WEAK EQUIVALENCE OF STATIONARY ACTIONS AND THE ENTROPY REALIZATION PROBLEM

PETER BURTON, MARTINO LUPINI, AND OMER TAMUZ

Abstract. We initiate the study of weak containment and weak equivalence for $\mu$-stationary actions for a given countable group $G$ endowed with a generating probability measure $\mu$. We show that Furstenberg entropy is a stable weak equivalence invariant, and furthermore is a continuous affine map on the space of stable weak equivalence classes. We prove the same for the associated stationary random subgroup (SRS). Applying these results to the entropy realization problem for ergodic stationary actions, we show that the set of values of the Furstenberg entropy of boundary actions is closed. This is obtained as an application of the omitting types theorem in first-order logic for metric structures.

1. Introduction

Suppose that $G$ is a countable discrete group, and $\mu$ is a generating probability measure on $G$. That is, the support of $\mu$ generates $G$ as a semigroup. A $\mu$-stationary action $\sigma$ of $G$ on a probability space $(X,\nu)$ is a group homomorphism $g \mapsto g^\sigma$ from $G$ to the group $\text{Aut}(X,\nu)^*$ of nonsingular transformations of $(X,\nu)$, with the property that $\nu = \sum_{g \in G} \mu(g)g^\sigma \nu$. Here $g^\sigma \nu$ denotes the pushforward of $\nu$ by the nonsingular transformation $g^\sigma$. It is clear that any measure-preserving action is, in particular, $\mu$-stationary. The notion of stationary dynamical system has received considerable interest in the last few years [8–11, 15, 22, 25, 31, 32].

Stationary actions of $G$ arise naturally from the study of random walks of $G$ with law $\mu$. Indeed suppose that $(X_n)$ is a sequence of independent identically distributed random variables on $G$ with law $\mu$, representing the steps of a random walk. One can then let $Z_n := X_1X_2 \cdots X_n$ be the random variables representing the locations of a random walk on $G$. The law $P$ of the random variable $(Z_n)_{n \in \mathbb{N}}$ is a probability distribution on $G^{\mathbb{N}}$ defined on its Borel $\sigma$-algebra $B$. The space $(G^{\mathbb{N}}, B, P)$ can be thought as the space of paths of the random walk, and an element of $G^{\mathbb{N}}$ as a random path.

In the study of random walks, tail events play a particularly important role. Briefly, tail events are the events that only depend on the asymptotic behavior of the random walk. Formally, these are the Borel subsets of $G^{\mathbb{N}}$ that are invariant under the shift map $\sigma : G^{\mathbb{N}} \to G^{\mathbb{N}}$ defined by $\sigma(g_1, g_2, \ldots) = (g_2, g_3, \ldots)$. Tail events form a $\sigma$-algebra $T$ called tail $\sigma$-algebra or shift-invariant $\sigma$-algebra. The group $G$ admits a natural action $\pi$ on the space $(G^{\mathbb{N}}, T, P)$ defined by letting $\pi_h$ for $h \in G$ be the nonsingular transformation $(g_1, g_2, \ldots) \mapsto (hg_1, hg_2, \ldots)$. Intuitively, $\pi_h$ maps a random path of the random walk starting at the identity, to a random path of the random walk starting at $h$. It is not difficult to verify that such an action of $G$ is indeed $\mu$-stationary. While the $\sigma$-algebra $T$ does not separate the points of $G^{\mathbb{N}}$—since it does not separate any two sequences that differ only on a finite initial segment—it follows from the Mackey point realization theorem [29, 34] that one can find a countably generated probability $G$-space $(\Pi, A, \eta)$ endowed with a $\sigma$-algebra that separates points that is $G$-equivariantly isomorphic to $(G^{\mathbb{N}}, T, P)$. Such a space is called the Poisson boundary of $(G, \mu)$.

The Poisson boundary is an important example of a class of $\mu$-stationary actions known as boundaries [21], also called $\mu$-proximal in [22]. Suppose that $g$ is a $\mu$-stationary action of $G$ on a countably separated probability space $(X, B, \nu)$. We can assume without loss of generality that $B$ is the Borel $\sigma$-algebra of a compact metrizable topology on $X$. Convergence theorems for martingales show that

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for $P$-almost every path $\omega \in G^N$ of a random walk on $G$, the sequence $(\omega^n_\omega)\nu$ of probability measures on $X$ has a w*-limit $\nu_\omega$. The measures $\nu_\omega$ are called conditional measures of the action $a$, and satisfy $\nu = \int \nu_\omega dP(\omega)$. The action $a$ is a boundary action if for $\nu$-a.e. $\omega \in G^N$ the corresponding conditional probability $\nu_\omega$ is a point mass. Intuitively this means that almost every $\mu$-random walk on $G$ converges to a point in $X$.

The related notion of strongly approximately transitive (SAT) action was introduced by Jaworski in [26]. A $\mu$-stationary action $a$ of $G$ on $(X,B,\nu)$ is SAT if and only if for any nonnull $\nu \in B$ and every $\varepsilon > 0$ there exists $g \in G$ such that $\nu(g \nu) > 1 - \varepsilon$. It follows from [26, Proposition 2.2] that a $\mu$-stationary action is SAT if and only if it is boundary. Such a characterization of boundary actions makes it apparent that any boundary action is, a fortiori, ergodic. The boundary actions are precisely the $\mu$-stationary actions that are factors of the Poisson boundary. The factor map is given $P$-a.e. by the assignment $\omega \mapsto \nu_\omega$ mapping a path of a random walk to the corresponding conditional probability, which in the case of a boundary action is a point mass for $P$-a.e. $\omega$.

The Poisson boundary is tightly connected with bounded $\mu$-harmonic functions on $G$. Recall that a real-valued function $f$ on $G$ is $\mu$-harmonic if $f(h) = \sum_{g \in G} f(gh) \mu(g)$ for any $h \in G$. Indeed the essentially bounded real-valued random variables on $(G^N,T,P)$ are in 1:1 $G$-equivariant correspondence with bounded $\mu$-harmonic functions on $G$. Such correspondence is obtained by assigning to a random variable $Y$ the bounded real-valued function $g \mapsto \int YdgP$. Particularly, one can assert that the Poisson boundary of $(G,\mu)$ is trivial if and only if the only bounded $\mu$-harmonic functions are the constant functions. (Recall that we are assuming that the measure $\mu$ is generating.)

Many geometric and combinatorial properties of a group $G$ can be characterized in terms of $\mu$-harmonic functions. For instance, a group $G$ is amenable if and only if for some symmetric generating probability measure $\mu$ on $G$ any bounded $\mu$-harmonic functions is constant [27, 35] or, equivalently, the Poisson boundary of $(G,\mu)$ is trivial. Such a characterization of amenability motivated the study of a strengthening of amenability known as the Liouville property. Such a property asks that for any (not necessarily generating) symmetric finitely supported probability measure on $G$, the corresponding Poisson boundary is trivial. This is equivalent to the assertion that any bounded $\mu$-harmonic on $G$ is constant on the subgroup of $G$ generated by the support of $\mu$.

It is therefore of interest to determine when the Poisson boundary is a trivial space. It turns out that this happens if and only if the Poisson boundary action is measure-preserving [22, Proposition 1.2]. This motivated the study of the Furstenberg entropy $h_\mu(a)$ of a $\mu$-stationary action $a$. The Furstenberg entropy is an invariant measuring how much the action $a$ fails to be measure-preserving [20, 31]. Formally, $h_\mu(a)$ is defined as $\sum_{g \in G} \mu(g) \int (-\log(d\nu/dg\nu)) d\nu$. The quantity $d\nu/dg\nu$ is the Radon-Nikodym derivative of $\nu$ with respect to $g\nu$, while $\int (-\log(d\nu/dg\nu)) d\nu$ is the relative entropy or Kullback-Leibler divergence of $\nu$ with respect to $g\nu$. By Jensen’s inequality, the Furstenberg entropy of a $\mu$-stationary action is zero if and only if the action is measure-preserving [22, Theorem 1.3]. The largest value of the Furstenberg entropy of a $\mu$-stationary action is the entropy $h_\Pi$ of the Poisson boundary $(\Pi,\eta)$ of $(G,\mu)$ [27]. A boundary $\mu$-stationary action is $G$-isomorphic to the Poisson boundary if and only if its Furstenberg entropy is equal to $h_\Pi$ [22, Theorem 1.3].

These results initiated the study of the Furstenberg entropy of $\mu$-stationary actions, and raised the so-called entropy realization problem. Such a problem asks, broadly speaking, which are the possible values of the Furstenberg entropy of ergodic or boundary $\mu$-stationary actions. In relation with this problem, it was proved by Bowen that when $G$ is a nonabelian free group and $\mu$ is the uniform measure on the canonical generators, then all the values of entropy between 0 and $h_\Pi$ are attained [10]. It has also been shown by Nevo that for a property (T) group the Furstenberg entropy has a gap around 0 [30], and in fact this holds for general measure class preserving actions. The existence of a gap for measure class preserving actions is in fact also sufficient for a group to have property (T), as was later shown in [12]; see also [7].

It is currently an open problem to determine whether the set of values of the Furstenberg entropy of ergodic $\mu$-stationary actions is a compact subset of $\mathbb{R}$. In this paper we answer a related problem, and show that the set of values of the Furstenberg entropy of boundary $\mu$-stationary actions is compact.
We also show that, assuming that the Poisson boundary action is free, then the set of possible values of the Furstenberg entropy of free boundary $\mu$-stationary actions is compact.

We prove these results as an application of the theory of weak containment and weak equivalence for $\mu$-stationary actions. The notion of weak containment and weak equivalence for measure-preserving actions has been initially introduced by Kechris in [28], inspired by the notion of weak containment in the sense of Zimmer for unitary representations [4]. Weak containment of unitary representations correspond to stable weak containment and stable weak equivalence of measure preserving actions as defined in [36]. The theory of weak containment for measure-preserving actions has attracted considerable interest in the last few years. Particularly it has been shown by Bowen and Tucker-Drob that the space of stable weak equivalence classes of measure-preserving actions form a metrizable Choquet simplex [13]. The extreme points of such a simplex are precisely the classes of ergodic actions, as proved by Burton [14] and, independently, by Bowen and Tucker-Drob [13]. The theory of stable weak equivalence of measure-preserving actions has a tight connection with the study of invariant random subgroups (IRS) of groups. Indeed, the IRS associated with a measure-preserving actions can be seen as a continuous affine function on the space of stable weak equivalence classes, which is 1:1 when the group is amenable [36, §7].

In this paper we generalize the theory of (stable) weak containment and (stable) weak equivalence to the more general setting of $\mu$-stationary actions. Furthermore we observe that weak equivalence can be seen as the first-order theory of the given $\mu$-stationary actions, when regarded as structures in first-order logic for metric structures. A good introduction to the logic for metric structures, also called continuous logic, can be found in [5]. We identify a $\mu$-stationary action with the corresponding action on the measure algebra. This allows one to regard a $\mu$-stationary action as a structure in the signature consisting of function symbols for the Boolean algebra operations, a unary function symbol for every element of the group, and a relation symbol for the probability measure on the space. In this signature, $\mu$-stationary actions form an axiomatizable class. Two actions are weakly equivalent if and only if they have the same universal theory, which is the portion of the first-order theory where only universal formulas are considered. Using this perspective, we show that the space of stable weak equivalence classes of $\mu$-stationary actions is a metrizable compact convex set. The space of stable weak equivalence classes of measure-preserving actions is a closed face of such a compact convex set. Furthermore, we consider the natural $\mu$-stationary analog of the IRS associated with a measure-preserving action, which we call $\mu$-stationary random subgroup ($\mu$-SRS). This can be seen as a particular case of a nonsingular random subgroup (NSRS) in the sense of [23]. We prove that the $\mu$-SRS of a $\mu$-stationary action can be seen as a continuous affine map on the space of stable weak equivalence classes. The same result is proved for the Furstenberg entropy of a stationary action. The fact that the values of Furstenberg entropy on boundary $\mu$-stationary actions form a compact set is obtained as a consequence with the latter result, together with the omitting types theorem in first-order logic for metric structures [6,17,19].

The rest of the present paper is divided into three sections. In Section 2 we introduce the notion of weak equivalence for stationary actions, define the topology on the space of weak equivalence classes, and characterize the continuous real-valued functions on such a space. In Section 3 we consider the related notion of stable weak equivalence, and prove that the space of stable weak equivalence classes is a metrizable compact convex set. Finally in Section 4 we prove that the Furstenberg entropy can be seen as a continuous affine function on the space of stable weak equivalence classes. We deduce from this together with the omitting types theorem that the set of values of the Furstenberg entropy on boundary actions is compact.

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2. Weak equivalence of stationary actions

2.1. Weak equivalence. In the rest of the paper we suppose that $G$ is a countable discrete group endowed with a probability measure $\mu$ with the property that the support of $\mu$ generates $G$ as a
An action \( a \) of \( G \) on a probability space \((X, \mu)\) is a map \( G \times X \to X \), denote by \( (g, x) \mapsto g^a x \).

A function \( \nu \) is called \( \mu \)-stationary if it satisfies \( \nu = \sum_{g \in G} \mu(g) g^a \).

Here \( g^a \nu \) denotes the push-forward of \( \nu \) under the map \( x \mapsto g^a x \).

**Definition 2.1.** If \( a, b \) are two \( \mu \)-stationary actions of \( G \) on probability spaces \((X_a, \mathcal{B}_a, \nu_a)\) and \((X_b, \mathcal{B}_b, \nu_b)\), then we say that \( a \) is weakly contained in \( b \) if any \( n \in \mathbb{N} \), \( p_1, \ldots, p_n \in \mathcal{B}_a \), and \( g_1, \ldots, g_n \in G \) (not necessarily distinct) and \( \varepsilon > 0 \), there exist \( q_1, \ldots, q_n \in \mathcal{B}_b \) such that for any function \( \sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \), one has that

\[
|\nu_a \left( g_{\sigma(1)} p_1 \cap \cdots \cap g_{\sigma(n)} p_n \right) - \nu_b \left( g_{\sigma(1)} q_1 \cap \cdots \cap g_{\sigma(n)} q_n \right)| < \varepsilon.
\]

We say that \( a \) is weakly equivalent to \( b \) if \( a \) is weakly contained in \( b \) and vice versa.

It is clear that such a definition coincides with the usual definition of weak containment for measure-preserving actions when \( a \) and \( b \) are measure preserving. It is also clear that if \( p_1, \ldots, p_n \) form a partition of \( X_a \), then it is not restrictive to require that \( q_1, \ldots, q_n \) form a partition of \( X_b \). As in the measure-preserving case, we will denote by \([a]_\sim\) the weak equivalence class of \( a \). We will observe below (Proposition 2.5) that Definition 2.1 is equivalent to the seemingly weaker condition where only pairwise intersections are considered.

We will also consider the strengthening of this notion called elementary equivalence (which coincides with the usual notion of elementary equivalence in model theory, as observed below). This is defined similarly as weak equivalence in terms of a group that is called Ehrenfeucht–Fraïssé game in model theory and pebble game in computer science. Such a game is played between two players: the Spoiler and the Duplicator. It has two parameters: the number of rounds \( k \), the tolerance \( \varepsilon \), and of the group \( g_1, \ldots, g_{kn} \in G \) (not necessarily distinct). At each round the Spoiler picks one between the actions \( a \) and \( b \) and \( a \) and \( a \) and an \( n \)-tuple of elements \( p_{nk+1}, \ldots, p_{nk+n} \) of \( \mathcal{B}_a \). The Duplicator has to respond with an \( n \)-tuple of elements \( q_{nk+1}, \ldots, q_{nk+n} \) of \( \mathcal{B}_b \). If, instead, the Spoiler chooses \( b \) and an \( n \)-tuple of elements \( q_{nk+1}, \ldots, q_{nk+n} \) of \( \mathcal{B}_b \), then the Duplicator has to answer with an \( n \)-tuple of elements \( p_{nk+1}, \ldots, p_{nk+n} \) of \( \mathcal{B}_a \). The game ends after \( k \) rounds. The Duplicator wins the game if, considering all the elements \( p_i \) of \( \mathcal{B}_a \) and \( q_i \) of \( \mathcal{B}_b \) for \( i = 1, 2, \ldots, kn \) chosen during the game, one has that

\[
\left| \nu_a \left( g_{\sigma(1)} p_1 \cap \cdots \cap g_{\sigma(kn)} p_{kn} \right) - \nu_b \left( g_{\sigma(1)} q_1 \cap \cdots \cap g_{\sigma(kn)} q_{kn} \right) \right| < \varepsilon
\]

for any function \( \sigma : \{1, 2, \ldots, kn\} \to \{1, 2, \ldots, kn\} \).

**Definition 2.2.** If \( a, b \) are two \( \mu \)-stationary actions of \( G \) on probability spaces \((X_a, \mathcal{B}_a, \nu_a)\) and \((X_b, \mathcal{B}_b, \nu_b)\), then we say that \( a \) is \( k \)-elementarily equivalent to \( b \) if the Duplicator has a winning strategy for the Ehrenfeucht–Fraïssé game for \( a \) and \( b \). We say that \( a \) and \( b \) are elementarily equivalent if \( a \) and \( b \) are \( k \)-elementarily equivalent for any \( k \in \mathbb{N} \). In this case we write \( a \approx b \).

It is clear that weak equivalence in the sense of Definition 2.1 coincides with 1-elementary equivalence in the sense of Definition 2.2, and hence elementary equivalence implies weak equivalence. We denote by \([a]_\approx\) the elementary equivalence class of \( a \).

**Example 2.3.** It follows from the Rokhlin lemma for amenable groups [33] that any two free measure-preserving actions of an amenable group \( G \) are elementarily equivalent. This was observed in [7] in the case of \( Z \)-actions.

As in the measure-preserving case, one can characterize weak equivalence and elementary equivalence in terms of ultrapowers. Suppose that \( I \) is an index set and \( \mathcal{U} \) is an ultrafilter on \( I \). The ultrapower \( \prod_{i \in I} a_i \) of an \( I \)-sequence \((a_i)\) of \( \mu \)-stationary actions on probability spaces \((X_{a_i}, \mathcal{B}_{a_i}, \nu_{a_i})\) can be defined analogously as in the measure-preserving case. It is not difficult to verify that such an ultraproduct is again a \( \mu \)-stationary action. When \((a_i)\) is the \( I \)-sequence constantly equal to \( 1 \), the corresponding ultraproduct is called the ultrapower of \( a \) with respect to \( \mathcal{U} \) and denoted by \( a^\mathcal{U} \).

One then has that \( a \) is weakly contained in \( b \) if and only if \( a \) is a factor of some (equivalently, any) ultrapower of \( b \) with respect to a nonprincipal ultrafilter \( \mathcal{U} \) over \( \mathbb{N} \).
A result in model theory known as the Keisler-Shelah theorem [5, Theorem 5.7] provides the following characterization of elementary equivalence: $a$ and $b$ are elementarily equivalent if and only if they have isomorphic ultrapowers. It follows from the Lowenheim-Skolem theorem [5, Proposition 7.3] that any action is elementarily equivalent to some action on a countably generated probability space. Therefore in the following we will assume, without loss of generality, that all probability spaces are countably generated.

In the following, if $\lambda, \nu$ are Borel measure on a Borel space $(X, \mathcal{B})$, and $\lambda$ is absolutely continuous with respect to $\nu$, then we denote by $d\lambda/d\nu$ the Radon-Nikodym derivative of $\lambda$ with respect to $\nu$ [16, Theorem 5.5.4]. Suppose that $g \in G$ and $a$ is a $\mu$-stationary action. Then we have $1 = (dg^a \nu/dg^b \nu) = \sum_{h \in G} \mu(h)(dg^ah^a \nu/dg^a \nu)$ and, in particular, $\mu(h)(dg^ah^a \nu/dg^a \nu) \leq 1$ for any $h \in G$. Taking $g = 1$ one obtains that $\mu(h)(dg^ah^a \nu/dg^a \nu) \leq 1$ and by taking $g = h^{-1}$ one obtains that $\mu(g^{-1})(dg^a \nu/dg^b \nu) \leq 1$. Therefore $\mu(g^{-1}) \leq dg^a \nu/dg^b \nu \leq \mu(g)^{-1}$ whenever $g$ belongs to the support of $\mu$; see also [27]. Since we are assuming that the support of $\mu$ generates $G$ as a semigroup, we can conclude that for any $g$ there exists a constant $M_g$ such that $dg^a \nu/dg^b \nu \leq M_g$ and $dg^a \nu/dg^b \nu \leq M_g$ for any $\mu$-stationary action $a$.

Remark 2.4. One can define the notion of weak containment, weak equivalence, and elementary equivalence for arbitrary nonsingular actions. All the results of this paper go through with no change where one considers, for some fixed constants $M_g > 0$ for $g \in G$, the class of nonsingular actions $a$ of $G$ on probability spaces with the property that $dg^a \nu/dg^b \nu \leq M_g$ for every $g \in G$. As remarked above, such a class of actions includes the class of $\mu$-stationary actions for a suitable choice of constants $M_g$.

2.2. Binary intersections. In this section we remark that the notion of weak containment for $\mu$-stationary actions as formulated in Definition 2.1 is equivalent to the apparently weaker definition where only pairwise intersections are considered.

Proposition 2.5. Suppose that $a, b$ are two $\mu$-stationary actions of $G$ on probability spaces $(X_a, \mathcal{B}_a, \nu_a)$ and $(X_b, \mathcal{B}_b, \nu_b)$. The following assertions are equivalent:

1. $a$ is weakly contained in $b$;
2. For any finite subset $F$ of $G$, $\varepsilon > 0$, and $p_1, \ldots, p_n \in \mathcal{B}_a$, there exist $q_1, \ldots, q_n \in \mathcal{B}_b$ such that $|\nu_a(g^aq_i \cap h^a q_j) - \nu_b(g^bq_i \cap h^b q_j)| < \varepsilon$ for any $g, h \in F$ and $i, j \in \{1, 2, \ldots, n\}$.
3. For any finite subset $F$ of $G$, $\varepsilon > 0$, and $p_1, \ldots, p_n \in \mathcal{B}_a$, there exist $q_1, \ldots, q_n \in \mathcal{B}_b$ such that $|\nu_a(g^aq_i \cap q_j) - \nu_b(g^bq_i \cap q_j)| < \varepsilon$ for any $g \in F$ and $i, j \in \{1, 2, \ldots, n\}$.

Proof. One can prove that (2)$\Rightarrow$(1) by induction on the number of $k$ of $k$-wise intersections. It remains to prove (3)$\Rightarrow$(1). Suppose that $F = \{g_0, \ldots, g_m\}$ is a finite subset of $G$, $n \in \mathbb{N}$, and $p_0, \ldots, p_n \in \mathcal{B}_a$. Without loss of generality, we can assume that $n = m$, $g_0$ is the identity of the group $G$, and $p_0 = X_a$. Fix $\varepsilon > 0$ and set $\delta := \varepsilon/7$. Set $p_{ij} := g_j p_i$ for $i, j \leq n$. In particular we have $p_{0i} = p_i$ and $p_{ij} = g_j p_0$ for $i, j \leq n$. By assumption there exist $q_{ij} \in \mathcal{B}_b$ such that $|\nu_b(g_j q_i \cap g_n q_{ik}) - \nu_b(g_j q_i \cap g_{n} p_{ik})| < \delta$ for any $i, j, k, \ell \leq n$. Since $p_{00} = p_0 = X_a$ and $g_0$ is the identity, we have that $\nu_b(g_0 q_{ij}) > 1 - \delta$. It follows that $|\nu_b(g^b q_{ij}) - \nu_b(g^b p_{ij})| < 2\delta$ for $i, j, k, \ell \leq n$. Therefore

\[
\nu_b(g^b k q_{00} \triangle q_{ij}) = \nu_b(g^b k q_{00}) + \nu_b(q_{ij}) - 2\nu_b(g^b k q_{00} \cap q_{ik}) \\
\leq 6\delta + \nu_b(g^b k p_0) + \nu_b(q_{ij}) - 2\nu_b(g^b k p_0 \cap p_{ik}) \\
= 6\delta + \nu_b(g^b k p_0 \triangle p_{ik}) = 6\delta.
\]

In conclusion

\[
|\nu_a(g^a k p_0 \cap g^a p_{ij}) - \nu_b(g^b k q_{00} \cap g^b p_{ij})| \leq |\nu_a(p_{ik} \cap p_{ij}) - \nu_b(g^b k q_{00} \cap q_{ij})| + 6\delta \leq 7\delta \leq \varepsilon.
\]

for every $i, j, k, \ell \leq n$. This proves that $q_{00}, \ldots, q_{n0}$ witness the fact that $a$ is weakly contained in $b$. \hfill \Box

2.3. Stationary actions as structures. In the following, we will identify an action of $G$ on a countably saturated probability space with the induced action on the associated measure algebra. More generally one can consider $\mu$-stationary actions of $G$ on (not necessarily separable) probability
algebras. A probability algebra is a Boolean algebra $\mathcal{A}$ with maximum element $1$ and minimum element $0$, endowed with a probability measure $\nu : \mathcal{A} \to [0,1]$. A probability algebra is endowed with a canonical metric defined by $d(A, B) = \nu(A \triangle B)$. An action of $G$ on $(\mathcal{A}, \nu)$ is stationary if $\nu(x) = \sum_{g \in G} \mu(g) \nu((g^{-1})^*x)$. This implies that the functions $x \mapsto g^*x$ are Lipschitz with constant $\mu(g^{-1})^{-1}$ for every $g$ in the support of $\mu$. Since, by assumption the support of $\mu$ generates $G$ as a semigroup, we can conclude that for any $g \in G$ the function $x \mapsto g^*x$ is Lipschitz with constant $L_g$ equal to the infimum of $\mu(g_1^{-1})^{-1} \cdots \mu(g_k^{-1})^{-1}$ where $k \in \mathbb{N}$ and $g_1 \cdots g_k = g$. In particular, $L_g$ depends only from $g$ and $\mu$ and not from the given $\mu$-stationary action $\alpha$.

It is a well-known fact that when $\mathcal{A}$ is the measure algebra of a countably generated probability space, any such an action has a point realization as a $\mu$-stationary action on the corresponding probability space [34, Theorem 3.3]; see also [29].

We will regard probability algebras and $\mu$-stationary actions of $G$ on probability algebras as structures in the logic for metric structures. A standard reference for this subject is [5]. The language for $\mu$-stationary actions of $G$ on probability algebras includes function symbols for the Boolean algebra operations and for the elements of the group, constant symbols for the maximum and the minimum element of the Boolean algebra, and a relation symbol $\nu$ for the measure on the space. It is not difficult to verify that the class of $\mu$-stationary actions of $G$ on probability algebras form a universally axiomatizable class. Furthermore the other Boolean algebras operations are definable in terms of the meet operation.

In this context, an atomic formula with free variables $x_1, \ldots, x_n$ is (equivalent to) and expression $\varphi(x_1, \ldots, x_n)$ of the form $\nu(g_1 x_1 \land \cdots \land g_n x_n)$ where $n \in \mathbb{N}$, $g_1, \ldots, g_n \in G$, and $x_1, \ldots, x_n$ are variables. A quantifier-free formula with free variables $x_1, \ldots, x_n$ is an expression of the form $f(\varphi_1, \ldots, \varphi_k)$ where $\varphi_1, \ldots, \varphi_k$ are atomic formulas with free variables $x_1, \ldots, x_n$ and $f : \mathbb{R}^k \to \mathbb{R}$ is a continuous function. Without loss of generality, one can assume that $f$ belongs to some fixed countable dense set of functions that is dense in the compact-open topology. A universal formula $\psi(x_1, \ldots, x_n)$ with free variables $x_1, \ldots, x_n$ is an expression $\sup_{y_1} \cdots \sup_{y_k} \psi(x_1, \ldots, x_n, y_1, \ldots, y_k)$ where $\psi(x_1, \ldots, x_n, y_1, \ldots, y_k)$ is a quantifier-free formula with free variables $x_1, \ldots, x_n, y_1, \ldots, y_k$. A existential formula is defined similarly as a universal formula, where suprema are replaced with infima. Finally, a formula is defined as a universal or existential formula, where both suprema and infima are allowed. A formula is called a sentence when it has no free variables.

If $\alpha$ is a $\mu$-stationary action of $G$ on a probability algebra $\mathcal{A}$ and $\varphi$ is a sentence, then one can define in the obvious way the interpretation of $\varphi$ in $\alpha$. This is the real number $\varphi^\alpha$ obtained by letting the suprema and infima in $\varphi$ range among the element of $\varphi$, and the interpreting $\nu(g_1 x_1 \land \cdots \land g_n x_n)$ for some $x_1, \ldots, x_n \in \mathcal{A}$ as $\nu(g_1^* x_1 \land \cdots \land g_n^* x_n)$. More generally, one can define the interpretation in $\alpha$ of a formula $\varphi(x_1, \ldots, x_n)$ with free variables $x_1, \ldots, x_n$. This will be a uniformly continuous function $\varphi^\alpha : \mathcal{A}^n \to \mathbb{R}$. We say that a collection $\mathcal{F}$ of formulas is uniformly equicontinuous if the collection of functions $\varphi^\alpha$ where $\varphi$ varies in $\mathcal{F}$ and $\alpha$ varies among the $\mu$-stationary actions of $G$, is uniformly equicontinuous. It is not difficult to see that a single formula is uniformly continuous. The universal theory $Th_{\nu}(\alpha)$ of an action $\alpha$ is the function $\varphi \mapsto \varphi^\alpha$ that assigns to a universal sentence $\varphi$ its interpretation in $\alpha$. The theory $Th(\alpha)$ of $\alpha$ is defined similarly, where arbitrary sentences are considered. One can thus identify the weak equivalence class $[\alpha]_\approx$ of $\alpha$ with the universal theory $Th_{\nu}(\alpha)$ of $\alpha$, and the elementary equivalence class $[\alpha]_{\approx 1}$ with the theory $Th(\alpha)$ of $\alpha$.

**Remark 2.6.** One could also define the interpretation of a formula by letting the variables range among the $L^\infty$-functions of norm at most $1$. It is not difficult to see that doing so does not change the notion of axiomatizable class, omitting type class, elementary equivalence invariant, and weak equivalence invariant defined below.

2.4. **The space of weak equivalence classes.** Suppose that $\mathcal{S}$ is a class of $\mu$-stationary actions on countably separated probability spaces.
Definition 2.7. The logic topology on $S$ is the weakest topology that makes the functions $a \mapsto \varphi^a$ continuous for any sentence $\varphi$. The weak logic topology on $S$ is the weakest topology that makes the function $a \mapsto \varphi^a$ continuous for any universal sentence $\varphi$.

Clearly, the weak logic topology is weaker than the logic topology. In general the logic topology is not separated. By definition, two actions are not separated in the logic topology if and only if they are elementarily equivalent. Similarly, two actions are not separated in the weak logic topology if and only if they are weakly equivalent. The observation that one can restrict, without loss of generality, to a countable collection of formulas shows that the logic topology and the weak logic topology on $S$ admit a countable basis. The weak logic topology on $S$ is induced by a countable collection of canonical pseudometrics $d_{n,F}$ for $n \in \mathbb{N}$ and $F \subset \mathbb{N}$ finite defined as follows. For a $\mu$-stationary action $a$ on $(X,B,\nu)$, let $C(a)$ be the closed subset of $[0,1]^{k \times k \times F}$ consisting of those elements of the form $(\nu(p_i \cap g p_j))_{i,j,g} \in k \times k \times F$ for some partition $p_0, \ldots, p_{k-1} \in B$. Then $d_{n,F}(a,b)$ is the Hausdorff distance between $C(a)$ and $C(b)$. It is clear from Proposition 2.5 that such a collection of pseudometrics is indeed compatible with the weak logic topology.

Let $S_\omega$ be the collection of weak equivalence classes of actions from $S$. As remarked above, we can identify $S_\omega$ with the space of universal theories of actions from $S$. The weak logic topology on $S$ induces a metrizable topology on $S_\omega$, which is compact whenever $S$ is compact in the weak logic topology.

2.5. Combinatorially rigid actions. The notions of rule and combinatorial rigid action have been considered in [1]. A $(k,F)$-rule for $k \in \mathbb{N}$ and a finite subset $F$ of $G$ is a subset $L \subseteq \{1, \ldots, k\}^F$. An action $a$ of $G$ on a probability space $(X,B,\nu)$ almost satisfies $L$ if for any $\varepsilon > 0$ there exists $p_1, \ldots, p_k \in B$ forming a partition of $X$ such that the set

$$q_L(p_1, \ldots, p_k) = \bigcup_{\sigma \in L} \bigcap_{g \in F} g^{-1}p_{\sigma(g)}$$

has $\nu$-measure at least $1 - \varepsilon$. The action $a$ properly satisfies $L$ if one can find a partition $p_1, \ldots, p_k \in B$ of $X$ such that $q_L(p_1, \ldots, p_k) = X$. An action $a$ is combinatorially rigid if whenever it almost satisfies a $(k,F)$-rule $L$ for $k \in \mathbb{N}$ and $F \subset G$ finite, then it properly satisfies it.

Consider for $L \subseteq \{1, 2, \ldots, k\}^F$ the existential sentence

$$\varphi_L \equiv \inf_{x_1, \ldots, x_n} \inf_{x_1, \ldots, x_n} \inf \{\nu(x_i \cap x_j), 1 - \nu(x_i \cup \cdots \cup x_n), 1 - \nu(q_L(p_1, \ldots, p_k))\}$$

where $q_L(p_1, \ldots, p_k)$ is defined as above. It is clear that a $\mu$-stationary action $a$ almost satisfies $L$ if and only if the evaluation of $\varphi_L$ at $a$ is equal to 0. Similarly a $\mu$-stationary action $a$ properly satisfies $L$ when the evaluation of $\varphi_L$ at $a$ is zero and moreover the value zero is attained by some choice of elements $x_1, \ldots, x_n$.

It follows from countable saturation of ultrapowers [5, Proposition 7.6] that whenever $U$ is a nonprincipal ultrafilter over $\mathbb{N}$, then the ultrapower $a^U$ is combinatorially rigid. Furthermore the downward Lowenheim-Skolem theorem can be applied to find a combinatorially rigid factor of $a^U$ acting on a countably separated probability space that is elementarily equivalent to $a^U$ and therefore to $a$. This gives a proof of the following proposition, that recovers [1, Theorem 2] in the measure-preserving case.

Proposition 2.8. Any $\mu$-stationary action is elementarily equivalent to a combinatorially rigid action.

2.6. Weak equivalence and orbits. In this section we consider the class of $\mu$-stationary actions of $G$ on the standard probability space $(X,B,\nu)$. Following [11], we denote such a class by $A_\mu(G,X,\nu)$. There are two natural topologies that one can put on the space $A_\mu(G,X,\nu)$. These are described in [11, §6.1.1] and [11, §6.1.2], and called the weak topology and the very weak topology. Suppose that $(a_n)$ is a sequence in $A_\mu(G,X,\nu)$. Then $(a_n)$ converges to $a$ in the very weak topology if, for any $p \in B$ and $g \in G$, $g^n p \rightarrow g^p$ for $n \rightarrow +\infty$ in the measure algebra $B$ endowed with its canonical metric.

The sequence $(a_n)$ converges to $a$ in the weak topology if it converges in the very weak topology and furthermore for any $g \in G$, $dg^n \nu/d\nu \rightarrow dg^p \nu/d\nu$ for $n \rightarrow +\infty$ in $L^1(\nu)$. Clearly, these two topologies agree for measure-preserving actions. The very weak topology on $A_\mu(G,X,\nu)$ coincides with the
topology of pointwise convergence with respect to the 2-norm on $L^\infty(\nu)$, so it is natural to consider it in this setting.

We denote by $\text{Aut}(X, \nu)$ the group of $\nu$-preserving Borel automorphisms $\gamma$ of $X$. The weak topology on $\text{Aut}(X, \nu)$ is defined as above, by letting $\gamma_n \to \gamma$ if and only if $\gamma_n(p) \to \gamma(p)$ in the measure algebra for any $p \in \mathcal{B}$. This makes $\text{Aut}(X, \nu)$ a Polish group. There is a natural continuous action of $\text{Aut}(X, \nu)$ on $A_\mu(G, X, \nu)$ by conjugation. The corresponding orbit equivalence relation is the relation of (measurable) conjugacy of $\mu$-stationary actions. We remark here that weak containment can be naturally described in terms of such an action. This recovers a well known fact in the measure-preserving case and can be proved in a similar way; see [28, Proposition 10.1].

**Proposition 2.9.** Suppose that $a, b \in A_\mu(G, X, \nu)$. The following assertions are equivalent:

1. $a$ is weakly contained in $b$;
2. there exists a sequence $(\gamma_n)$ in $\text{Aut}(X, \nu)$ such that $\gamma_n b \gamma_n^{-1} \to a$ in the very weak topology.

2.7. **Weak equivalence invariants.** Suppose that $\mathcal{S}$ is the class of all $\mu$-stationary actions on countably separated probability spaces.

**Definition 2.10.** The function $f : \mathcal{S} \to \mathbb{R}$ is a continuous weak equivalence invariant if $f(a) = f(b)$ whenever $a \sim b$ and $f$ is continuous when $\mathcal{S}$ is endowed with the weak logic topology. A continuous weak equivalence invariant $f : \mathcal{S} \to \mathbb{R}$ is nondecreasing if $f(a) \leq f(b)$ whenever $a$ is weakly contained in $b$, and nonincreasing if $f(a) \geq f(b)$ whenever $a$ is weakly contained in $b$ for $i = 1, 2, \ldots, d$.

Similarly one can define continuous elementary equivalence invariants by considering the logic topology instead of the weak logic topology. The following syntactic characterization of elementary equivalence invariants is a well known model-theoretic fact. We present a sketch of the proof for convenience of the reader. If $\mathcal{F}$ is a class of sentences, we say that a function $f$ is $\mathcal{F}$-approximable if for any $\varepsilon > 0$ there exists a sentence $\varphi$ in $\mathcal{F}$ such that $|\varphi^a - f(a)| < \varepsilon$ for any $a \in \mathcal{S}$.

**Proposition 2.11.** Fix a function $f : \mathcal{S} \to \mathbb{R}$.

1. $f$ is a continuous elementary equivalence invariant if and only if $f$ is $\mathcal{F}$-approximable, where $\mathcal{F}$ is the class of all sentences.
2. $f$ is a continuous weak equivalence invariant if and only if $f$ is $\mathcal{F}$-approximable, where $\mathcal{F}$ is the class of sentences of the form $\varphi - \psi$, where $\varphi, \psi$ are universal sentences.
3. $f$ is a nondecreasing continuous weak equivalence invariant if and only if $f$ is $\mathcal{F}$-approximable, where $\mathcal{F}$ is the class of universal sentences.
4. $f$ is a nonincreasing continuous weak equivalence invariant if and only if $f$ is $\mathcal{F}$-approximable, where $\mathcal{F}$ is the class of existential sentences.

**Proof.** For (1), necessity is obvious, while sufficiency can be seen by applying the Stone-Weierstrass theorem. The same argument gives (2). Finally, (3) can be proved similarly as [18, Proposition 2.4.6], and (4) follows by applying (3) to $-f$. \qed

The continuous elementary equivalence invariants are also called *definable predicates* in the model theory literature [5, §9].

**Definition 2.12.** A collection $\mathcal{S}$ of $\mu$-stationary actions on countably separated probability spaces is an axiomatizable class if there exists a collection $\mathcal{F}$ of continuous elementary equivalence invariants such that, given a $\mu$-stationary action $a$ on a countably separated probability space, one has that $a$ belongs to $\mathcal{S}$ if and only if $\varphi^a = 0$ for any $\varphi \in \mathcal{F}$. We say that $\mathcal{S}$ is universally axiomatizable if one can take $\mathcal{F}$ to be a collection of nondecreasing continuous weak equivalence invariants.

For instance, the collection of measure-preserving actions is an axiomatizable class. Similarly, the collection of $\mu$-stationary actions on the standard probability space is axiomatizable. Clearly, the intersection of a collection of axiomatizable classes is still axiomatizable. It follows from Proposition 2.11 that, whenever $\mathcal{S}$ is an axiomatizable class, then the logic topology and the weak logic topology on $\mathcal{S}$ are compact.
3. Stable weak equivalence and convexity

3.1. Stable weak equivalence. In parallel with the corresponding notion in the measure-preserving case, we consider now stable weak equivalence of \( \mu \)-stationary actions. Recall that we only consider \( \mu \)-stationary actions on countably separated probability spaces. We denote such a class by \( S \). Let \( \iota \) the trivial action of \( G \) on the standard (atomless) probability space.

**Definition 3.1.** Suppose that \( a, b \) are \( \mu \)-stationary actions. We say that \( a \) is stably weakly contained in \( b \) if \( a \times \iota \) is weakly contained in \( b \times \iota \). Similarly, \( a \) is stably weakly equivalent to \( b \) if \( a \times \iota \) is weakly equivalent to \( b \times \iota \).

One can also think about stable weak containment as the *restriction* of weak containment to the class of stable actions. We say that a \( \mu \)-stationary action \( a \) of \( G \) is *stable* if it weakly equivalent to \( a \times \iota \). This implies that \( a \) is an action on the standard probability space.

Given actions \( a, b \) are \( \mu \)-stationary actions on the standard probability space and \( t \in [0, 1] \) one can define their convex combination \( ta + (1 - t) b \) similarly as in the measure-preserving case. One can think about \( a \) as acting on \([0, t]\) and \( b \) acting on \([t, 1]\), and then let \( ta + (1 - t) b \) be the action on \([0, 1]\) that agrees with \( a \) on \([0, t]\) and with \( b \) on \([t, 1]\). The definition in the general case is analogous.

Observe that, when \( a, b \) are stable, then \( ta + (1 - t) b \) is stable as well. It is easy to see that an action \( a \) is stable if and only if it is weakly equivalent to \( \frac{1}{2} a + \frac{1}{2} a \). One can then deduce from this that the class of stable \( \mu \)-stationary actions of \( G \) is axiomatizable. We denote such a class by \( S^st \). It then follows from the remarks from §2.4 that the space \( S^st \) of weak equivalence classes of stable \( \mu \)-stationary actions of \( G \) is a compact metrizable space. The space \( S^st \) can be equivalently identified with the space of stable weak equivalence classes of arbitrary \( \mu \)-stationary actions on countably separated probability spaces.

The notion of convex combination of actions defines above induces a convex structure on \( S^st \), obtained by setting \( t [a] + (1-t) [b] = [ta + (1-t) b] \) for \( a, b \in S^st \). The fact that convex combinations commute with ultrapowers, together with the ultrapower characterization of weak equivalence, shows that such a notion of convex combination is well defined. This can also be seen directly from the definition of weak equivalence.

**Definition 3.2.** An *affine continuous weak equivalence invariant* is a continuous weak equivalence invariant \( f : S \rightarrow \mathbb{R} \) such that \( f(ta + (1-t)b) = tf(a) + (1-t)f(b) \) for any stable action \( a, b \) and \( t \in [0, 1] \).

Observe that, if \( f \) is an affine continuous weak equivalence invariant, then \( f(a) = f(b) \) whenever \( a \) and \( b \) are stably weakly equivalent. Thus it induces a continuous affine map from the space of stable weak equivalence classes to \( \mathbb{R} \).

We say that a quantifier free formula \( f(\psi_1, \ldots, \psi_n) \), where \( \psi_1, \ldots, \psi_n \) are atomic formulas, is affine if \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) is an affine map. An affine universal sentence is a universal sentence of the form \( \sup_{\pi} \varphi(\pi) \), where \( \varphi(\pi) \) is an affine quantifier-free formula.

**Proposition 3.3.** For a function \( f : S \rightarrow \mathbb{R} \), the following statements are equivalent:

1. \( f \) is an affine continuous weak equivalence invariant;
2. \( f \) is \( F \)-approximable, where \( F \) is the class of sentences of the form \( \varphi - \psi \), where \( \varphi, \psi \) are affine universal sentences.

We now present the proof of Proposition 3.3. If \( \sup_{\pi} \varphi(\pi) \) is an affine universal sentence and \( \lambda \) is a positive real number, then we identify \( \lambda \sup_{\pi} \varphi(\pi) \) with \( \sup_{\pi} \lambda \varphi(\pi) \) and \( -\sup_{\pi} \varphi(\pi) \) with \( \inf_{\pi} (-\varphi(\pi)) \). Similar conventions will apply to an existential sentence. In this way we can regard the space \( F \) of sentences of the form \( \varphi - \psi \), where \( \varphi \) and \( \psi \) are universal sentences, as a vector space. We consider a seminorm on \( F \) defined by \( \| \varphi - \psi \|_{S^st} = \sup_{\pi} |\varphi^{a} - \psi^{a}| \) where \( a \) varies among all the stable \( \mu \)-stationary actions on the standard probability space. We let \( V \) be the completion of \( F \) with respect to such a norm. We can canonically identify \( V \) with a subspace of the space \( C(S^st) \) of continuous functions on the compact space \( S^st \). Furthermore, it contains the function \( 1 \) constantly equal to 1. This makes \( V \) a (real) function system with order unit 1.
The state space \( S(V) \) of \( V \) is the space of linear functionals of norm 1 that are \textit{unital}, in the sense that map the order unit to 1. The state space \( S(V) \) is a compact convex set when endowed with the \textit{w*}-topology. Kadison’s representation theorem for function systems [2, Theorem II.8.1] shows that the canonical map from \( V \) to the space of continuous affine functions on \( S(V) \) is a surjective linear isometry.

We now observe that \( S(V) \) is affinely homeomorphic to \( S_{\text{aff}}^G \). Indeed any \( a \in S_{\text{aff}}^G \) induces a state \( s_a \) on \( V \) that extend the linear map \( \varphi \mapsto \varphi^a \) defined on \( F \). By definition of weak equivalence, such a linear functional only depends from the weak equivalence class of \( a \), and thus it induces a function from \( S_{\text{aff}}^G \) to \( S(V) \). Such a function is continuous by definition of the topology on \( S_{\text{aff}}^G \), affine since we are only considering affine formulas, and onto since the range separates the points of \( V \). It remains to prove that such a maps is injective. This will be a consequence of the following:

**Lemma 3.4.** Suppose that \( a, b \) are stable \( \mu \)-stationary actions on the standard probability space. Then \( a \) is weakly contained in \( b \) if and only if \( \varphi^a \leq \varphi^b \) for any affine universal sentence.

**Proof.** We have already observed that if \( a \) is weakly contained in \( b \), then \( \varphi^a \leq \varphi^b \) for any universal sentence. Conversely, suppose that \( a \) is not weakly contained in \( b \). Then there exist \( d \in \mathbb{N} \) and \( \delta > 0 \) and a finite subset \( F \) of \( G \) such that, letting \( A \) be the (finite) set of atomic formulas \( \psi(x_1, \ldots, x_d) \) in \( d \) free variables only involving group elements from \( F \), the following holds. Let \( C(a) \) be the set of elements of \([0,1]^A\) of the form \( \varphi \mapsto \psi^a(p_1, \ldots, p_d) \) for \( p_1, \ldots, p_d \in B \), and similarly define \( C(b) \). Then \( C(a) \) is not contained in \( C(b) \). Since \( a \) and \( b \) are stable actions, \( C(a) \) and \( C(b) \) are compact convex sets. Therefore by the geometric form of the Hahn-Banach theorem, there exist \( \varepsilon > 0 \) and an affine function \( f : \mathbb{R}^A \rightarrow \mathbb{R} \) such that \( f|_{C(a)} \geq \varepsilon \) and \( f|_{C(b)} \leq 0 \). Thus one can define the universal affine sentence \( \varphi = \sup_{\psi \in A} f((\psi)_x) \). By construction we have \( \varphi^a \geq \varepsilon \) and \( \varphi^b \leq 0 \). This concludes the proof.

The above discussion shows that the canonical map from \( V \) to the space \( A(S_{\text{aff}}^G) \) of continuous affine functions on \( S_{\text{aff}}^G \) is a surjective isometric isomorphism. In particular this provides a proof of Proposition 3.3. We define a class \( S \) of \( \mu \)-stationary stable actions to be convex axiomatizable if there exists a collection \( \mathcal{F} \) of continuous affine weak equivalence invariants such that a stable action \( a \) belongs to \( S \) if and only if \( f(a) \geq 0 \) for any \( f \in \mathcal{F} \). This is equivalent to the assertion that \( S_{\text{aff}} \) is a compact convex subset of \( S_{\text{aff}}^G \). In this case, \( A(S_{\text{aff}}) \) can be canonically identified with the quotient of \( A(S_{\text{aff}}^G) \) with respect to the subspace \( \{ f \in A(S_{\text{aff}}^G) : f|_{S_{\text{aff}}^G} = 0 \} \).

For instance the class \( S_{\text{mp}} \) of stable measure-preserving actions is convex axiomatizable by the universal affine sentences \( \sup_x (\mu(x) - \mu(gx)) \) for \( g \in G \). As a consequence, we obtain that \( S_{\text{mp}} \) is a metrizable compact convex set, and \( A(S_{\text{mp}}) \) is isometrically isomorphic to the completion of the space of differences of affine universal formulas with respect to the seminorm \( ||\varphi - \psi||_{S_{\text{mp}}} = \sup |\varphi^a - \psi^a| \) where \( a \) ranges among all the measure preserving stable actions of \( G \). This recovers the result from [13, 14] that \( S_{\text{mp}} \) is affinely homeomorphic to a metrizable compact convex subset of a Banach space.

### 3.2. The stationary random subgroup of a stationary action.

Let \( S(G) \) be the space of subgroups of \( G \) endowed with the Fell topology [23].

**Definition 3.5.** A \( \mu \)-stationary random subgroup or \( \mu \)-SRS is a Borel measure \( \nu \) on \( S(G) \) such that the action of \( G \) on \( (S(G), \nu) \) by conjugation is \( \mu \)-stationary.

Suppose that \( a \) is a \( \mu \)-stationary action of \( G \) on \((X_a, B_a, \nu_a)\). There is a Borel map from \( X_a \) to \( S(G) \) mapping \( x \) to its stabilizer subgroup \( st_a(x) \). In analogy with the measure-preserving case, we define the \( \mu \)-stationary random subgroup \( SRS(a) \) associated with \( a \) to be the push-forward measure of \( \nu_a \) under the map \( x \mapsto st_a(x) \). It is not difficult to verify that this is indeed a \( \mu \)-SRS. When \( a \) is measure-preserving, then \( SRS(a) \) is in fact an invariant random subgroup. For nonsingular automorphisms \( \alpha_1, \ldots, \alpha_n \) of \((X, \nu)\), we let \( \text{Fix} (\alpha_1, \ldots, \alpha_n) \) be the set of common fixed points of \( \alpha_1, \ldots, \alpha_n \), and \( \text{Free}(\alpha_1, \ldots, \alpha_n) \) be the set of \( x \in X \) such that \( \alpha_i x \neq x \) for every \( i = 1, 2, \ldots, n \). Suppose that \( g_1, \ldots, g_n, h_1, \ldots, h_m \in G \). Let \( A(g_1, \ldots, g_n, h_1, \ldots, h_m) \) be the set

\[ \{ H \in S(G) : g_1, \ldots, g_n \in H, h_1, \ldots, h_m \notin H \} \]
The measure of $\mathcal{A} (g_1, \ldots, g_n, h_1, \ldots, h_m)$ according to SRS $(a)$ is $\nu_a (\text{Fix} (g_1^a, \ldots, g_n^a) \cap \text{Free} (h_1^a, \ldots, h_m^a))$.

**Proposition 3.6.** The function $a \mapsto \text{SRS} (a) (\mathcal{A} (g_1, \ldots, g_n, h_1, \ldots, h_m))$ is an affine continuous weak equivalence invariant.

**Proof.** Suppose that $g \in G$ and $p \in B_a$. Fix a constant $M_g$ such that $dg^a \nu/d\nu \leq M_g$ and $dv/dg^a \nu \leq M_g$ for any $\mu$-stationary action $a$. Observe that $\nu_a (p \setminus \text{Fix} (g^a)) > t$ if and only if there exists $q \subset p$ such that $\nu_a ((q \cup g^a q \cup (g^n a)^{-1} q) \cap p) > t + 3M_g \nu_a (gq \cap q)$. It follows that $\nu_a (p \setminus \text{Fix} (g^a))$ is the supremum of $\nu_a ((q \cup g^a q \cup (g^n a)^{-1} q) \cap p) - 3M_g \nu_a (gq \cap q)$ where $q$ ranges among the subsets of $p$. Furthermore we have that $\nu_a (p \triangle \text{Fix} (g^a)) = 1 + 2\nu (p \setminus \text{Fix} (g)) - \nu (p) - \nu (X_a \setminus \text{Fix} (g))$. Define $\psi_g (x)$ to be the formula

$$\sup_y (\nu ((y_1 \cap x) \cup (g y_1 \cap x) \cup (g^{-1} y_1 \cap x)) \cap x) - 3M_g \nu (g(x \cap y) \cap (x \cap y))$$

and $\varphi_g (x)$ to be the formula $2\varphi_g (x) - \nu (x) + 1 - \psi_g (1)$. Then in view of the discussion above we have that $\varphi_g^a (p) = \nu_a (p \triangle \text{Fix} (g^a))$. Finally consider the sentence $\theta$ defined by

$$\sup_{x_1, \ldots, x_n \in X_a} \nu (x_1 \cap \cdots \cap x_n \cap (1 - y_1) \cap \cdots \cap (1 - y_m)) - (\varphi_{g_1} (x_1) + \cdots + \varphi_{g_n} (x_n) + \varphi_{h_1} (y_1) + \cdots + \varphi_{h_m} (x_m)).$$

Observe that $\vartheta^a = \nu_a (\text{Fix} (g_1^a, \ldots, g_n^a) \cap \text{Free} (h_1^a, \ldots, h_m^a))$ for any $\mu$-stationary action. In view of Proposition 2.11., this concludes the proof that the $a \mapsto \text{SRS} (a) ((\mathcal{A} (g_1, \ldots, g_n, h_1, \ldots, h_m)))$ is a continuous weak equivalence invariant. It is immediate to verify that such an invariant is affine. \hfill $\Box$

**Corollary 3.7.** The $\mu$-stationary random subgroup associated with a $\mu$-stationary action $a$ depends only on the stable weak equivalence class of $a$. Furthermore the map $[a \times e]_\omega \mapsto \text{SRS} (a)$ is a continuous affine map from the space of stable weak equivalence classes of $\mu$-stationary actions onto the space of $\mu$-stationary random subgroups of $G$.

### 3.3. Weak containment and $\mu$-stationary factors.

Suppose that $W$ is a compact metrizable space and $W^G$ is the corresponding topological Bernoulli shift. Following [11], we let $\mathcal{P}_\mu (W^G)$ be the space of $\mu$-stationary Borel probability measures on $W^G$. These are the Borel probability measures on $W^G$ that make the Bernoulli action $\mu$-stationary. The space $\mathcal{P}_\mu (W^G)$ is a Choquet simplex, whose extreme points are the ergodic $\mu$-stationary measures [3]. Given a $\mu$-stationary action $a$ on $(X_a, \mathcal{B}_a, \nu_a)$, we let $E (a, W)$ be the subspace of $\mathcal{P}_\mu (W^G)$ consisting of measures of the form $\phi \nu_a$, where $\phi : X_a \to W$ is a $G$-invariant Borel map. The same argument as for measure-preserving actions—see [36, §3]—shows that a $\mu$-stationary action $a$ is weakly contained in $b$ iff $E (a, W)$ is contained in the closure of $E (b, W)$ for some (equivalently, every) nontrivial compact metrizable space $W$. Similarly, $a$ is stably weakly contained in $b$ iff $E (a, W)$ is contained in the closed convex hull of $E (b, W)$ for some (equivalently, every) nontrivial compact metrizable space $W$. Therefore one can reformulate [11, Theorem 4.1] as follows:

**Theorem 3.8.** Any $\mu$-stationary action is weakly contained in any essentially free extension of the Poisson boundary of $(G, \mu)$.

### 4. FURSTENBERG ENTROPY AND BOUNDARY ACTIONS

#### 4.1. Furstenberg entropy as a weak equivalence invariant.

The Furstenberg entropy $h_\mu (a)$ of $\mu$-stationary action $a$ on $(X_a, \mathcal{B}_a, \nu_a)$ is given by

$$h_\mu (a) = \sum_{g \in G} \mu (g) \int_{X_a} - \log (dv/dg^a \nu) (x) dg^a \nu (x).$$

The goal of this section is to prove that the distribution of the Radon-Nikodym derivatives $dg_a \nu/d\nu$ and the Furstenberg entropy $h_\mu (a)$ are continuous weak-equivalence invariants.

**Theorem 4.1.** For $\mu$-stationary action $a$ of $G$, let $(X_a, \mathcal{B}_a, \nu_a)$ be the space on which $a$ acts. The function $a \mapsto \nu_a (x \in X_a : (dg^a \nu/d\nu) (x) > c)$ is an affine continuous weak equivalence invariant for $\mu$-stationary actions.
Proof. Let $M$ be a positive constant such that $dg^n \nu / dv \leq M$ and $dv / dg^n \nu \leq M$ for any $\mu$-stationary action $a$. Suppose that $(a_n)$ is a sequence of $\mu$-stationary actions and $a$ is a $\mu$-stationary action such that the sequence $(a_n)$ converges in the weak logic topology to $a$. For simplicity we will assume that $a_n$ and $a$ are actions on the standard probability space $X = [0, 1]$ endowed with the Borel $\sigma$-algebra and the Lebesgue measure $\nu$. Let $\omega = dg^n \nu / dv$, $\omega_n = dg^{a_n} \nu / dv$ for $n \in \mathbb{N}$, $C = \{ x \in X : \omega(x) > \epsilon \}$, and $C_n = \{ x \in X : \omega_n(x) > \epsilon \}$. We will prove that $\nu(C) \leq \lim \inf_n \nu(C_n)$. The proof that $\nu(C) \geq \lim \sup_n \nu(C_n)$ is analogous. Suppose by contradiction $\nu(C) > \lim \inf_n \nu(C_n)$. Thus, after passing to a subsequence, we can assume that there is $\delta > 0$ such that $\nu(C_n) \leq \nu(C) - \delta$ for every $n \in \mathbb{N}$. Define the Borel linear order $\prec$ on $X$ by letting $t \prec s$ if $\omega(t) < \omega(s)$ or $\omega(t) = \omega(s)$ and $t < s$. Similarly define $\prec_n$ in terms of $\omega_n$. Note that if $D$ is a terminal segment of $\prec$ then we have $\nu(g^n D) \geq \nu(g^n E)$ for any $E$ with $\nu(E) = \nu(D)$. For $n \in \mathbb{N}$ write $D_n$ for the terminal segment of $\prec$ such that $\nu(D_n) = \nu_n(C_n)$ and write $E_n$ for the terminal segment of $\prec_n$ such that $\mu(C) = \mu_n(E_n)$. Let also $F_n$ be the terminal segment of $\prec$ such that $\nu(F_n) = \nu(C_n) + \delta$ and let $K_n$ be the terminal segment of $\prec_n$ such that $\mu_n(K_n) = \mu(C_n) + \delta$. Clearly $D_n \subseteq F_n \subseteq C$ and $C_n \subseteq K_n \subseteq E_n$. We have
\[ \mu(F_n \setminus D_n) = \mu(F_n) - \mu(D_n) = \delta = \mu_n(K_n) - \mu_n(C_n) = \mu_n(K_n \setminus C_n) \] and similarly
\[ \mu(C \setminus F_n) = \mu_n(E_n \setminus K_n). \] Note that since $\omega(x) > \epsilon \geq \omega_n(y)$ if $x \in C$ but $y \in X \setminus C_n$, (2) implies
\[ \mu(g^n(C \setminus F_n)) \geq \mu_n(g^n(E_n \setminus K_n)). \] Let $H$ be the terminal segment of $\prec$ such that $\mu(H) = \mu(C) - \delta$ so that by (1) we have $\delta = \mu(C \setminus H) = \mu(F_n \setminus D_n)$. Since $F_n \setminus D_n \subseteq C$ and $C \setminus H$ has the lowest Radon-Nikodym derivative of any subset of $C$ with measure $\delta$ this implies
\[ \nu(g^n(C \setminus H)) \leq \nu(g^n(F_n \setminus D_n)). \] For $n \in \mathbb{N}$ from (1), (3) and (4) we have
\[
\nu(g^n(C \setminus D_n)) - \nu(g^n(E_n \setminus C_n)) \\
= \nu(g^n(C \setminus F_n)) + \mu(g^n(C \setminus D_n)) - \nu(g^n(E_n \setminus K_n)) - \nu(g^n(K_n \setminus C_n)) \\
\geq \nu(g^n(F_n \setminus D_n)) - \nu(g^n(K_n \setminus C_n)) \geq \nu(g^n(F_n \setminus D_n)) - c \cdot \nu(K_n \setminus C_n) \\
= \nu(g^n(F_n \setminus D_n)) - c \delta \geq \nu(g^n(C \setminus H)) - c \delta.
\] For $x \in C$ we have $\omega(x) > \epsilon$ so the last quantity is strictly positive. Choose
\[ 0 < \epsilon < \frac{1}{2(4 + M)} \cdot (\nu(g^n(C \setminus H)) - c \delta). \] Since by assumption $\varphi^a \leq \lim \inf_n \varphi^{a_n}$ for any universal sentence $\varphi$, there exists $n \in \mathbb{N}$ with the following property: for every Borel partition $A_1, \ldots, A_k$ of $X$ there is a partition $B_1, \ldots, B_k$ of $X$ such that $|\nu(A_i) - \nu(B_i)| < \epsilon$ and $|\nu(g^n A_i \cap A_j) - \nu(g^n B_i \cap B_j)| < \epsilon$ for all $i, j \in \{1, \ldots, k\}$. Fixing $n$, write $C' = C_n$, $D = D_n$, $E = E_n$ and $\epsilon' = \epsilon$. Note that from (5) and (6) we have
\[ 2(4 + M) \epsilon < \nu(g^n(C \setminus D)) - \nu(g^{a_n}(E \setminus C')). \] Let $A_1 = X \setminus C$ and $A_2 = C$. Find $B_1, B_2 \subseteq X$ such that $|\nu(A_i) - \nu(B_i)| < \epsilon$ and $|\nu(g^n A_i \cap A_j) - \nu(g^n B_i \cap B_j)| < \epsilon$ for each $i, j \in \{1, 2\}$. Note that
\[ \nu(X \setminus (B_1 \cup B_2)) \leq 2 \epsilon. \]
We have
\begin{align*}
\nu(g^n A_1) &= \nu(g^n A_1 \cap A_1) + \nu(g^n A_1 \cap A_2) \\
&\geq \nu(g^n B_1 \cap B_1) + \nu(g^n B_1 \cap B_2) - 2\varepsilon \\
&\geq \nu(g^n B_1 \cap B_1) + \nu(g^n B_1 \cap B_2) + \nu(g^n B_1 \setminus (B_1 \cup B_2)) - 4\varepsilon \\
&\geq \nu(g^n B_1) - 4\varepsilon.
\end{align*}
(8)

Note that
\[ \nu(B_1) \geq \nu(A_1) - \varepsilon = \nu(X \setminus C) - \varepsilon. \]
Write \( L \) for the initial segment of \( <' \) such that \( \nu(L) = \nu(X \setminus C) - \varepsilon \). Note that \( \nu(X \setminus E) = \nu(X \setminus C) \) and so \( \nu(X \setminus (E \cup L)) = \varepsilon \). We have
\[ \nu(g^n(X \setminus E)) = \nu(g^n L) + \nu(g^n (X \setminus (E \cup L))) \]
and therefore
\[ \nu(g^n L) \geq \nu(g^n(X \setminus E)) - M\varepsilon. \]
(9)

Since \( \nu(B_1) \geq \nu(L) \) and \( \nu(g^n L) \leq \nu(g^n J) \) for any \( J \subseteq X \) with \( \nu(J) \geq \nu(L) \) from (9) we see
\[ \nu(g^n B_1) \geq \nu(g^n(X \setminus E)) - M\varepsilon. \]

From (8) we have
\[ \nu(g^n(X \setminus C)) \geq \nu(g^n(X \setminus E)) - (4 + M)\varepsilon. \]
(10)

Now write \( A_1 = C' \) and \( A_2 = X \setminus C' \). Find \( B_1, B_2 \subseteq X \) such that
\[ |\nu(A_i \cap A_j) - \nu(B_i \cap B_j)| < \varepsilon \]
and
\[ |\nu(g^n A_i \cap A_j) - \nu(g^n B_i \cap B_j)| < \varepsilon \]
for each \( i, j \in \{1, 2\} \). Arguing as before we have \( \nu(g^n A_1) \leq \nu(g^n B_1) + 4\varepsilon \) and \( \nu(g^n B_1) \leq \nu(g^n D) + M\varepsilon \) so that
\[ \nu(g^n C') \leq \nu(g^n D) + (4 + M)\varepsilon. \]
(11)

From (10) and (11) we have
\[ \nu(g^n(X \setminus C \cup D)) \geq \nu(g^n((X \setminus E) \cup C')) - 2(4 + M)\varepsilon. \]
(12)

Note that
\[ D \cup (C \setminus D) \cup (X \setminus C) = X \]
and
\[ C' \cup (E \setminus C') \cup (X \setminus E) = X. \]

Thus from (7) and (12) we have
\[ 1 = \nu(g^n(D \cup (X \setminus C))) + \nu(g^n(C \setminus D)) \]
\[ \geq \nu(g^n(C' \cup (X \setminus E))) - 2(4 + M)\varepsilon + \nu(g^n(C \setminus D)) \]
\[ > \nu(g^n(C' \cup (X \setminus E))) + \nu(g^n(E \setminus C')) = 1 \]

which is the desired contradiction. This concludes the proof. \( \square \)

Theorem 4.1 implies the following:

Corollary 4.2. The Furstenberg entropy of a \( \mu \)-stationary action is an affine continuous weak equivalence invariant.

Corollary 4.2 implies, for instance, that any nontrivial boundary action is not weakly equivalent to a measure-preserving action.

Corollary 4.3. Suppose that \( G \) has property (T). Then the set of stable weak equivalence classes of ergodic \( \mu \)-stationary actions is not dense in the compact convex set of stable weak equivalence classes of \( \mu \)-stationary actions of \( G \).
Proof. As recalled in the introduction, it is proved in [30] that whenever $G$ has property (T), the set of values of the Furstenberg entropy of ergodic $\mu$-stationary actions has a gap around 0. The conclusion then follows from Corollary 4.2.

The same conclusions of Corollary 4.3 hold more generally for any group with entropy gap, as it is clear from the proof. Such a class of groups strictly contains the class of property (T) groups, as remarked in [11, Proposition 7.5].

Remark 4.4. The same proof as above shows that Theorem 4.1 and Corollary 4.2 hold more generally for any class of nonsingular actions $\alpha$ with the property that the Radon-Nikodym derivatives $dg^\alpha \nu/d\nu$ are uniformly bounded by a constant $M_\alpha$ depending only on $g$ and not on $\alpha$; see also Remark 2.4.

It is proved in [36, Theorem 1.8] that, in the measure preserving case, the weak equivalence class of an action of an amenable group $G$ is completely determined by its associated invariant random subgroup. In other words, the map $a \mapsto \text{IRS}(a)$ that assigns to a measure preserving action its associated invariant random subgroup induces an affine homeomorphism from the compact convex set of stable weak equivalence classes of measure preserving actions of $G$ onto the simplex of invariant random subgroups of $G$.

This is no longer true in the stationary case, since by Theorem 4.1 the distributions of the Radon-Nikodym derivatives $dg^\alpha \nu/d\nu$ provide a stable weak equivalence invariant. It is natural to conjecture that, for amenable groups, this is the only new piece of information needed, together with the associated stationary random subgroup, to completely determine the stable weak equivalence class of a given stationary action.

Problem 4.5. Suppose that $G$ is amenable group and $a, b$ are two $\mu$-stationary actions of $G$. Assume that $\text{SRS}(a) = \text{SRS}(b)$ and, for any $g \in G$ and $c \in \mathbb{R}$,

$$\mu\{x : (dg^a \nu/(d\nu))(x) > c\} = \mu\{x : (dg^b \nu/(d\nu))(x) > c\}.\]

Does it follow that $a$ and $b$ are stably weakly equivalent?

4.2. Furstenberg entropy of boundary actions. A $\mu$-stationary action of $G$ on a probability space $(X, \mathcal{B}, \nu)$ is strongly approximately transitive (SAT) as defined in [26] if and only if for any nonnull $p \in \mathcal{B}$ and $\varepsilon > 0$ there exists $g \in G$ such that $\nu(gp) > 1 - \varepsilon$. As remarked in the introduction, it is a consequence of [26, Proposition 2.2] together with the description of the Poisson boundary in terms of $\mu$-harmonic functions that a $\mu$-stationary action of $G$ on a probability space $(X, \mathcal{B}, \nu)$ is a boundary if and only if it is SAT.

In this section we use this fact to address the entropy realization problem for boundary actions.

Theorem 4.6. The set of values of the Furstenberg entropy of boundary $\mu$-stationary actions is a compact subset of $\mathbb{R}$.

We will prove Theorem 4.6 as an application of the omitting types theorem for the logic for metric structures, as stated in [19, Theorem 4.3]; see also [5, 17, 24]. We will first recall some notions from first order logic. Fix $k \in \mathbb{N}$ and suppose that $\varphi_{i,k}(x_1, \ldots, x_k)$ for $i \in \mathbb{N}$ are uniformly equicontinuous formulas. Let for every $n, k \in \mathbb{N}$, $t_{n,k}$ be the set of conditions $\{\varphi_{i,k}(x_1, \ldots, x_k) \geq 2^{-n} : i \in \mathbb{N}\}$. Such a set of conditions is a $k$-type in model-theoretic jargon, and the sequence $(t_{n,k})_{n \in \mathbb{N}}$ is a uniform sequence of $k$-types in the terminology of [19, §4]. Suppose that $a$ is a $\mu$-stationary action of $G$ on $(X_a, \mathcal{B}_a, \nu_a)$. A $k$-tuple $(p_1, \ldots, p_k)$ is a realization in $a$ of the type $t_{n,k}$ defined above if $\varphi_{i,k}(p_1, \ldots, p_k) \geq 2^{-n}$ for every $i \in \mathbb{N}$. The action $a$ omits the type $t_{n,k}$ if it contains no realization of $t_{n,k}$.

Definition 4.7. A class $\mathcal{S}$ of $\mu$-stationary actions of $G$ is an omitting types class if there exist formulas $\varphi_{i,k}(x_1, \ldots, x_k)$ for $i, k \in \mathbb{N}$ such that, letting $t_{n,k} = \{\varphi_{i,k}(x_1, \ldots, x_k) \geq 2^{-n} : i \in \mathbb{N}\}$ for every $n, k \in \mathbb{N}$, one has that a $\mu$-stationary action belongs to $\mathcal{S}$ if and only it omits the type $t_{n,k}$ for any $n, k \in \mathbb{N}$.

For instance, the class of $\mu$-stationary boundary actions is an omitting types class. Indeed fix an enumeration $\{g_i : i \in \mathbb{N}\}$ of $G$. Consider the formulas $\varphi_i(x_1, \ldots, x_k) := \max\{\nu(x_1), 1 - \nu(g_ix_1)\}$ for
Suppose that the Poisson boundary action of Theorem 4.9. Henceforth there exists a \( \phi \in n \) such that for any \( T \) be the Poisson boundary action, and \( \psi \) converges to \( v \). Without loss of generality we can assume that \( v \) is different from zero and from the Furstenberg entropy of the Poisson boundary \( \mu \). We can also assume, after passing to a subsequence, that the sequence \( \psi_n \) is monotone. Our goal is to prove that there exists a \( \mu \)-stationary boundary action \( \beta \) such that \( \mu (\beta) = v \).

By compactness, we can assume that the sequence \( (a_n) \) converges in the logic topology to a \( \mu \)-stationary action \( \alpha \). We also let \( \pi \) be the Poisson boundary action, and \( \nu \) be the trivial action on the standard probability space.

Recall that from Corollary 4.2 the Furstenberg entropy is an affine continuous weak equivalence invariant. Consider now the theory \( T \) consisting of the single condition \( \mu = v \). We let also \( \varphi_{i,k}(x_1,\ldots,x_k) \) for \( i,k \in N \) be the formulas described above witnessing that the class of boundary actions is an omitting types class. It remains to prove that the assumptions of Theorem 4.8 are satisfied.

Suppose that \( F \subset N \) is a finite subset, \( \varepsilon,\delta > 0 \), and \( \psi_{m}(x_1,\ldots,x_k) \) for \( m \in F \) are formulas such that for any \( \mu \)-stationary action \( \beta \) with Furstenberg entropy \( \mu \) one has that there exist \( p_1,\ldots,p_k \in B \) such that \( \psi_{k}(p_1,\ldots,p_k) \) is decreasing. Set \( t_n = 1 - \frac{\varepsilon}{2n} \), and \( b_n = (1 - t_n)a_n + t_n \). Observe that \( \mu (b_n) = v \) for every \( n \in N \) and \( t_n \rightarrow 0 \). Since \( a_n \) is a boundary action for every \( n \in N \), it follow from our assumption about the formulas \( \psi_{k}(p_1,\ldots,p_k) \) that, for \( n \in N \) large enough, one has that the action \( b_n \) has the property that for some \( k \in F \) and some \( i \in N \), there exist \( p_1,\ldots,p_k \in B \) such that \( \psi_{k}(p_1,\ldots,p_k) < \varepsilon \) and \( \varphi_{i,k}(p_1,\ldots,p_k) < \delta \). One can obtain a similar conclusion in the case when the sequence \( (\psi_{m}) \) is increasing by letting \( t_n = \frac{\varepsilon}{2n} \) and \( b_n = (1 - t_n)a_n + t_n \pi \). In either case this shows that the assumptions of Theorem 4.8 are satisfied. Henceforth there exists a \( \mu \)-stationary action of \( G \) with Furstenberg entropy \( v \).

The same argument as the one above with the extra ingredient of Corollary 3.7 proves the following fact:

**Theorem 4.9.** Suppose that the Poisson boundary action of \( (G,\mu) \) is free. Then the set of values of the Furstenberg entropy on free stationary \( \mu \)-stationary actions is a compact subset of \( \mathbb{R} \).

**References**


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