} \in C$, then the line integral

$$\int_C f \, ds,$$

is the total mass of the curve. Clearly this is always positive, whichever way you parametrise the curve.

Example 7. If $C$ is an oriented path and $\vec{F}(\vec{r})$ is a force field, then the line integral

$$\oint_C \vec{F} \cdot d\vec{s},$$

is the work done when moving along $C$. If we reverse the orientation, then the sign flips. For example, imagine $C$ is a spiral staircase and $\vec{F}$ is the force due to gravity. Going up the staircase costs energy and going down we gain energy.