Curvature and Torsion

Based on lecture notes by James McKernan

Blackboard 1. Let \( \vec{r}: I \rightarrow \mathbb{R}^n \) be a \( C^2 \) regular curve (i.e., \( \vec{r}'(t) \neq \vec{0} \) for all \( t \)).

The curvature \( \kappa(s) \) of \( \vec{r}(s) \) is the magnitude of the vector

\[
\vec{T}'(s) = \frac{d\vec{T}(s)}{ds},
\]

and the unit normal vector \( \vec{N} \) is the unit vector pointing in the direction of \( \vec{T}'(s) \)

\[
\vec{N}(s) = \frac{\vec{T}'(s)}{||\vec{T}'(s)||}.
\]

One can try to calculate the curvature using the parameter \( t \). By the chain rule,

\[
\frac{d\vec{T}(t)}{dt} = \frac{d\vec{T}(s)}{ds} \frac{ds}{dt}.
\]

So

\[
\frac{d\vec{T}(s)}{ds} = \frac{\frac{d\vec{T}(t)}{dt}}{\frac{ds}{dt}}.
\]

The denominator is the speed. It follows that \( \vec{n} \) and \( d\vec{T}/dt \) point in the same direction.

Note that the normal vector and the unit tangent vector are always orthogonal. Indeed, more generally

**Proposition 2.** Let \( \vec{v}: I \rightarrow \mathbb{R}^n \). Then

\[
\frac{d(\vec{v} \cdot \vec{v})}{dt} = 2\vec{v}' \cdot \vec{v},
\]

and in particular if \( |\vec{r}(t)| \) is constant then \( \vec{v}' \) and \( \vec{v} \) are orthogonal.

Now,

\[
||\vec{T}(s)|| = 1.
\]

and so, as \( \vec{N}(s) \) points in the same direction as \( \vec{T}'(s) \), it follows that the tangent vector and the normal vector are orthogonal.

Blackboard 3.

\[
\vec{B}(s) = \vec{T}(s) \times \vec{N}(s).
\]

is called the binormal vector.

The three vectors \( \vec{T}(s) \), \( \vec{N}(s) \), and \( \vec{B}(s) \) are unit vectors and pairwise orthogonal, that is, these vectors are an orthonormal basis of \( \mathbb{R}^3 \). Notice that \( \vec{T}(s) \), \( \vec{N}(s) \), and \( \vec{B}(s) \) are a right handed set.

We call these vectors a **moving frame** or the Frenet-Serret frame. Now

\[
\frac{d\vec{B}(s)}{ds} \cdot \vec{B}(s) = 0,
\]

as

\[
\vec{B}(s) \cdot \vec{B}(s) = 1.
\]

It follows that

\[
\frac{d\vec{B}(s)}{ds} = 1.
\]
lies in the plane spanned by $\vec{T}(s)$ and $\vec{N}(s)$.

$$\frac{d\vec{B}}{ds}(s) \cdot \vec{T}(s) = \frac{d(\vec{T} \times \vec{N})}{ds}(s) \cdot \vec{T}(s)$$

$$= \left( \frac{d\vec{T}}{ds}(s) \times \vec{N}(s) + \vec{T}(s) \times \frac{d\vec{N}}{ds}(s) \right) \cdot \vec{T}(s)$$

$$= \kappa(s)(\vec{N}(s) \times \vec{N}(s)) \cdot \vec{T}(s) + (\vec{T}(s) \times \frac{d\vec{N}}{ds}(s)) \cdot \vec{T}(s)$$

$$= 0 + (\vec{T}(s) \times \vec{T}(s)) \cdot \frac{d\vec{N}}{ds}(s)$$

$$= 0.$$

It follows that

$$\frac{d\vec{B}}{ds}(s) \text{ and } \vec{T}(s),$$

are orthogonal, and so

$$\frac{d\vec{B}}{ds}(s) \text{ is parallel to } \vec{N}(s).$$

**Blackboard 4.** The torsion of the curve $\vec{r}(s)$ is the unique scalar $\tau(s)$ such that

$$\frac{d\vec{B}}{ds}(s) = -\tau(s)\vec{N}(s).$$

If we have a helix, the sign of the torsion distinguishes between a right handed helix and a left handed helix. The magnitude of the torsion measures how spread out the helix is (the curvature measures how tight the turns are). Now

$$\frac{d\vec{N}}{ds}(s)$$

is orthogonal to $\vec{N}(s)$, and so it is a linear combination of $\vec{T}(s)$ and $\vec{B}(s)$. In fact,

$$\frac{d\vec{N}}{ds}(s) = \frac{d(\vec{B} \times \vec{T})}{ds}(s)$$

$$= \frac{d\vec{B}}{ds}(s) \times \vec{T}(s) + \vec{B}(s) \times \frac{d\vec{T}}{ds}(s)$$

$$= -\tau(s)\vec{N}(s) \times \vec{T}(s) + \kappa(s)\vec{B}(s) \times \vec{N}(s)$$

$$= \tau(s)\vec{B}(s) - \kappa(s)\vec{T}(s)$$

$$= -\kappa(s)\vec{T}(s) + \tau(s)\vec{B}(s).$$

**Blackboard 5.** We say that $\vec{r}(t)$ is smooth if $\vec{r}(t)$ is $C^\infty$.

**Theorem 6** (Frenet Formulae). Let $\vec{r}: I \longrightarrow \mathbb{R}^3$ be a regular smooth parametrised curve. Then

$$\begin{pmatrix} \vec{T}'(s) \\ \vec{N}'(s) \\ \vec{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \vec{T}(s) \\ \vec{N}(s) \\ \vec{B}(s) \end{pmatrix}. $$
Of course, $s$ represents the arclength parameter and primes denote derivatives with respect to $s$. Notice that the $3 \times 3$ matrix $A$ appearing in (6) is skew-symmetric, that is $A^t = -A$. The way we have written the Frenet formulae, it appears that we have two $3 \times 1$ vectors; strictly speaking these are the rows of two $3 \times 3$ matrices.

**Theorem 7.** Let $I \subset \mathbb{R}$ be an open interval and suppose we are given two smooth functions

$$\kappa: I \rightarrow \mathbb{R} \quad \text{and} \quad \tau: I \rightarrow \mathbb{R},$$

where $\kappa(s) > 0$ for all $s \in I$.

Then there is a regular smooth curve $\vec{r}: I \rightarrow \mathbb{R}^3$ parametrised by arclength with curvature $\kappa(s)$ and torsion $\tau(s)$. Further, any two such curves are congruent, that is, they are the same up to translation and rotation.

Let’s consider the example of the helix:

**Example 8.**

$$\vec{r}(s) = (a \cos \frac{s}{c}, a \sin \frac{s}{c}, \frac{bs}{c}),$$

where

$$c^2 = a^2 + b^2.$$  

Let’s assume that $a > 0$. By convention $c > 0$. Then

$$\vec{T}(s) = \frac{1}{c}(-a \sin \frac{s}{c}, a \cos \frac{s}{c}, b).$$

Hence

$$\frac{dT}{ds}(s) = \frac{-a}{c^2} (\cos \frac{s}{c}, \sin \frac{s}{c}, 0) = \frac{a}{c^2} (-\cos \frac{s}{c}, -\sin \frac{s}{c}, 0) = \frac{a}{c^2} \vec{N}(s).$$

It follows that

$$\kappa(s) = \frac{a}{c^2} \quad \text{and} \quad \vec{N}(s) = (-\cos \frac{s}{c}, -\sin \frac{s}{c}, 0).$$

Finally,

$$\vec{B}(s) = \left| \begin{array}{ccc} \hat{i} & \frac{\hat{j}}{c} & \frac{\hat{k}}{c} \\ -\frac{a}{c} \sin \frac{s}{c} & \frac{a}{c} \cos \frac{s}{c} & \frac{b}{c} \\ -\cos \frac{s}{c} & -\sin \frac{s}{c} & 0 \end{array} \right|$$

It follows that

$$\vec{B}(s) = (\frac{b}{c} \sin \frac{s}{c}, -\frac{b}{c} \cos \frac{s}{c}, \frac{a}{c}).$$

Finally, note that

$$\frac{d\vec{B}}{ds}(s) = \frac{b}{c^2} (\cos \frac{s}{c}, \sin \frac{s}{c}, 0) = -\frac{b}{c^2} \vec{N}.$$  

Using this we can compute the torsion:

$$\tau(s) = \frac{b}{c^2}.$$  

It is interesting to use the torsion and curvature to characterise various geometric properties of curves. Let’s say that a parametrised differentiable curve $\vec{r}: I \rightarrow \mathbb{R}^3$ is planar if there is a plane $\Pi$ which contains the image of $\vec{r}$.

**Theorem 9.** A regular smooth curve $\vec{r}: I \rightarrow \mathbb{R}^3$ is planar if and only if the torsion is zero.
Proof. We may assume that the curve passes through the origin.

Suppose that $\vec{r}$ is planar. Then the image of $\vec{r}$ is contained in a plane $\Pi$. As the curve passes through the origin, $\Pi$ contains the origin as well. Note that the unit tangent vector $\vec{T}(s)$ and the unit normal vector $\vec{N}(s)$ are contained in $\Pi$. It follows that $\vec{B}(s)$ is a normal vector to the plane; as $\vec{B}(s)$ is a unit vector, it must be constant. But then

$$\frac{d\vec{B}}{ds}(s) = \vec{0} = 0\vec{N}(s),$$

so that the torsion is zero.

Now suppose that the torsion is zero. Then

$$\frac{dB}{ds}(s) = 0\vec{N} = \vec{0},$$

so that $\vec{B}(s) = B_0$, is a constant vector. Consider the function

$$f(s) = \vec{r}(s) \cdot \vec{B}(s) = \vec{r}(s) \cdot \vec{B}_0.$$

Then

$$\frac{df}{ds}(s) = \frac{d(\vec{r} \cdot \vec{B}_0)}{ds}(s) = \vec{T}(s) \cdot \vec{B}_0 = 0.$$

So $f(s)$ is constant. It is zero when $\vec{r}(a) = \vec{0}$ (the curve passes through the origin) so that $f(s) = 0$. But then $\vec{r}(s)$ is always orthogonal to a fixed vector, so that $\vec{r}$ is contained in a plane, that is, $C$ is planar.

It is interesting to try to figure out how to characterise curves which are contained in spheres or cylinders.