Weak equivalence of stationary actions and the entropy realization problem

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Abstract

We introduce the notion of weak containment for stationary actions of a countable group and define a natural topology on the space of weak equivalence classes. We prove that Furstenberg entropy is an invariant of weak equivalence, and moreover that it descends to a continuous function on the space of weak equivalence classes.

1 Introduction

Let G be a countable discrete group and let m be a probability measure on G. Let also (X, μ) be a standard probability space. A measurable action of G on (X, μ) is said to be m-stationary if the corresponding convolution of m with μ is equal to μ. More explicitly, this means $\sum_{g \in G} m(g) \cdot \mu(gA) = \mu(A)$ for all measurable subsets A of X. Stationary actions are automatically nonsingular, and form a natural intermediate class between measure-preserving actions and general nonsingular actions. We will write $\mathrm{Stat}(G, m, X, \mu)$ for the set of m-stationary actions of G on (X, μ). Given an action $a \in \mathrm{Stat}(G, m, X, \mu)$ we will write g^a for the nonsingular transformation of (X, μ) corresponding to g.

In [9], Kechris defined a notion of weak containment for measure-preserving actions of countable groups analogous to the standard notion of weak containment for unitary representations. The same definition can be given for stationary actions.

Definition 1.1. *Let $a, b \in \mathrm{Stat}(G, m, X, \mu)$. We say that a is weakly contained in b, in symbols $a \preceq b$, if the following condition holds. For every $\epsilon > 0$, every finite $F \subseteq G$ and every finite collection $A_1, \ldots, A_n$ of measurable subsets of X, there are measurable subsets $B_1, \ldots, B_n$ of X such that*

$$|\mu(g^a A_i \cap A_j) - \mu(g^b B_i \cap B_j)| < \epsilon$$

for all $g \in F$ and all $i, j \in \{1, \ldots, n\}$. We say that a is weakly equivalent to b, in symbols $a \sim b$, if $a \preceq b$ and $b \preceq a$.

Thus a is weakly contained in b if the statistics of a on finite partitions can be simulated arbitrary well in the action b. Weak equivalence is a much coarser relation than isomorphism; for example in [5] it is shown that all free measure-preserving actions of an amenable group are weakly equivalent. It is also better behaved from the perspective of descriptive set theory: there is in general no standard Borel structure on the set of isomorphism classes of m-stationary actions, whereas in Section 3 we will define a natural Polish topology on the set of weak equivalence classes of m-stationary actions for any pair (G, m).

In [6], Furstenberg introduced an invariant $h_m(X, \mu, a)$ which quantifies how far an m-stationary action a is from being measure-preserving. Later termed Furstenberg entropy, this is defined by

$$h_m(X, \mu, a) = -\sum_{g \in G} m(g) \cdot \int_X \log \frac{\mathrm{d}g^a\mu}{\mathrm{d}\mu}(x)\, \mathrm{d}\mu(x).$$

By Jensen's inequality, we have that $h_m(X, \mu, a)$ is nonnegative, and it is zero if and only if a is measure-preserving. The following problem has been studied in articles such as [1], [2], [4], [7], [8] and [10].

Problem 1.2 (Furstenberg entropy realization problem)**.** *For a fixed pair (G, m), describe the possible values of Furstenberg entropy on ergodic ν-stationary systems.*

The goal of this note is to establish the following theorem, which shows that the above problem can be regarded as a problem about the structure of the space of weak equivalence classes.

Theorem 1.3. *Furstenberg entropy is an invariant of weak equivalence and descends to a continuous function on the space of weak equivalence classes.*

2 A characterization of weak containment

In this section we verify that one obtains an equivalent notion if one alters the definition of weak containment to allow shifts on both sides of the intersections.

Proposition 2.1. *Let $a, b \in \mathrm{Stat}(G, m, X, \mu)$. Then the following are equivalent.*

(i) a is weakly contained b.

(ii) For any finite subset F of G, $\epsilon > 0$, and measurable subsets $A_1, \ldots, A_n$ of X, there exist measurable subsets $B_1, \ldots, B_n$ of X such that

$$\left|\mu(g^a A_i \cap h^a A_j) - \mu(g^b B_i \cap h^b B_j)\right| < \epsilon \tag{2.1}$$

for all $g, h \in F$ and $i, j \in \{1, \ldots, n\}$.

Proof. Taking $h = 1_G$ it is clear that (ii) implies (i). We now show (i) implies (ii). Suppose that $F = \{g_0, \ldots, g_m\}$ is a finite subset of G, n is a natural number, and $A_0, \ldots, A_n$ are measurable subsets of X. Without loss of generality, we can assume that $n = m$, $g_0 = 1_G$ and $A_0 = X$. Fix $\epsilon > 0$ and choose $0 < \delta < \epsilon/7$. Set $A_{i,j} = g_j^a A_i$ for $i, j \in \{1, \ldots, n\}$. In particular we have $A_{i,0} = A_i$ and $A_{i,j} = g_j^a A_{i,0}$ for $i, j \in \{1, \ldots, n\}$. By assumption there exist measurable subsets $B_{i,j}$ of X such that

$$|\mu(A_{i,j} \cap g_m^a A_{l,k}) - \mu(B_{i,j} \cap g_m^b B_{l,k})| < \delta$$

for all $i, j, k, l, m \in \{1, \ldots, n\}$. Since $A_{0,0} = X$ and $g_0 = 1_G$, we have that $\mu(B_{0,0}) > 1 - \delta$. It follows that

$$|\mu(g_m^a A_{l,k}) - \mu(g_m^b B_{l,k})| < 2\delta$$

for $m, l, k \in \{1, \ldots, n\}$. Therefore

$$\begin{aligned}\mu(B_{j,m}\triangle g_m^b B_{j,0}) &= \mu(B_{j,m})+\mu(g_m^b B_{j,0})-2\mu(B_{j,m}\cap g_m^b B_{j,0})\\ &\leq 6\delta+\mu(A_{j,m})+\mu_(g_m^a A_{j,0})-2\mu(A_{j,m}\cap g_m^a A_{j,0})\\ &= 6\delta+\mu(A_{j,m}\triangle g_m^a A_{j,0})=6\delta.\end{aligned}$$

In conclusion

$$|\mu(g_k^a A_i\cap g_m^a A_j)-\mu(g_k^b B_{i,0}\cap g_m^b B_{j,0})|\leq|\mu(g_k^a A_{i,0}\cap A_{j,m})-\mu(g_k^b B_{i,0}\cap B_{j,m})|+6\delta\leq 7\delta<\epsilon.$$

for every $i,j,k,m\in\{1,\ldots,n\}$. Thus we can take $B_i=B_{i,0}$ to obtain (2.1). $\square$

3 The space of weak equivalence classes

For $a\in\mathrm{Stat}(G,m,X,\mu)$ we will write $\tilde{a}$ for the weak equivalence class of a. Let $(g_k)_{k=1}^{\infty}$ be an enumeration of G. For a natural number m and an ordered finite partition $\overline{A}=\{A_1,\ldots,A_n\}$ of X, we will write $M_{m,\overline{A}}(a)$ for the point in $[0,1]^{m\times n\times n}$ whose (k,i,j)-coordinate is $\mu(g_k^a A_i\cap A_j)$. Let then $C_{m,n}(a)$ be the closure in $[0,1]^{m\times n\times n}$ of the set

$$\Big\{M_{m,\overline{A}}:\overline{A}\text{ is a partition of }X\text{ into }n\text{ pieces.}\Big\}.$$

Clearly we have $a\preceq b$ if and only if $C_{m,n}(a)\subseteq C_{m,n}(b)$ for all natural numbers m,n. Let

$$\delta(a,b)=\sum_{m,n=1}^{\infty}\frac{1}{2^{m+n}}\cdot d_H\big(C_{m,n}(a),C_{m,n}(b)\big),$$

where d_H is the Hausdorff distance on the space of compact subsets of $[0,1]^{m\times n\times n}$. Then for any $a,b,c,d\in\mathrm{Stat}(G,m,X,\mu)$ with $a\sim c$ and $b\sim d$ we have $\delta(a,b)=\delta(c,d)$. Thus the quantity $\tilde{\delta}(\tilde{a},\tilde{b}):=\delta(a,b)$ is a well-defined metric on the space of weak equivalence classes. The corresponding topology is easily seen to be Polish. We denote this space by $\widetilde{\mathrm{Stat}}(G,m,X,\mu)$. As in the measure-preserving case, an ultraproduct construction shows that $\widetilde{\mathrm{Stat}}(G,m,X,\mu)$ is compact.

In addition to its topology, $\widetilde{\mathrm{Stat}}(G,m,X,\mu)$ carries a convex structure. Given $a,b\in\mathrm{Stat}(G,m,X,\mu)$, and $t\in(0,1)$ one can realize a as an action on $[0,t)$ and realize b as an action on $[t,1]$. One then defines $ta+(1-t)b$ to be the action on $[0,1]$ which agrees with a on $[0,t)$ and b on $[t,1]$. It is easy to see that this procedure gives a well-defined operation on $\widetilde{\mathrm{Stat}}(G,m,X,\mu)$. As in the measure-preserving case discussed in [3], the convex structure is better behaved if one instead considers the relation $\preceq_s$ of stable weak containment. This is defined by letting $a\preceq_s b$ if and only if $a\preceq b\times\iota$, where ι is the trivial action of G on a standard probability space. Write $\widetilde{\mathrm{Stat}}_s(G,m,X,\mu)$ for the space of stable weak equivalence classes. $\tilde{\delta}$ gives a Polish topology on $\widetilde{\mathrm{Stat}}_s(G,m,X,\mu)$ and since $h_m(X,\mu,a\times\iota)=h_m(X,\mu,a)$, Theorem 1.3 continues to hold if we replace weak equivalence by stable weak equivalence. The arguments from [3] carry over to show that $\widetilde{\mathrm{Stat}}_s(G,m,X,\mu)$ is isomorphic to a compact convex subset of a Banach space, and that its extreme points are exactly those stable weak equivalence classes containing an ergodic action. Moreover, the map $a\mapsto h_m(X,\mu,a)$ respects the convex combination operation. Thus understanding the convex structure of $\widetilde{\mathrm{Stat}}_s(G,m,X,\mu)$ could give new understanding of Problem 1.2.

4 Proof of Theorem 1.3

For each n, let $a_n \in \mathrm{Stat}(G, m, X, \mu)$; let also $a \in \mathrm{Stat}(G, m, X, \mu)$. Assume that $\tilde{a}_n$ converges to $\tilde{a}$ in $\widetilde{\mathrm{Stat}}(G, m, X, \mu)$. Fixing $g \in G$, it is enough to show the following: for any $c \geq 0$ we have

$$\lim_{n\to\infty} \mu\left(\left\{x \in X : \frac{\mathrm{d}g^{a_n}\mu}{\mathrm{d}\mu}(x) > c\right\}\right) = \mu\left(\left\{x \in X : \frac{\mathrm{d}g^{a}\mu}{\mathrm{d}\mu}(x) > c\right\}\right).$$

Let M be a positive constant such that $\frac{\mathrm{d}g^b\mu}{\mathrm{d}\mu} \leq M$ for any m-stationary action a. Let $\omega_n = \frac{\mathrm{d}g^a\mu}{\mathrm{d}\mu}$ and $\omega_n = \frac{\mathrm{d}g^{a_n}\mu}{\mathrm{d}\mu}$. Write $C = \{x \in X : \omega(x) > c\}$, and $C_n = \{x \in X : \omega_n(x) > c\}$. We will prove that $\mu(C) \leq \liminf_n \mu(C_n)$. The proof that $\mu(C) \geq \limsup_n \mu(C_n)$ is analogous. Suppose by contradiction $\mu(C) > \liminf_n \mu(C_n)$. Thus, after passing to a subsequence, we can assume that there is $\delta > 0$ such that $\mu(C_n) \leq \mu(C) - \delta$ for every $n \in \mathbb{N}$. Identify X with $[0,1]$, so that we have a Borel linear order on X. Define the Borel linear order $\sqsubseteq$ on X by letting $t \sqsubseteq s$ iff $\omega(t) < \omega(s)$ or $\omega(t) = \omega(s)$ and $t < s$. Similarly define $\sqsubseteq_n$ in terms of ω_n. Note that if D is a terminal segment of $\sqsubseteq$ then we have $\mu(g^a D) \geq \mu(g^a E)$ for any E with $\mu(E) = \mu(D)$. For $n \in \mathbb{N}$ write D_n for the terminal segment of $\sqsubseteq$ such that $\mu(D_n) = \mu_n(C_n)$ and write E_n for the terminal segment of $\sqsubseteq_n$ such that $\mu(C) = \mu_n(E_n)$. Let also F_n be the terminal segment of $\sqsubseteq$ such that $\mu(F_n) = \mu(C_n) + \delta$ and let K_n be the terminal segment of $\sqsubseteq_n$ such that $\mu_n(K_n) = \mu(C_n) + \delta$. Clearly $D_n \subseteq F_n \subseteq C$ and $C_n \subseteq K_n \subseteq E_n$. We have

$$\mu(F_n \setminus D_n) = \mu(F_n) - \mu(D_n) = \delta = \mu_n(K_n) - \mu_n(C_n) = \mu_n(K_n \setminus C_n) \tag{4.1}$$

and similarly

$$\mu(C \setminus F_n) = \mu_n(E_n \setminus K_n). \tag{4.2}$$

Note that since $\omega(x) > c \geq \omega_n(y)$ if $x \in C$ but $y \in X \setminus C_n$, (4.2) implies

$$\mu\left(g^a(C \setminus F_n)\right) \geq \mu_n\left(g^{a_n}(E_n \setminus K_n)\right). \tag{4.3}$$

Let H be the terminal segment of $\sqsubseteq$ such that $\mu(H) = \mu(C) - \delta$ so that by (4.1) we have $\delta = \mu(C \setminus H) = \mu(F_n \setminus D_n)$. Since $F_n \setminus D_n \subseteq C$ and $C \setminus H$ has the lowest Radon-Nikodym derivative of any subset of C with measure δ this implies

$$\mu\left(g^a(C \setminus H)\right) \leq \mu\left(g^a(F_n \setminus D_n)\right). \tag{4.4}$$

For $n \in \ltimes$ from (4.1), (4.3) and (4.4) we have

$$\begin{aligned}
&\mu\left(g^a(C \setminus D_n)\right) - \mu\left(g^{a_n}(E_n \setminus C_n)\right) \\
&= \mu\left(g^a(C \setminus F_n)\right) + \mu\left(g^a(F_n \setminus D_n)\right) - \mu\left(g^{a_n}(E_n \setminus K_n)\right) - \mu\left(g^{a_n}(K_n \setminus C_n)\right) \\
&\geq \mu\left(g^a(F_n \setminus D_n)\right) - \mu\left(g^{a_n}(K_n \setminus C_n)\right) \geq \mu\left(g^a(F_n \setminus D_n)\right) - c \cdot \mu\left(K_n \setminus C_n\right) \\
&= \mu\left(g^a(F_n \setminus D_n)\right) - c\delta \geq \mu\left(g^a(C \setminus H)\right) - c\delta.
\end{aligned} \tag{4.5}$$

For $x \in C$ we have $\omega(x) > c$ so the last quantity is strictly positive. Choose

$$0 < \varepsilon < \frac{1}{2(4+M)} \cdot \left(\mu\left(g^a(C \setminus H)\right) - c\delta\right). \tag{4.6}$$

Since $\tilde{a}_n \to \tilde{a}$, for every Borel partition $A_1, \ldots, A_k$ of X there is a partition $B_1, \ldots, B_k$ of X such that $|\mu(A_i) - \mu(B_i)| < \varepsilon$ and $|\mu(g^a A_i \cap A_j) - \mu(g^{a_n} B_i \cap B_j)| < \varepsilon$ for all $i, j \in \{1, \ldots, k\}$. Fixing n, write $C' = C_n$, $D = D_n$, $E = E_n$ and $\sqsubseteq' = \sqsubseteq_n$. Note that from (4.5) and (4.6) we have

$$2(4+M)\varepsilon < \mu\left(g^a(C \setminus D)\right) - \mu\left(g^{a_n}(E \setminus C')\right). \tag{4.7}$$

Let $A_1 = X \setminus C$ and $A_2 = C$. Find $B_1, B_2 \subseteq X$ such that $|\mu(A_i) - \mu(B_i)| < \varepsilon$ and

$$|\mu(g^a A_i \cap A_j) - \mu(g^{a_n} B_i \cap B_j)| < \varepsilon$$

for each $i, j \in \{1, 2\}$. Note that

$$\mu\left(X \setminus (B_1 \cup B_2)\right) \leq 2\varepsilon.$$

We have

$$\begin{aligned}
\mu(g^a A_1) &= \mu(g^a A_1 \cap A_1) + \mu(g^a A_1 \cap A_2) \\
&\geq \mu(g^{a_n} B_1 \cap B_1) + \mu(g^{a_n} B_1 \cap B_2) - 2\varepsilon \\
&\geq \mu(g^{a_n} B_1 \cap B_1) + \mu(g^{a_n} B_1 \cap B_2) + \mu\left(g^{a_n} B_1 \setminus (B_1 \cup B_2)\right) - 4\varepsilon \\
&\geq \mu(g^{a_n} B_1) - 4\varepsilon.
\end{aligned} \tag{4.8}$$

Note that

$$\mu(B_1) \geq \mu(A_1) - \varepsilon = \mu(X \setminus C) - \varepsilon.$$

Write L for the initial segment of $\sqsubseteq'$ such that $\mu(L) = \mu(X \setminus C) - \varepsilon$. Note that $\mu(X \setminus E) = \mu(X \setminus C)$ and so $\mu\left(X \setminus (E \cup L)\right) = \varepsilon$. We have

$$\mu\left(g^{a_n}(X \setminus E)\right) = \mu(g^{a_n} L) + \mu\left(g^{a_n}\left(X \setminus (E \cup L)\right)\right)$$

and therefore

$$\mu(g^{a_n} L) \geq \mu\left(g^{a_n}(X \setminus E)\right) - M\varepsilon. \tag{4.9}$$

Since $\mu(B_1) \geq \mu(L)$ and $\mu(g^{a_n} L) \leq \mu(g^{a_n} J)$ for any $J \subseteq X$ with $\mu(J) \geq \mu(L)$ from (4.9) we see

$$\mu(g^{a_n} B_1) \geq \mu\left(g^{a_n}(X \setminus E)\right) - M\varepsilon.$$

From (4.8) we have

$$\mu\left(g^a(X \setminus C)\right) \geq \mu\left(g^{a_n}(X \setminus E)\right) - (4+M)\varepsilon. \tag{4.10}$$

Now write $A_1 = C'$ and $A_2 = X \setminus C'$. Find $B_1, B_2 \subseteq X$ such that

$$|\mu(A_i \cap A_j) - \mu(B_i \cap B_j)| < \varepsilon$$

and

$$|\mu(g^{a_n} A_i \cap A_j) - \mu(g^a B_i \cap B_j)| < \varepsilon$$

for each $i, j \in \{1, 2\}$. Arguing as before we have $\mu(g^{a_n} A_1) \leq \mu(g^a B_1) + 4\varepsilon$ and $\mu(g^a B_1) \leq \mu(g^a D) + M\varepsilon$ so that

$$\mu(g^{a_n} C') \leq \mu(g^a D) + (4+M)\varepsilon. \tag{4.11}$$

From (4.10) and (4.11) we have

$$\mu\left(g^a(X \setminus C) \cup D)\right) \geq \mu\left(g^{a_n}\left((X \setminus E) \cup C'\right)\right) - 2(4+M)\varepsilon. \tag{4.12}$$

Note that

$$D \sqcup (C \setminus D) \sqcup (X \setminus C) = X$$

and

$$C' \sqcup (E \setminus C') \sqcup (X \setminus E) = X.$$

Thus from (4.7) and (4.12) we have

$$\begin{aligned} 1 &= \mu\left(g^a\left(D \cup (X \setminus C)\right)\right) + \mu\left(g^a(C \setminus D)\right) \\ &\geq \mu\left(g^{a_n}\left(C' \cup (X \setminus E)\right)\right) - 2(4+M)\varepsilon + \mu\left(g^a(C \setminus D)\right) \\ &> \mu\left(g^{a_n}\left(C' \cup (X \setminus E)\right)\right) + \mu\left(g^{a_n}(E \setminus C')\right) = 1 \end{aligned}$$

which is the desired contradiction. This concludes the proof of Theorem 1.3.

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