Maxima and minima: II
Based on lecture notes by James McKernan

To see how to maximize and minimize a function on the boundary, let’s consider a concrete example.

Let
\[ K = \{ (x, y) \mid x^2 + y^2 \leq 2 \} \]

Then \( K \) is compact. Let
\[ f : K \to \mathbb{R} \]
be the function \( f(x, y) = xy \). Then \( f \) is continuous and so \( f \) achieves its maximum and minimum.

I. Let’s first consider the interior points. Then
\[ \nabla f(x, y) = (y, x) \]
so that \((0, 0)\) is the only critical point. The Hessian of \( f \) is
\[ Hf(x, y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \]

\( d_1 = 0 \) and \( d_2 = -1 \neq 0 \) so that \((0, 0)\) is a saddle point.

It follows that the maxima and minima of \( f \) are on the boundary, that is, the set of points
\[ C = \{ (x, y) \mid x^2 + y^2 = 2 \} . \]

II. Let \( g : \mathbb{R}^2 \to \mathbb{R} \) be the function \( g(x, y) = x^2 + y^2 \). Then the circle \( C \) is a level curve of \( g \). The original problem asks to maximize and minimize
\[ f(x, y) = xy \quad \text{subject to} \quad g(x, y) = x^2 + y^2 = 2 . \]

One way to proceed is to use the second equation to eliminate a variable. The method of Lagrange multipliers does exactly the opposite. Instead of eliminating a variable we add one more variable, traditionally called \( \lambda \).

In general, say we want to maximize \( f(x, y) \) subject to \( g(x, y) = c \). Then at a maximum point \( p \) it won’t necessarily be the case that \( \nabla f(p) = 0 \), but it will be the case that the directional derivative \( \nabla f(p) \cdot \hat{n} \) will be zero for any \( \hat{n} \) that is in the direction of the level set \( g(x, y) = c \). Since \( \nabla g \) is orthogonal to this level set, at a maximum point \( p \) it will be the case that \( \nabla f(p) \) and \( \nabla g(p) \) will be at the same direction, or that \( \nabla f(p) = \lambda \nabla g(p) \) for some \( \lambda \).

Consider the function
\[ h(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c) . \]

Let’s see what happens at points where \( \nabla h = 0 \). Taking the derivatives with respect to \( x \) and \( y \) and equating to zero yields
\[ \nabla f(x, y) - \lambda \nabla g(x, y) = 0 , \]
which is what we’re looking for. Taking the derivative with respect to \( \lambda \) and equating to zero yields
\[ g(x, y) = c , \]
which is the second condition we need. Hence finding a point in which \( \nabla h = 0 \) is the same as solving our problem.

So now let’s maximize and minimize
\[ h(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - 2) = xy - \lambda(x^2 + y^2 - 2) . \]
We find the critical points of $h(x, y, \lambda)$:

\[
\begin{align*}
y &= 2 \lambda x \\
x &= 2 \lambda y \\
2 &= x^2 + y^2.
\end{align*}
\]

First note that if $x = 0$ then $y = 0$ and $x^2 + y^2 = 0 \neq 2$, impossible. So $x \neq 0$. Similarly one can check that $y \neq 0$ and $\lambda \neq 0$. Divide the first equation by the second:

\[
\frac{y}{x} = \frac{x}{y},
\]

so that $y^2 = x^2$. As $x^2 + y^2 = 2$ it follows that $x^2 = y^2 = 1$. So $x = \pm 1$ and $y = \pm 1$. This gives four potential points $(1, 1), (-1, 1), (1, -1), (-1, -1)$. Then the maximum value of $f$ is 1, and this occurs at the first and the last point. The minimum value of $f$ is $-1$, and this occurs at the second and the third point.

One can also try to parametrize the boundary:

\[
\vec{r}(t) = \sqrt{2}(\cos t, \sin t).
\]

So we maximize the composition

\[
h : [0, 2\pi] \longrightarrow \mathbb{R},
\]

where $h(t) = 2 \cos t \sin t$. As $I = [0, 2\pi]$ is compact, $h$ has a maximum and minimum on $I$. When $h'(t) = 0$, we get

\[
\cos^2 t - \sin^2 t = 0.
\]

Note that the LHS is $\cos 2t$, so we want

\[
\cos 2t = 0.
\]

It follows that $2t = \pi/2 + 2m\pi$, so that

\[
t = \pi/4, \quad 3\pi/4, \quad 5\pi/4, \quad \text{and} \quad 7\pi/4.
\]

These give the four points we had before.

What is the closest point to the origin on the surface

\[
F = \{ (x, y, z) \in \mathbb{R}^3 \mid x \geq 0, y \geq 0, z \geq 0, xyz = p \}.
\]

So we want to minimize the distance to the origin on $F$. The first trick is to minimize the square of the distance. In other words, we are trying to minimize $f(x, y, z) = x^2 + y^2 + z^2$ on the surface

\[
F = \{ (x, y, z) \in \mathbb{R}^3 \mid x \geq 0, y \geq 0, z \geq 0, xyz = p \}.
\]

In words, given three numbers $x \geq y \geq 0$ and $z \geq 0$ whose product is $p > 0$, what is the minimum value of $x^2 + y^2 + z^2$?

Now $F$ is closed but it is not bounded, so it is not even clear that the minimum exists.

Let’s use the method of Lagrange multipliers. Let

\[
h : \mathbb{R}^4 \longrightarrow \mathbb{R},
\]

be the function

\[
h(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(xyz - p).
\]
We look for the critical points of $h$:

$$
2x = \lambda yz \\
2y = \lambda xz \\
2z = \lambda xy \\
p = xyz.
$$

Once again, it is not possible for any of the variables to be zero. Taking the product of the first three equations, we get

$$8(xyz) = \lambda^3(x^2y^2z^2).$$

So, dividing by $xyz$ and using the last equation, we get

$$8 = \lambda^3 p,$$

that is

$$\lambda = \frac{2}{p^{1/3}}.$$  

Taking the product of the first two equations, and dividing by $xy$, we get

$$4 = \lambda^2 z^2,$$

so that

$$z = p^{1/3}.$$  

So $h(x, y, z, \lambda)$ has a critical point at

$$(x, y, z, \lambda) = \left(\frac{p^{1/3}}{3}, \frac{p^{1/3}}{3}, \frac{p^{1/3}}{3}, \frac{2}{p^{1/3}}\right).$$

We check that the point

$$(x, y, z) = \left(\frac{p^{1/3}}{3}, \frac{p^{1/3}}{3}, \frac{p^{1/3}}{3}\right),$$

is a minimum of $x^2 + y^2 + z^2$ subject to the constraint $xyz = p$. At this point the sum of the squares is

$$3p^{2/3}.$$  

Suppose that $x \geq 2p^{1/3}$. Then the sum of the squares is at least $4p^{2/3}$. Similarly if $y \geq 2p^{1/3}$ or $z \geq 2p^{1/3}$. On the other hand, the set

$$K = \{ (x, y, z) \in \mathbb{R}^3 \mid x \in [0, 2p^{1/3}], y \in [0, 2p^{1/3}], z \in [0, 2p^{1/3}], xyz = p \},$$

is closed and bounded, so that $f$ achieves it minimum on this set, which we have already decided is at

$$(x, y, z) = \left(\frac{p^{1/3}}{3}, \frac{p^{1/3}}{3}, \frac{p^{1/3}}{3}\right),$$

since $f$ is larger on the boundary. Putting all of this together, the point

$$(x, y, z) = \left(\frac{p^{1/3}}{3}, \frac{p^{1/3}}{3}, \frac{p^{1/3}}{3}\right),$$

is a point where the sum of the squares is a minimum.

Here is another such problem. Find the closest point to the origin which also belongs to the cone

$$x^2 + y^2 = z^2,$$

and to the plane

$$x + y + z = 3.$$
As before, we minimize \( f(x, y, z) = x^2 + y^2 + z^2 \) subject to \( g_1(x, y, z) = x^2 + y^2 - z^2 = 0 \) and \( g_2(x, y, z) = x + y + z = 3 \). Introduce a new function, with two new variables \( \lambda_1 \) and \( \lambda_2 \),

\[
h : \mathbb{R}^5 \rightarrow \mathbb{R},
\]

given by

\[
h(x, y, z, \lambda_1, \lambda_2) = f(x, y, z) - \lambda_1 g_1(x, y, z) - \lambda_2 g_2(x, y, z)
\]

\[
= x^2 + y^2 + z^2 - \lambda_1(x^2 + y^2 - z^2) - \lambda_2(x + y + z - 3).
\]

We find the critical points of \( h \):

\[
2x = 2\lambda_1 x + \lambda_2
\]

\[
2y = 2\lambda_1 y + \lambda_2
\]

\[
2z = -2\lambda_1 z + \lambda_2
\]

\[
z^2 = x^2 + y^2
\]

\[
3 = x + y + z.
\]

Suppose we subtract the first equation from the second:

\[
y - x = \lambda_1(y - x).
\]

So either \( x = y \) or \( \lambda_1 = 1 \). Suppose \( x \neq y \). Then \( \lambda_1 = 1 \) and \( \lambda_2 = 0 \). In this case \( z = -z \), so that \( z = 0 \). But then \( x^2 + y^2 = 0 \) and so \( x = y = 0 \), which is not possible.

It follows that \( x = y \), in which case \( z = \pm\sqrt{2}x \) and

\[
(2 \pm \sqrt{2})x = 3.
\]

So

\[
x = \frac{3}{2 \pm \sqrt{2}} = \frac{3(2 \mp \sqrt{2})}{2}.
\]

This gives us two critical points:

\[
p = \left( \frac{3(2 - \sqrt{2})}{2}, \frac{3(2 - \sqrt{2})}{2}, \frac{3\sqrt{2}(2 - \sqrt{2})}{2} \right)
\]

\[
q = \left( \frac{3(2 + \sqrt{2})}{2}, \frac{3(2 + \sqrt{2})}{2}, -\frac{3\sqrt{2}(2 + \sqrt{2})}{2} \right).
\]

Of the two, clearly the first is closest to the origin.

To finish, we had better show that this point is the closest to the origin on the whole locus

\[
F = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2, x + y + z = 3 \}.
\]

Let

\[
K = \{ (x, y, z) \in F \mid x^2 + y^2 + z^2 \leq 25 \}.
\]

Then \( K \) is closed and bounded, whence compact. So \( f \) achieves its minimum somewhere on \( K \), and so it must achieve its minimum at \( p \). Clearly outside \( f \) is at least 25 on \( F \setminus K \), and so \( f \) is a minimum at \( p \) on the whole of \( F \).