Blackboard 1. A function \( f : U \longrightarrow V \) between two open subsets of \( \mathbb{R}^n \) is called a \textbf{diffeomorphism} if:

1. \( f \) is a bijection,
2. \( f \) is differentiable, and
3. \( f^{-1} \) is differentiable.

By definition of the inverse function, \( f \circ f^{-1} : V \longrightarrow V \) and \( f^{-1} \circ f : U \longrightarrow U \) are both the identity function, so that
\[
(f \circ f^{-1})(\vec{y}) = \vec{y} \quad \text{and} \quad (f^{-1} \circ f)(\vec{x}) = \vec{x}.
\]
It follows that
\[
Df(\vec{x})Df^{-1}(\vec{y}) = I_n \quad \text{and} \quad Df^{-1}(\vec{y})Df(\vec{x}) = I_n,
\]
by the chain rule. It follows that
\[
\det(Df) \neq 0.
\]
and
\[
Df^{-1} = (Df)^{-1}.
\]

Example 2. Let \( g(r, \theta) = (r \cos \theta, r \sin \theta) \). Then
\[
Dg(r, \theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{pmatrix},
\]
so that
\[
\det Dg(r, \theta) = r.
\]

Theorem 3. Let \( g : U \longrightarrow V \) be a diffeomorphism between open subsets of \( \mathbb{R}^2 \),
\[
g(u, v) = (x(u, v), y(u, v)).
\]
Let \( D^* \subset U \) be a region and let \( D = f(D^*) \subset V \). Let \( f : D \longrightarrow \mathbb{R} \) be a function. Then
\[
\int_D f(x, y) \, dx \, dy = \int_{D^*} f(x(u, v), y(u, v)) |\det Dg(u, v)| \, du \, dv.
\]
It is convenient to use the following notation:
\[
\frac{\partial(x, y)}{\partial(u, v)}(u, v) = \det Dg(u, v).
\]
The LHS is called the \textbf{Jacobian}. Note that
\[
\frac{\partial(x, y)}{\partial(u, v)}(u, v) = \left( \frac{\partial(u, v)}{\partial(x, y)}(x, y) \right)^{-1}.
\]

Example 4. Let \( g(r, \theta) = (r \cos \theta, r \sin \theta) \). Then
\[
\det Dg(r, \theta) = r,
\]
and so
\[
\int_D f(x, y) \, dx \, dy = \int_{D^*} f(x(u, v), y(u, v)) r \, du \, dv.
\]
Example 5. There is no simple expression for the integral of $e^{-x^2}$. However it is possible to compute the following integral

$$I = \int_{-\infty}^{\infty} e^{-x^2} \, dx.$$  

(In what follows, we will ignore issues relating to the fact that the integrals are improper; in practice all integrals converge). Instead of computing $I$, we compute $I^2$,

$$I^2 = \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right) \left( \int_{-\infty}^{\infty} e^{-y^2} \, dy \right)$$

$$= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{-x^2-y^2} \, dx \right) \, dy$$

$$= \iint_{\mathbb{R}^2} e^{-x^2-y^2} \, dx \, dy$$

$$= \iint_{\mathbb{R}^2} r e^{-r^2} \, dr \, d\theta$$

$$= \int_{0}^{\infty} \left( \int_{0}^{2\pi} r e^{-r^2} \, d\theta \right) \, dr$$

$$= \int_{0}^{\infty} e^{-r^2} \left( \int_{0}^{2\pi} \, d\theta \right) \, dr$$

$$= 2\pi \int_{0}^{\infty} e^{-r^2} \, dr$$

$$= 2\pi \left[ -\frac{e^{-r^2}}{2} \right]_{0}^{\infty}$$

$$= \pi.$$

So $I = \sqrt{\pi}$.

Example 6. Find the area of the region $D$ bounded by the four curves

$xy = 1, \quad xy = 3, \quad y = x^3, \quad$ and $\quad y = 2x^3$.

Define two new variables,

$$u = \frac{x^3}{y} \quad \text{and} \quad v = xy.$$

Then $D$ is a rectangle in $uv$-coordinates,

$$D^* = [1/2, 1] \times [1, 3]$$

Now for the Jacobian we have

$$\frac{\partial (u, v)}{\partial (x, y)}(x, y) = \begin{vmatrix} 3y^2 & -x^3 \cr y & x \cr \end{vmatrix} = \frac{4x^3}{y} = 4u.$$

It follows that

$$\frac{\partial (x, y)}{\partial (u, v)}(u, v) = \frac{1}{4u}.$$
This is nowhere zero. In fact note that we can solve for $x$ and $y$ explicitly in terms of $u$ and $v$.

$$uv = x^4 \quad \text{and} \quad y = \frac{x}{v}.$$  

So

$$x = (uv)^{1/4} \quad \text{and} \quad y = u^{-1/4}v^{3/4}.$$  

Therefore

$$\text{area}(D) = \int\int_{D} dx \, dy = \int\int_{D^*} \frac{1}{4u} \, du \, dv$$

$$= \frac{1}{4} \int_{1}^{3} \left( \int_{1/2}^{1} \frac{1}{u} \, du \right) \, dv$$

$$= \frac{1}{4} \int_{1}^{3} [\ln u]_{1/2}^{1} \, dv$$

$$= \frac{1}{4} \int_{1}^{3} \ln 2 \, dv$$

$$= \frac{1}{2} \ln 2.$$

**Theorem 7.** Let $g: U \rightarrow V$ be a diffeomorphism between open subsets of $\mathbb{R}^3$, 

$$g(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w)).$$

Let $W^* \subset U$ be a region and let $W = f(W^*) \subset V$. Let $f: W \rightarrow \mathbb{R}$ be a function.

Then

$$\int\int\int_{W} f(x, y, z) \, dx \, dy \, dz = \int\int\int_{W^*} f(x(u, v, w), y(u, v, w), z(u, v, w)) |\det Dg(u, v, w)| \, du \, dv \, dw.$$  

As before, it is convenient to introduce more notation:

$$\frac{\partial (x, y, z)}{\partial (u, v, w)}(u, v, w) = \det Dg(u, v, w).$$