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CHAPTER 1

Introduction

1.1. Modeling opinion exchange

The exchange of opinions between individuals is a fundamental social interaction that plays a role in nearly any social, political and economic process. While it is unlikely that a simple mathematical model can accurately describe the exchange of opinions between two persons, one could hope to gain some insights on emergent phenomena that affect large groups of people.

Moreover, many models in this field are an excellent playground for mathematicians, especially those working in probability, algorithms and combinatorics. One of the goals of this survey is to introduce such models to mathematicians, and especially to those working in discrete mathematics, information theory, optimization and probability and statistics.

1.1.1. Modeling approaches. Many of the models we discuss in the survey comes from the literature in theoretical economics. In microeconomic theory, the main paradigm of modeling human interaction is by a game, in which participants are rational agents, choosing their moves optimally and responding to the strategies of their peers. A particularly interesting class of games is that of probabilistic Bayesian games, in which players also take into account the uncertainty and randomness of the world.

Another class of models, which have a more explicit combinatorial description, are what we refer to as heuristic models. These consider the dynamics that emerge when agents are assumed to utilize some (usually simple) update rule or algorithm when interacting with each other. Economists often justify such models as describing agents with bounded rationality.

It is interesting that both of these approaches are often justified by an Occam’s razor argument. To justify the heuristic models, the argument is that assuming that people use a simple heuristic satisfies Occam’s razor. Indeed, it is undeniable that the simpler the heuristic, the weaker the assumption. On the other hand, the Bayesian argument
is that even by choosing a simple heuristic one has too much freedom to reverse engineer any desired result. Bayesians therefore opt to only assume that agents are rational. This, however, may result in extremely complicated behavior.

There exists several other natural dichotomies and sub-dichotomies. In rational models, one can assume that agents tell each other their opinions. A more common assumption in Economics is that agents learn by observing each other’s actions; these are choices that an individual makes that not only reflect its belief, but also carry potential gain or penalty. For example, in financial markets one could assume that traders tell each other their value estimates, but a perhaps more natural setting is that they learn about these values by seeing which actual bids their peers place, since the latter are costly to manipulate. Hence the adage “actions speak louder than words.”

Some actions can be more revealing than others. A bid by a trader could reveal the value the trader believes the asset carries, but in a different setting it could perhaps just reveal whether the trader thinks that the asset is currently overpriced or underpriced. In other models an action could perhaps reveal all that an agent knows. We shall see that widely disparate outcomes can result in models that differ only by how revealing the actions are.

Although the distinction between opinions, beliefs and actions is sometimes blurry, we shall follow the convention of having agents learn from each other’s actions. While in some models this will only be a matter of nomenclature, in others this will prove to be a pivotal choice. The term belief will be reserved for a technical definition (see below), and we shall not use opinion, except informally.

1.2. Mathematical Connections

Many of the models of information exchange on networks are intimately related to nice mathematical concepts, often coming from probability, discrete mathematics, optimization and information theory. We will see how the theories of Markov chains, martingales arguments, influences and graph limits all play a crucial role in analyzing the models we describe in these notes. Some of the arguments and models we present may fit well as classroom materials or exercises in a graduate course in probability.

1.3. Related Literature

It is impossible to cover the huge body of work related to information exchange in networks. We will cite some relevant papers at
each section. However, some general comments apply to the economics literature:

- It is well accepted that many of the papers in the area have serious mathematical mistakes either in statements or in proofs. From our experience it is better to check all details in papers carefully to see what they prove and if the proofs are correct.
- Since the focus in economics is often more about the justification of the model and the interpretation of the results, many papers discuss very small variants of the same model using essentially the same proofs.
- For mathematicians who are used to models coming from natural sciences, the models in the economics literature will often look as very rough approximation and the conclusions drawn in terms of real life networks unjustified. Our view is that the models have very limited implication towards real life and can serve as most as allegories. We refer the readers who are interested in this point to Rubinstein’s book ”Economic Fables” [36].

1.4. Framework

The majority of models we consider share the a common underlying framework, which describes a set of agents, a state of the world, and the information the agents have regarding this state. We describe it formally in Section 1.5 bellow, and shall note explicitly whenever we depart from it.

We will take a probabilistic / statistical point of view in studying models. In particular we will assume that the model includes a random variable $S$ which is the true state of the world. It is this $S$ that all agents want to learn. For some of the models, and in particular the rational, economic models, this is a natural and even necessary modeling choice. For some other models - the voter model, for example (Section 2.2), this is a somewhat artificial choice. However, it helps us take a single perspective by asking, for each model, how well it performs as a statistical procedure aimed at estimating $S$. Somewhat surprisingly, we will reach similarly flavored conclusions in widely differing settings. In particular, a repeated phenomenon that we observe is that egalitarianism, or decentralization facilitates the flow of information in social networks, in both game-theoretical and heuristic models.
1.5. General definitions

1.5.1. Agents, state of the world and private signals. Let $V$ be a countable set of agents, which we take to be $\{1, 2, \ldots, n\}$ in the finite case and $\mathbb{N} = \{1, 2, \ldots\}$ in the infinite case. Let $\{0, 1\}$ be the set of possible values of the state of the world $S$.

Let $\Omega$ be a compact metric space equipped with the Borel sigma-algebra. For example, and without much loss of generality, $\Omega$ could be taken to equal the closed interval $[0, 1]$. Let $W_i \in \Omega$ be agent $i$'s private signal, and denote $\bar{W} = (W_1, W_2, \ldots)$.

Fix $\mu_0$ and $\mu_1$, two mutually absolutely continuous measures on $\Omega$. We assume that $S$ is distributed uniformly, and that conditioned on $S$, the $W_i$'s are i.i.d. $\mu_S$: when $S = 0$ then $\bar{W} \sim \mu_0^V$, and when $S = 1$ then $\bar{W} \sim \mu_1^V$.

More formally, let $\delta_0$ and $\delta_1$ be the distributions on $\{0, 1\}$ such that $\delta_0(0) = \delta_1(1) = 1$. We consider the probability space $\{0, 1\} \times \Omega^V$, with the measure $\mathbb{P}$ defined by

$$\mathbb{P} = \mathbb{P}_{\mu_0, \mu_1, V} = \frac{1}{2} \delta_0 \times \mu_0^V + \frac{1}{2} \delta_1 \times \mu_1^V,$$

and let

$$(S, \bar{W}) \sim \mathbb{P}.$$ 

1.5.2. The social network. A social network $G = (V, E)$ is a directed graph, with $V$ the set of agents. The set of neighbors of $i \in V$ is $\partial i = \{j : (i, j) \in E\} \cup \{i\}$ (i.e., $\partial i$ includes $i$). The out-degree of $i$ is given by $|\partial i|$. The degree of $G$ is give by $\sup_{i \in V} |\partial i|$.

We make the following assumption on $G$.

ASSUMPTION 1.5.1. We assume throughout that $G$ is simple and strongly connected, and that each out-degree is finite.

We recall that a graph is strongly connect if for every two nodes $i, j$ there exists a directed path from $i$ to $j$. Finite out-degrees mean that an agent observes the actions of a finite number of other agents. We do allow infinite in-degrees; this corresponds to agents whose actions are observed by infinitely many other agents. In the different models that we consider we impose various other constraints on the social network.

1.5.3. Time periods and actions. We consider the discrete time periods $t = 0, 1, 2, \ldots$, where in each period each agent $i \in V$ has to choose an action $A_i^t \in \{0, 1\}$. This action is a function of agent $i$’s private signal, as well as the actions of its neighbors in previous time periods, and so can be thought of as a function from $\Omega \times \{0, 1\}^{\partial i \times t}$ to $\{0, 1\}$. The exact functional dependence varies among the models.
1.5.4. Extensions, generalizations, variations and special cases. The framework presented above admits some natural extensions, generalizations and variations. Conversely, some special cases deserve particular attention. Indeed, some of the results we describe apply more generally, while others do not apply more generally, or apply only to special cases. We discuss these matters when describing each model.

- **The state of the world** can take values from sets larger than \( \{0, 1\} \), including larger finite sets, countably infinite sets or continuums.

- **The agents’ private signals** may not be i.i.d. conditioned on \( S \): they may be independent but not identical, they may be identical but not independent (for example, we consider a case when they are merely, pairwise independent), or they may have a general joint distribution.

  An interesting special case is when the space of private signals is equal to the space of the states of the world. In this case one can think of the private signals as each agent’s initial guess of \( S \).

- A number of models consider only undirected social networks, that is, symmetric social networks in which \((i, j) \in E \iff (j, i) \in E\).

- More general network model include weighted directed models where different directed edges have different weights.

- **Time** can be continuous. In this case we assume that each agent is equipped with an i.i.d. Poisson clock according to which it “wakes up” and acts. In the finite case this is equivalent to having a single, uniformly chosen random agent act in each discrete time period. It is also possible to define more general continuous time processes.

- **Actions** can take more values than \( \{0, 1\} \). In particular we shall consider the case that actions take values in \([0, 1]\).

  In order to model randomized behavior of the agents, we shall also consider actions that are not measurable in the private signal, but depend also on some additional randomness. This will require the appropriate extension of the measure \( \mathbb{P} \) to a larger probability space.

1.6. Questions

The main phenomena that we shall study are convergence, agreement, unanimity, learning and more.
1. INTRODUCTION

- **Convergence.** We say that agent $i$ converges when $\lim_{t \to \infty} A_i^t$ exists. We say that the entire process converges when all agents converge.

  The question of convergence will arise in all the models we study, and its answer in the positive will often be a requirement for subsequent treatment. When we do have convergence we define

  $$A_i^\infty = \lim_{t \to \infty} A_i^t.$$  

- **Agreement and unanimity.** We say that agents $i$ and $j$ agree when $\lim_{t \to \infty} A_i^t = \lim_{t \to \infty} A_j^t$. Unanimity is the event that $i$ and $j$ agree for all pairs of agents $i$ and $j$. In this case we can define

  $$A_{\infty} = A_i^\infty,$$

  where the choice of $i$ on the r.h.s. is immaterial.

- **Learning.** We say that agent $i$ learns $S$ when $A_i^\infty = S$, and that learning occurs in a model when all agents learn. In cases where we allow actions in $[0, 1]$, we will say that $i$ learns whenever $\text{round}(A_i^\infty) = S$, where $\text{round}(\cdot)$ denotes rounding to the nearest integer, with $\text{round}(1/2) = 1/2$.

  We will also explore the notion of asymptotic learning. This is said to occur for a sequence of graph $\{G_n\}_{n=1}^\infty$ if the agents on $G_n$ learn with probability approaching one as $n$ tends to infinity.

A recurring theme will be the relation between these questions and the geometry or topology of the social network. We shall see that indeed different networks may exhibit different behaviors in these regards, and that in particular, and across very different settings, decentralized or egalitarian networks tend to promote learning.

1.7. Acknowledgments

Allan Sly is our main collaborator in this field. We are grateful to him for allowing us to include some of our joint results, as well as for all that we learned from him. The manuscript was prepared for the 9th Probability Summer School in Cornell, which took place in July 2013. We are grateful to Laurent Saloff-Coste and Lionel Levine for organizing the school and for the participants for helpful comments and discussions. We would like to thank Shachar Kariv for introducing us to this field, and Eilon Solan for encouraging us to continue working in it. The research of Elchanan Mossel is partially supported by NSF grants
DMS 1106999 and CCF 1320105, and by ONR grant N000141110140. Omer Tamuz was supported by a Google Europe Fellowship in Social Computing.
CHAPTER 2

Heuristic Models

2.1. The DeGroot model

The first model we describe was pioneered by Morris DeGroot in 1974 \[13\]. DeGroot’s contribution was to take standard results in the theory of Markov Processes (See, e.g., Doob \[15\]) and apply them in the social setting. The basic idea for these models is that people repeatedly average their neighbors’ actions. This model has been studied extensively in the economics literature.

2.1.1. Definition. Following our general framework (Section \[1.5\]), we shall consider a state of the world \(S \in \{0, 1\}\) with conditionally i.i.d. private signals. The distribution of private signals is what we shall henceforth refer to as Bernoulli private signals: for some \(\frac{1}{2} > \delta > 0\), \(\mu_i(S) = \frac{1}{2} + \delta\) and \(\mu_i(1 - S) = \frac{1}{2} - \delta\), for \(i = 0, 1\). Obviously this is equivalent to setting \(P[W_i = S] = \frac{1}{2} + \delta\).

In the DeGroot model, we let the actions take values in \([0, 1]\). In particular, we define the actions as follows:

\[A_0^i = W_i\]

and for \(t > 0\)

\[A_t^i = \sum_{j \in \partial i} w(i, j)A_{t-1}^j,\]

where we make the following three assumptions:

(1) \(\sum_{j \in \partial i} w(i, j) = 1\) for all \(i \in V\).
(2) \(i \in \partial i\) for all \(i \in V\).
(3) \(w(i, j) > 0\) for all \((i, j) \in E\).

The last two assumptions are non-standard, and, in fact, not strictly necessary. We make them to facilitate the presentation of the results for this model.

We assume that the social network \(G\) is finite. We consider both the general case of a directed strongly connected network, and the special case of an undirected network.
2.1.2. Questions and answers. We shall ask, with regards to the DeGroot model, the same three questions that appear in Section 1.6.

(1) **Convergence.** Is it the case that agents’ actions converge? That is, does for each agent \( i \), the limit \( \lim_{t \to \infty} A_i^t \) exist almost surely? We shall show that this is indeed the case.

(2) **Agreement.** Do all agents eventually reach agreement? That is, does \( A_i^\infty = A_j^\infty \) for all \((i,j) \in V\)? Again, we answer this question in the positive.

(3) **Learning.** Do all agents learn? In the case of continuous actions we say that agent \( i \) has learned \( S \) if round \( (A_i^\infty) = S \). Since we have agreement in this model, it follows that either all agents learn or all do not learn. We will show that the answer to this question depends on the topology of the social network, and that, in particular, a certain form of egalitarianism is a sufficient condition for learning with high probability.

2.1.3. Results. The key to the analysis of the DeGroot model is the realization that Eq. 1 describes a transformation from the actions at time \( t - 1 \) to the actions at time \( t \) that is the *Markov operator* \( P_w \) of the a random walk on the graph \( G \). However, while usually the analysis of random walks deals with action of \( P_w \) on distributions from the right, here we act on functions from the left \[16\]. While this is an important difference, it is still easy to derive properties of the DeGroot process from the theory of Markov chains (see, e.g., Doob \[15\]).

Note first, that assumptions (2) and (3) on Eq. 1 make this Markov chain irreducible and a-periodic. Since, for a node \( j \)

\[
A_j^t = \mathbb{E} \left[ W_{X_j^t} \right],
\]

where \( X_j^t \) is the Markov chain started at \( j \) and run for \( t \) steps, if follows that \( A_j^\infty := \lim_t A_j^t \) is nothing but the expected value of the private signals, according to the stationary distribution of the chain. We thus obtain

**Theorem 2.1.1 (Convergence and agreement in the DeGroot model).**

*For each* \( j \in V \),

\[
A_\infty := \lim_t A_j^t = \sum_{i \in V} \alpha_i W_i,
\]

*where* \( \alpha = (\alpha_1, \ldots, \alpha_n) \) *is the stationary distribution of the Markov chain described by* \( P_w \).

Recall that \( \alpha \) is the left eigenvector of \( P_w \) corresponding to eigenvalue 1, normalized in \( \ell^1 \). In the internet age, the vector \( \alpha \) is also
known as the PageRank vector \[34\]. It is the asymptotic probability of finding a random walker at a given node after infinitely many steps of the random walk. Note that \(\alpha\) is not random; it is fixed and depends only the weights \(w\). Note also that Theorem \[2.1.1\] holds for any real valued starting actions, and not just ones picked from the distribution described above. To gain some insight into the result, let us consider the case of undirected graphs and simple (lazy) random walks. For these, it can be shown that

\[
\alpha_i = \frac{|\partial i|}{\sum_j |\partial j|}.
\]

Recall that \(\mathbb{P}[A_h^i = S] = \frac{1}{2} + \delta\). We observe the following.

**Proposition 2.1.2 (Learning in the DeGroot model).** For a set of weights \(w\), let \(p_w(\delta) = \mathbb{P}[\text{round}(A_{\infty}) = S]\). Then:

- \(p_w\) is a monotone function of \(\delta\) with \(p_w(0) = 1/2\) and \(p_w(1/2) = 1\).
- For a fixed \(0 < \delta < 1/2\), among all \(w\)’s on graphs of size \(n\), \(p_w(\delta)\) is maximized when the stationary distribution of \(G\) is uniform.

**Proof.**

- The first part follows by coupling. Note that we can couple the processes with \(\delta_1\) and \(\delta_2\) such that the value is \(S\) is the same and moreover, whenever \(W_1 = S\) in the \(\delta_1\) process we also have \(W_1 = S\) in the \(\delta_2\) process. Now, since the vector \(\alpha\) is independent of \(\delta\) and \(A_{\infty} = \sum_i \alpha_i W_i\), the coupling above results in \(|A_{\infty} - S|\) being smaller in the \(\delta_2\) process than it is in the \(\delta_1\) process.
- The second part follows from the Neyman-Peason lemma in statistics. This lemma states that among all possible estimators, the one that maximizes the probability that \(S\) is reconstructed correctly is given by

\[
\hat{S} = \text{round} \left( \frac{1}{n} \sum_i W_i \right)
\]

\[\square\]

We note that an upper bound on \(p_w(\delta)\) can be obtained using Hoeffding’s inequality \[22\]. We leave this as an exercise to the reader.

Finally, the following proposition is again a consequence of well known results on Markov chains. See the books by Saloff-Coste \[37\] or Levin, Peres and Wilmer \[25\] for basic definitions.
Proposition 2.1.3 (Rate of Convergence in the Degroot Model). Suppose that at time \( t \), the total variation distance between the chain started at \( i \) and run for \( t \) steps and the stationary distribution is at most \( \epsilon \) then a.s.:

\[
\max_i |A^i_t - A^i_\infty| \leq 2\epsilon \delta.
\]

Proof. Note that

\[
A^i_t - A^i_\infty = \mathbb{E} \left[ W_{X^i_t} - W_{X^\infty} \right].
\]

Since we can couple the distributions of \( X^i_t \) and \( X^\infty \) so that they disagree with probability at most \( \epsilon \) and the maximal difference between any two private signals is at most \( \delta \), the proof follows.

2.1.4. Degroot with cheaters and bribes. A cheater is an agent who plays a fixed action.

- **Exercise.** Consider the DeGroot model with a single cheater who picks some fixed action. What does the process converge to?
- **Exercise.** Consider the DeGroot model with \( k \) cheaters, each with some (perhaps different) fixed action. What does the model converge to?
- **Research problem.** Consider the following zero sum game. A and B are two companies. Each company’s strategy is a choice of \( k \) cheaters (cheaters chosen by both play honestly), for whom the company can choose a fixed value in \([0, 1]\). The utility of company \( A \) is the sum of the players’ limit actions, and the utility of company \( B \) is minus the utility of \( A \). What are the equilibria of this game?

2.1.5. The case of infinite graphs. Consider the DeGroot model on an infinite graph, with a simple random walk.

- **Easy exercise.** Give an example of specific private signals for which the limit \( A^\infty \) doesn’t exist.
- **Easy exercise.** Prove that \( A^\infty \) exists and is equal to \( S \) on non-amenable graphs a.s.
- **Harder exercise.** Prove that \( A^\infty \) exists and is equal to \( S \) on general infinite graphs.

2.2. The voter model

This model was described by P. Clifford and A. Sudbury [10] in the context of a spatial conflict where animals fight over territory (1973) and further analyzed by A. Holley and T.M. Liggett [23].
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2.2.1. Definition. As in the DeGroot model above, we shall consider a state of the world $S \in \{0, 1\}$ with conditionally i.i.d. Bernoulli private signals, so that $\mathbb{P}[W_i = S] = \frac{1}{2} + \delta$.

We consider binary actions and define them in a way that resembles our definition of the DeGroot model. We let:

$$A_0^i = W_i$$

and for $t > 0$, all $i$ and all $j \in \partial i$,

$$\mathbb{P}[A_t^i = A_{t-1}^j] = w(i, j),$$

so that in each round each agent chooses a neighboring agent to emulate. We make the following assumptions:

1. All choices are independent.
2. $\sum_{j \in \partial i} w(i, j) = 1$ for all $i \in V$.
3. $i \in \partial i$ for all $i \in V$.
4. $w(i, j) > 0$ for all $(i, j) \in E$.

As in the DeGroot model, the last two assumptions are non-standard, and are made to facilitate the presentation of the results for this model.

We assume that the social network $G$ is finite. We consider both the general case of a directed strongly connected network, and the special case of an undirected network.

2.2.2. Questions and answers. We shall ask, with regards to the voter model, the same three questions that appear in Section 1.6.

1. **Convergence.** Does, for each agent $i$, the limit $\lim_{t \to \infty} A_t^i$ exist almost surely? We shall show that this is indeed the case.
2. **Agreement.** Does $A_{i_{\infty}} = A_{j_{\infty}}$ for all $(i, j) \in V$? Again, we answer this question in the positive.
3. **Learning.** In the case of discrete actions we say that agent $i$ has learned $S$ if $A_{i_{\infty}} = S$. Since we have agreement in this model, it follows that either all agents learn or all do not learn. Unlike other models we’ve discussed, we will show that the answer here is no. Even for large egalitarian networks, learning doesn’t necessarily holds. We will later discuss a variant of the voter model where learning holds.

2.2.3. Results. We first note that

**Proposition 2.2.1.** In the voter model with assumptions (2) all agents converge to the same action.

**Proof.** The voter model is a Markov chain. Clearly the states where $A_t^i = 0$ for all $i$ and the state where $A_t^i = 1$ for all $i$ are absorbing states of the chain (once you’re there you never move). Moreover, it is
We next wish to ask what is the probability that the agents learned $S$? For the voter model this chance is never very high as the following proposition shows:

**Theorem 2.2.2 ((Non) Learning in the Voter model).** Let $A_\infty$ denote the limit action for all the agents in the voter model. Then:

\begin{align}
\mathbb{P}[A_\infty = 1|W] &= \sum_{i \in V} \alpha_i W_i,
\end{align}

and

\begin{align}
\mathbb{P}[A_\infty = S|W] &= \sum_{i \in V} \alpha_i 1(W_i = S).
\end{align}

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is the stationary distribution of the Markov chain described by $P_w$. Moreover,

\begin{align}
\mathbb{P}[A_\infty = S] &= \frac{1}{2} + \delta.
\end{align}

**Proof.** Note that Eq. 4 follows immediately from Eq. 3 and that Eq. 5 follows from Eq. 4 by taking expectation over $W$. To prove Eq. 3 we build upon a connection to the DeGroot model. Let $D_i$ denote the action of agent $i$ in the DeGroot model at time $t$. We are assuming that the DeGroot model is defined using the same $w(i, j)$ and that the private signals are identical for the voter and DeGroot model. Under these assumption it is easy to verify by induction on $i$ and $t$ that

\[ \mathbb{P}[A_i^t = 1] = D_i^t. \]

Thus

\[ \mathbb{P}[A_\infty^i = 1] = D_\infty^i = \sum_{i \in V} \alpha_i W_i, \]

as needed. □

In the next subsection we will discuss a variant of the voter model that does lead to learning.

We next briefly discuss the question of the convergence rate of the voter model. Here again the connection to the Markov chain of the DeGroot model is paramount (see, e.g., Holley and Liggett [23]). We won’t discuss this beautiful theory in detail. Instead, we will just discuss the case of undirected graphs where all the weights are 1.
Exercise: Consider the voter model on an undirected graph. This is equivalent to letting $w(i, j) = 1/d_i$ for all $i$, where $d_i = |\partial i|$.

- Show that $X_t = \sum d_i A_i^t$ is a martingale.
- Let $T$ be the stopping time where $A_i^t = 0$ for all $i$ or $A_i^t = 1$ for all $i$. Show that $\mathbb{E} [X_T] = \mathbb{E} [X_0]$ and use this to deduce that
  \[ \mathbb{P} [A_\infty = 1 | W] = \frac{\sum_{i \in V} d_i W_i}{\sum_{i \in V} d_i} \]
- Let $d = \max_i d_i$. Show that
  \[ \mathbb{E} [(X_t - X_{t-1})^2 | t < T] \geq 1/(2d). \]
  Use this to conclude that
  \[ \mathbb{E} [T] / (2d) \leq \mathbb{E} [(X_T - X_0)^2] \leq n^2, \]
  so
  \[ \mathbb{E} [T] \leq 2dn^2. \]

2.2.4. A variant of the voter model. As we just saw, the voter model does not lead to learning even on large egalitarian networks. It is natural to ask if there are variants of the model that do. We will now describe such a variant (see e.g. [5, 29]). For simplicity we consider a undirected graph $G = (V, E)$ and the following asynchronous dynamics.

- At time $t = 0$, let $A_i^0 = (W_i, 1)$.
- At each time $t \geq 1$ choose an edge $e = (i, j)$ of the graph at random and continue as follows:
  - For all $k \notin \{i, j\}$, let $A_k^t = A_k^{t-1}$.
  - Denote $(a_i, w_i) = A_i^{t-1}$ and $(a_j, w_j) = A_j^{t-1}$.
  - If $a_i \neq a_j$ and $w_i = w_j = 1$, let $a_i' = a_i, a_j' = a_j$ and $w_i' = w_j' = 0$.
  - If $a_i \neq a_j$ and $w_i = 1 > w_j = 0$, let $a_i' = a_j = a_i$ and $w_i' = w_j = 0$.
  - Similarly, if $a_i \neq a_j$ and $w_j = 1 > w_i = 0$, let $a_i' = a_j' = a_j$ and $w_i' = w_j = w_j$.
  - If $a_i \neq a_j$ and $w_j = w_i = 0$, let $a_i' = a_j' = 0(1)$ with probability 1/2 each. Let $w_i' = w_j' = 0$.
  - Otherwise, if $a_i = a_j$, let $a_i' = a_i, a_j' = a_j, w_i' = w_i, w_j' = w_j$.
  - With probability 1/2 let $A_i^t := (a_i', w_i')$ and $A_j^t := (a_j', w_j')$.
  - With probability 1/2 let $A_i^t := (a_j', w_j')$ and $A_j^t := (a_i', w_i')$.

Here is a useful way to think about this dynamics. The $n$ players all being with opinions given by $W_i$. Moreover these opinions are all strong (this is indicated by the second coordinate of the action being 1). At each round a random edge is chosen and the two agents sharing the
edge declare their opinions regarding $S$. If their opinions are identical, then nothing changes except that with probability $1/2$ the agents swap their location on the edge. If the opinions regarding $S$ differ and one agent is strong (second coordinate is 1) while the second one is weak (second coordinate is 0) then the weak agent is convinced by the strong agent. If the two agents are strong, then they keep their opinion but become weak. If the two of them are weak, then they both choose the same opinion at random. At the end of the exchange, the agents again swap their positions with probability $1/2$. We leave the following as an exercise:

**Proposition 2.2.3.** Let $A^t_i = (X^t_i, Y^t_i)$. Then a.s.

\[
\lim X^t_i = X,
\]

where

- $X = 1$ if $\sum_i W_i > n/2$,
- $X = 0$ if $\sum_i W_i < n/2$ and
- $\mathbb{P}[X = 1] = 1/2$ if $\sum_i W_i = n/2$.

Thus this variant of the voter model yields optimal learning.

### 2.3. Deterministic iterated dynamics

A natural deterministic model of discrete opinion exchange dynamics is **majority dynamics**, in which each agent adopts, at each time period, the opinion of the majority of its neighbors. This is a model that has been studied since the 1940’s in such diverse fields as biophysics [26], psychology [9] and combinatorics [21].

**2.3.1. Definition.** In this section, let $A^t_i$ take values in $\{-1, +1\}$, and let

\[
A^t_{i+1} = \text{sgn} \sum_{j \in \partial i} A^t_j.
\]

we assume that $|\partial i|$ is odd, so that there are never cases of indifference and $A^t_i \in \{-1, +1\}$ for all $t$ and $i$. We assume also that the graph is undirected.

A classical combinatorial result (that has been discovered independently repeatedly; see discussion and generalization in [21]) is the following.

**Theorem 2.3.1.** Let $G$ be a finite undirected graph. Then

\[
A^t_{i+1} = A^t_{i-1}
\]

for all $i$, for all $t \geq |E|$, and for all initial opinion sets $\{A^0_j\}_{j \in V}$. 
That is, each agent (and therefore the entire dynamical system) eventually enters a cycle of period at most two. We prove this below.

A similar result applies to some infinite graphs, as discovered by Moran [27] and Ginosar and Holzman [20]. Given an agent \( i \), let \( n_r(G, i) \) be the number of agents at distance exactly \( r \) from \( i \) in \( G \). Let \( g(G) \) denote the asymptotic growth rate of \( G \) given by

\[
g(G) = \limsup_r n_r(G, i)^{1/n}.
\]

This can be shown to indeed be independent of \( i \). Then

**Theorem 2.3.2 (Ginosar and Holzman, Moran).** If \( G \) has degree at most \( d \) and \( g < \frac{d+1}{d-1} \), then for each initial opinion set \( \{A^i_j\} \) and each \( i \in V \) there exists a time \( T_i \) such that

\[
A^i_{t+1} = A^i_{t-1}
\]

for all \( t \geq T_i \).

That is, each agent (but not the entire dynamical system) eventually enters a cycle of period at most two. We will not give a proof of this theorem.

In the case of graphs satisfying \( g(G) < (d + 1)/(d - 1) \), and in particular in finite graphs, we shall denote

\[
A^i_\infty = \lim_t A^i_{2t}.
\]

This exists surely, by Theorem 2.3.2 above.

In this model we shall consider a state of the world \( S \in \{-1, +1\} \) with conditionally i.i.d. Bernoulli private signals in \( \{-1, +1\} \), so that \( \mathbb{P}[W_i = S] = \frac{1}{2} + \delta \). As above, we set \( A^i_0 = W_i \).

**2.3.2. Questions and answers.** We ask the usual questions with regards to this model.

1. **Convergence.** While it is easy to show that agents’ opinions do not necessarily converge in the usual sense, they do converge to sequences of period at most two. Hence we will consider the limit action \( A^i_\infty = \lim_t A^i_{2t} \) as defined above to be the action that agent \( i \) converges to.

2. **Agreement.** This is easily not the case in this model that \( A^i_\infty = A^j_\infty \) for all \( i, j \in V \). However, in [28] it is shown that agreement is reached, with high probability, for good enough expanders.

3. **Learning.** Since we do not have agreement in this model, we will consider a different notion of learning. This notion may actually be better described as *retention of information*. We
2. Heuristic Models

define it below. Condorcet’s Jury Theorem \[11\], in an early version of the law of large numbers, states that given \( n \) conditionally i.i.d. private signals, one can estimate \( S \) correctly, except with probability that tends to zero with \( n \). The question of retention of information asks whether this still holds when we introduce correlations “naturally” by the process of majority dynamics.

Let \( G \) be finite, undirected graphs. Let

\[
\hat{S} = \arg\max_{s \in \{-1, +1\}} \mathbb{P}[S = s | A^1_{\infty}, \ldots, A^{|V|}_{\infty}].
\]

This is the maximum a-posteriori (MAP) estimator of \( S \), given the limit actions. Let

\[
\iota(G, \delta) = \mathbb{P}[\hat{S} \neq S],
\]

where \( G \) and \( \delta \) appear implicitly in the right hand side. This is the probability that the best possible estimator of \( S \), given the limit actions, is not equal to \( S \).

Finally, let \( \{G_n\}_{n \in \mathbb{N}} \) be a sequence of finite, undirected graphs. We say that we have retention of information on the sequence \( \{G_n\} \) if \( \iota(G_n, \delta) \to_n 0 \) for all \( \delta > 0 \). This definition was first introduced, to the best of our knowledge, in Mossel, Neeman and Tamuz \[28\].

Is information retained on all sequences of growing graphs? The answer, as we show below, is no. However, we show that information is retained on sequences of transitive graphs \[28\].

2.3.3. Convergence. To prove convergence to period at most two for finite graphs, we define the Lyapunov functional

\[
L_t = \sum_{(i,j) \in E} (A^i_{t+1} - A^i_t)^2.
\]

We prove Theorem \[2.3.1\] by showing that \( L_t \) is monotone decreasing, that \( A^i_{t+1} = A^i_{t-1} \) whenever \( L_t - L_{t-1} = 0 \), and that \( L_t = L_{t-1} \) for all \( t > |E| \). This proof appears (for a more general setting) in Goles and Olivos \[21\]. For this we will require the following definitions:

\[
J^i_t = (A^i_{t+1} - A^i_{t-1}) \sum_{j \in \partial i} A^j_t
\]

and

\[
J_t = \sum_{i \in V} J^i_t.
\]

Claim 2.3.3. \( J^i_t \geq 0 \) and \( J^i_t = 0 \) iff \( A^i_{t+1} = A^i_{t-1} \).
Proof. This follows immediately from the facts that
\[ A_{t+1}^i = \text{sgn} \sum_{j \in \partial i} A_j^i, \]
and that \( \sum_{j \in \partial i} A_j^i \) is never zero. \( \square \)

It follows that

Corollary 2.3.4. \( J_t \geq 0 \) and \( J_t = 0 \) iff \( A_{t+1}^i = A_{t-1}^i \) for all \( i \in V \).

We next that \( L_t \) is monotone decreasing.

Proposition 2.3.5. \( L_t - L_{t-1} = -J_t \).

Proof. By definition,
\[ L_t - L_{t-1} = \sum_{(i,j) \in E} (A_{t+1}^i - A_t^i)^2 - \sum_{(i,j) \in E} (A_t^i - A_{t-1}^j)^2. \]
Opening the parentheses and canceling identical terms yields
\[ L_t - L_{t-1} = -2 \sum_{(i,j) \in E} A_{t+1}^i A_t^j + 2 \sum_{(i,j) \in E} A_t^i A_{t-1}^j. \]

Since the graph is undirected we can change variable on the right sum and arrive at
\[ L_t - L_{t-1} = -2 \sum_{(i,j) \in E} A_{t+1}^i A_t^j - A_{t-1}^j A_t^i \]
\[ = -2 \sum_{(i,j) \in E} (A_{t+1}^i - A_{t-1}^i) A_t^j. \]
Finally, applying the definitions of \( J_t^i \) and \( J_t \) yields
\[ L_t - L_{t-1} = - \sum_{i \in V} J_t^i = -J_t. \]
\( \square \)

Proof of Theorem 2.3.1. Since \( L_0 \leq |E| \), \( L_t \leq L_{t-1} \) and \( L_t \) is integer, it follows that \( L_t \neq L_{t-1} \) at most \( |E| \) times. Hence, by Proposition 2.3.5, \( J_t > 0 \) at most \( |E| \) times. But if \( J_t = 0 \), then the state of the system at time \( t + 1 \) is the same as it was at time \( t - 1 \), and so it has entered a cycle of length at most two. Hence \( J_t = 0 \) for all \( t > |E| \), and the claim follows. \( \square \)
2.3.4. Retention of information. In this section we prove that

1. There exists a sequence of finite, undirected graphs \( \{ G_n \}_{n \in \mathbb{N}} \) of size tending to infinity such that \( \iota(G, \delta) \) does not tend to zero for any \( 0 < \delta < \frac{1}{2} \).

2. Let \( \{ G_n \}_{n \in \mathbb{N}} \) be a sequence of finite, undirected, connected transitive graphs of size tending to infinity. Then \( \iota(G_n, \delta) \to_n 0 \), and, furthermore, if we let \( G_n \) have \( n \) vertices, then

\[
\iota(G_n, \delta) \leq C n^{-C \delta \log(1/\delta)}
\]

for some universal constant \( C > 0 \).

A transitive graph is a graph for which, for every two vertices \( i \) and \( j \) there exists a graph homomorphism \( \sigma \) such that \( \sigma(i) = j \). A graph homomorphism \( h \) is a permutation on the vertices such that \( (i,j) \in E \) iff \( (\sigma(i), \sigma(j)) \in E \). Equivalently, the group \( \text{Aut}(G) \leq S_{|V|} \) acts transitively on \( V \).

Berger [7] gives a sequence of graphs \( \{ H_n \}_{n \in \mathbb{N}} \) with size tending to infinity, and with the following property. In each \( H_n = (V, E) \) there is a subset of vertices \( W \) of size 18 such that if \( A_i^t = -1 \) for some \( t \) and all \( i \in W \) then \( A_{\infty}^j = -1 \) for all \( j \in V \). That is, if all the vertices in \( W \) share the same opinion, then eventually all agents acquire that opinion.

**Proposition 2.3.6.** \( \iota(H_n, \delta) \geq (1 - \delta)^{18} \).

**Proof.** With probability \( (1 - \delta)^{18} \) we have that \( A_i^0 = -S \) for all \( i \in W \). Hence \( A_{\infty}^j = -S \) for all \( j \in V \), with probability at least \( (1 - \delta)^{18} \). Since the MAP estimator \( \hat{S} \) can be shown to be a symmetric and monotone function of \( A_{\infty}^j \), it follows that in this case \( \hat{S} = -S \), and so

\[
\iota(H_n, \delta) = \mathbb{P} \left[ \hat{S} \neq S \right] \geq (1 - \delta)^{18}.
\]

We next turn to prove the following result

**Theorem 2.3.7.** Let \( G \) a finite, undirected, connected transitive graph with \( n \) vertices, \( n \) odd. then

\[
\iota(G, \delta) \leq C n^{-C \delta \log(1/\delta)}
\]

for some universal constant \( C > 0 \).

Let \( \hat{S} = \text{sgn} \sum_{i \in V} A_i^\infty \) be the result of a majority vote on the limit actions. Since \( n \) is odd then \( \hat{S} \) takes values in \( \{-1, +1\} \). Note that
\( \hat{S} \) is measurable in the initial private signals \( W_i \). Hence there exists a function \( f : \{-1,+1\}^n \to \{-1,+1\} \) such that
\[
\hat{S} = f(W_1, \ldots, W_n).
\]

**Claim 2.3.8.** \( f \) satisfies the following conditions.

1. **Symmetry.** For all \( x = (x_1, \ldots, x_n) \in \{-1,+1\}^n \) it holds that \( f(-x_1, \ldots, -x_n) = -f(x_1, \ldots, x_n) \).
2. **Monotonicity.** \( f(x_1, \ldots, x_n) = 1 \) implies that \( f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) = 1 \) for all \( i \in [n] \).
3. **Anonymity.** There exists a subgroup \( G \leq S_n \) that acts transitively on \([n]\) such that \( f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n) \) for all \( x \in \{-1,+1\}^n \) and \( \sigma \in G \).

This claim is straightforward to verify, with anonymity a consequence of the fact that the graph is transitive.

**2.3.4.1. Influences, Russo’s formula, the KKL theorem and Talagrand’s theorem.** To prove Theorem 2.3.7 we use Russo’s formula, a classical result in probability that we prove below.

Let \( X_1, \ldots, X_n \) be random variables taking values in \( \{-1,+1\} \). For \(-\frac{1}{2} < \delta < \frac{1}{2}\), let \( \mathbb{P}_\delta \) be the distribution such that \( \mathbb{P}_\delta [X_i = +1] = \frac{1}{2} + \delta \) independently. Let \( g : \{-1,+1\}^n \to \{-1,+1\} \) be a monotone function (as defined above in Claim 2.3.8). Let \( Y = g(X) \), where \( X = (X_1, \ldots, X_n) \).

Denote by \( \tau_i : \{-1,+1\}^n \to \{-1,+1\}^n \) the function given by \( \tau_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n) \). We define the influence \( I_i^\delta \) of \( i \in [n] \) on \( Y \) as the probability that \( i \) is pivotal:
\[
I_i^\delta = \mathbb{P}_\delta [g(\tau_i(X)) \neq g(X)].
\]

That is \( I_i^\delta \) is the probability that the value of \( Y = g(X) \) changes, if we change \( X_i \).

**Theorem 2.3.9 (Russo’s formula).**
\[
\frac{d\mathbb{P}_\delta [Y = +1]}{d\delta} = \sum_i I_i^\delta,
\]

**Proof.** Let \( \mathbb{P}_{\delta_1, \ldots, \delta_n} \) be the distribution on \( X \) such that
\[
\mathbb{P}_{\delta_1, \ldots, \delta_n} [X_i = +1] = \delta_i.
\]

We prove the claim by showing that
\[
\frac{\partial \mathbb{P}_{\delta_1, \ldots, \delta_n} [Y = +1]}{\partial \delta_i} = \mathbb{P}_{\delta_1, \ldots, \delta_n} [g(\tau_i(X)) \neq g(X)],
\]

for all \( i \in [n] \).
and noting that $P_{\delta_1, \ldots, \delta_n} = P_\delta$, and that for general differentiable $h : \mathbb{R}^n \to \mathbb{R}$ it holds that
\[
\frac{\partial h(x, \ldots, x)}{\partial x} = \sum_i \frac{\partial h(x_1, \ldots, x_n)}{\partial x_i}.
\]

Indeed, if we denote $E = E_{\delta_1, \ldots, \delta_n}$ and $P = P_{\delta_1, \ldots, \delta_n}$, then
\[
\frac{\partial}{\partial \delta_i} P[Y = +1] = \frac{\partial}{\partial \delta_i} \frac{1}{2} E[g(X)].
\]

Denote $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Then
\[
E[g(X)] = \sum_x P[X_{-i} = x_{-i}, X_i = x_i] g(x)
\]
\[
= \sum_x P[X_{-i} = x_{-i}] P[X_i = x_i] g(x),
\]
where the second equality follows from the independence of the $X_i$'s. Hence
\[
\frac{\partial}{\partial \delta_i} P_{\delta_1, \ldots, \delta_n}[Y = +1] = \frac{\partial}{\partial \delta_i} \frac{1}{2} \sum_x P[X_{-i} = x_{-i}] P[X_i = x_i] g(x)
\]
\[
= \frac{1}{2} \sum_x P[X_{-i} = x_{-i}] x_i g(x),
\]
where the second equality follows from the fact that $P[X = +1] = \delta_i$ and $P[X = -1] = 1 - \delta_i$. Now, $\sum x_i g(x)$ is equal to zero when $g(\tau_i(x)) = g(x)$, and to two otherwise, since $g$ is monotone. Hence
\[
\frac{\partial}{\partial \delta_i} P_{\delta_1, \ldots, \delta_n}[Y = +1] = \sum_x P[X_{-i} = x_{-i}] \mathbb{1}(g(\tau_i(x)) \neq g(x))
\]
\[
= P[g(\tau_i(X)) \neq g(X)].
\]

Kahn, Kalai and Linial [24] prove a deep result on Boolean functions on the hypercube (i.e., functions from $\{-1, +1\}^n$ to $\{-1, +1\}$), which was later generalized by Talagrand [40]. Their theorem states that there must exist an $i$ with influence at least $O(\log n/n)$.

**Theorem 2.3.10 (Talagrand).** Let $\epsilon_\delta = \max_i I_i^\delta$ and $q_\delta = P_\delta [Y = 1]$. Then
\[
\sum_i I_i^\delta \geq K \log \left(1/\epsilon_\delta\right) q_\delta (1 - q_\delta).
\]
for some universal constant $K$. 
Using this result, the proof of Theorem 2.3.7 is straightforward, and we leave it as an exercise to the reader.
CHAPTER 3

Bayesian Models

In this chapter we study Bayesian agents. We call an agent Bayesian when its actions maximize the expectation of some utility function. This is a model which comes from Economics, where, in fact, its use is the default paradigm. We will focus on the case in which an agent’s utility depends only on the state of the world \( S \) and on its actions, and is the same for all agents and all time periods.

3.0.5. Toy model: continuous actions. Before defining general Bayesian models, we consider the following simple model on an undirected connected graph. Let \( S \in \{0, 1\} \) be a binary state of the world, and let the private signals be i.i.d. conditioned on \( S \).

We denote by \( H_i^t \) the information available to agent \( i \) at time \( t \). This includes its private signal, and the actions of its neighbors in the previous time periods:

\[
H_i^t = \{ W_i, A_{ij}^{t'} : j \in \partial i, t' < t \}.
\]

The actions are given by

\[
A_i^t = \mathbb{P} [ S = 1 | H_i^t ].
\]

That is, each agent’s action is its belief, or the probability that it assigns to the event \( S = 1 \), given what it knows.

For this model we prove the following results:

- **Convergence.** The actions of each agents converge almost surely to some \( A_i^\infty \). This is a direct consequence of the observation that \( \{H_i^t\}_{t \in \mathbb{N}} \) is a filtration, and so \( \{A_i^t\}_{t \in \mathbb{N}} \) is a bounded martingale. Note that this does not use the independence of the signals.

- **Agreement.** The limit actions \( A_i^\infty \) are almost surely the same for all \( i \in V \). This follows from the fact that if \( i \) and \( j \) are connected then \( A_i^\infty + A_j^\infty \in H_i^\infty \cap H_j^\infty \) and if \( A_i^\infty \) and \( A_j^\infty \) are not a.s. equal then:

\[
\mathbb{E} \left[ (\frac{1}{2}(A_i^\infty + A_j^\infty) - S)^2 \right] < \max \left( \mathbb{E} \left[ (A_i^\infty - S)^2 \right], \mathbb{E} \left[ (A_j^\infty - S)^2 \right] \right).
\]
Note again that this argument does not use the independence of the signals. We will show this in further generality in Section 3.2 below. This is a consequence of a more general agreement theorem that applies to all Bayesian models, which we prove in Section 3.1.

- **Learning.** When $|V| = n$, we show in Section 3.4 that $A_\infty^i = \mathbb{P}[S = 1|W_1, \ldots, W_n]$. This is the strongest possible learning result; the agents’ actions are the same as they would be if each agent knew all the others’ private signals. In particular, it follows that $\mathbb{P}\left[\text{round}(A_\infty^i) \neq S\right]$ is exponentially small in $n$. This result crucially relies on the independence of the signals as the following example shows.

**Example 3.0.11.** Consider two agents 1, 2 with $W_i = 0$ or 1 with probability 1/2 each and independently, and $S = W_1 + W_2 \mod 2$. Note that here $A_t^i = 1/2$ for $i = 1, 2$ and all $t$, while it is trivial to recover $S$ from $W_1, W_2$.

**3.0.6. Definitions and some observations.** Following our general framework (see Section 1.5) we shall (mostly) consider a state of the world $S \in \{0, 1\}$ chosen from the uniform distribution, with conditionally i.i.d. private signals. We will consider both discrete and continuous actions, and each shall correspond to a different utility function. We shall denote by $U^i_t$ agent $i$’s utility function at time $t$, and deal with myopic agents, or agents who strive to maximize, at each period $t$, the expectation of $U^i_t$. We will assume that $U^i_t = u(S, A^i_t)$ for some continuous function $u : \{0, 1\} \times [0, 1] \rightarrow [0, 1]$ that is independent of $i$ and $t$.

As in the toy model above, we denote by $H^i_t$ the information available to agent $i$ at time $t$, including its private signal, and the actions of its neighbors in the previous time periods:

$$H^i_t = \{W_i, A^j_{t'} : j \in \partial i, t' < t\}.$$

Given a utility function $U^i_t = u(S, A^i_t)$, a Bayesian agent will choose

$$A^i_t = \text{argmax}_s \mathbb{E}\left[u(S, s)|H^i_t\right].$$

Equivalently, one can define $A^i_t$ as a random variable which, out of all $\sigma(H^i_t)$-measurable random variables, maximizes the expected utility:

$$A^i_t = \text{argmax}_{A \in \sigma(H^i_t)} \mathbb{E}[u(S, A)].$$

We assume that in cases of indifference (i.e., two actions that maximize the expected utility) the agents chooses one according to some known deterministic rule.
We consider two utility functions; a discrete one that results in discrete actions, and a continuous one that results in continuous actions. The first utility function is

\[ U^i_t = \mathbb{1}(A^i_t = S). \]  

Although this function is not continuous as a function from \([0, 1]\) to \([0, 1]\), we will, in this case, consider the set of allowed actions to be \(\{0, 1\}\), and so \(u : \{0, 1\} \times \{0, 1\} \rightarrow \mathbb{R}\) will be continuous again.

To maximize the expectation of \(U^i_t\) conditioned on \(H^i_t\), a myopic agent will choose the action

\[ A^i_t = \arg\max_{s \in \{0, 1\}} \mathbb{P}[S = s | H^i_t], \]  

which will take values in \(\{0, 1\}\).

We will also consider the following utility function, which corresponds to continuous actions:

\[ U^i_t = 1 - (A^i_t - S)^2. \]  

To maximize the expectation of this function, an agent will choose the action

\[ A^i_t = \mathbb{P}[S = 1 | H^i_t]. \]  

This action will take values in \([0, 1]\).

An important concept in the context of Bayesian agents is that of belief. We define agent \(i\)'s belief at time \(t\) to be

\[ B^i_t = \mathbb{P}[S = 1 | H^i_t]. \]  

This is the probability that \(S = 1\), conditioned on all the information available to \(i\) at time \(t\). It is easy to check that, in the discrete action case, the action is the rounding of the belief. In the continuous action case the action equals the belief.

An important distinction is between bounded and unbounded private signals [39]. We say that the private signal \(W^i\) is unbounded when the private belief \(B^i_0 = \mathbb{P}[S = 1 | W^i]\) can be arbitrarily close to both 1 and 0; formally, when the convex closure of the support of \(B^i_0\) is equal to \([0, 1]\). We say that private signals are bounded when there exists an \(\epsilon > 0\) such \(B^i_0\) is supported on \([\epsilon, 1 - \epsilon]\).

Unbounded private signals can be thought of as being “unboundedly strong”, and therefore could be expected to promote learning. This is indeed the case, as we show below.

The following claim follows directly from the fact that the sequence of sigma-algebras \(\sigma(H^i_t)\) is a filtration.
Claim 3.0.12. The sequence of beliefs of agent $i$, $\{B^i_t\}_{t \in \mathbb{N}}$, is a bounded martingale.

It follows that a limiting belief almost surely exists, and we can define

$$B^i_\infty = \lim_{t \to \infty} B^i_t. \tag{16}$$

Furthermore, if we let $H^i_\infty = \bigcup_t H^i_t$, then

$$B^i_\infty = \mathbb{P} [S = 1 | H^i_\infty]. \tag{17}$$

We would like to also define the limiting action of agent $i$. However, it might be the case that the actions of an agent do not converge. We therefore define $A^i_t$ to be an action set, given by the set of accumulation points of the sequence $A^i_t$. In the case that $A^i_\infty$ is a singleton $\{x\}$, we denote $A^i_\infty = x$, in a slight abuse of notation. Note that in the case that actions take values in $\{0, 1\}$ (as we will consider below), $A^i_\infty$ is either equal to 1, to 0, or to $\{0, 1\}$.

The following claim is straightforward.

Claim 3.0.13. Fix a continuous utility function $u$. Then

$$\lim_{t} \mathbb{E} [u(S, A^i_t) | H^i_t] = \mathbb{E} [u(S, a) | H^i_\infty] \geq \mathbb{E} [u(S, b) | H^i_\infty]$$

for all $a \in A^i_\infty$ and all $b$.

That is, any action in $A^i_\infty$ is optimal (that is, maximizes the expected utility), given what the agent knows at the limit $t \to \infty$. It follows that

$$\mathbb{E} [u(S, a) | H^i_\infty] = \mathbb{E} [u(S, b) | H^i_\infty]$$

for all $a, b \in A^i_\infty$. It follows that in the case of actions in $\{0, 1\}$, $A^i_\infty = \{0, 1\}$ only if $i$ is asymptotically indifferent, or expects the same utility from both 0 and 1.

We will show that an oft-occurring phenomenon in the Bayesian setting is agreement on limit actions, so that $A^i_\infty$ is indeed a singleton, and $A^i_\infty = A^j_\infty$ for all $i, j \in V$. In this case we can define $A_\infty$ as the common limit action.

3.1. Agreement

In this section we show that regardless of the utility function, and, in fact, regardless of the private signal structure, Bayesian agents always reach agreement, except in cases of indifference. This theorem originated in the work of Aumann [2], with contributions by Geanakoplos and others [19, 38]. It first appeared as below in Gale and Kariv [17].
Rosenberg, Solan and Vieille [35] correct an error in the proof and extend this result to the even more general setting of strategic agents, which we shall not discuss.

**Theorem 3.1.1** (Gale and Kariv). Fix a utility function \( U_t^i = u(S, A_t^i) \), and consider \((i, j) \in E\). Then
\[
E \left[ u(S, a^i) \mid H^i_t \right] = E \left[ u(S, a^j) \mid H^j_t \right]
\]
for any \( a^i \in A^i_\infty \) and \( a^j \in A^j_\infty \).

That is, any action in \( A^j_\infty \) is optimal, given what \( i \) knows, and so has the same expected utility as any action in \( A^i_\infty \). Note that this theorem applies even when private signals are not conditionally i.i.d., and when \( S \) is not necessarily binary.

Eq. [10] is a particularly useful way to think of the agents’ actions, as the proof of the following claim shows.

**Claim 3.1.2.** For all \((i, j) \in E\) it holds that
1. \( E \left[ U^i_{t+1} \right] \geq E \left[ U^i_t \right] \).
2. \( E \left[ U^j_{t+1} \right] \geq E \left[ U^j_t \right] \).

**Proof.**
(1) Since \( \sigma(H^i_t) \) is included in \( \sigma(H^i_{t+1}) \), the maximum in Eq. [10] is taken over a larger space for \( A^i_{t+1} \) than it is for \( A^i_t \), and therefore a value at least as high is achieved.
(2) Since \( A^j_t \) is \( \sigma(H^i_{t+1}) \)-measurable, it follows from Eq. [10] that
\[
E \left[ u(S, A^i_{t+1}) \right] \geq E \left[ u(S, A^j_t) \right].
\]

The proof of the following corollary is left as exercise to the reader.

**Corollary 3.1.3.** For all \( i, j \in V \),
\[
\lim_t E \left[ U^i_t \right] = \lim_t E \left[ U^j_t \right]
\]

The proof of Theorem 3.1.1 follows directly from Corollary 3.1.3, Claim 3.0.13, and the fact that \( A^i_\infty \) is \( \sigma(H^i_\infty) \)-measurable whenever \((i, j) \in E\).

### 3.2. Continuous utility models

As mentioned above, in the case that the utility function is
\[
U_t^i = 1 - (A_t^i - S)^2,
\]
it follows readily that
\[
A^i_t = B^i_t = P \left[ S = 1 \mid H^i_t \right].
\]
and so, by Claim 3.0.12, the actions of each agent form a martingale, and furthermore each converge to a singleton $A^i_{\infty}$. Aumann’s celebrated Agreement Theorem from the paper titled “Agreeing to Disagree” [2], as followed-up by Geanakoplos and Polemarchakis in the paper titled “We can’t disagree forever” [19], implies that all these limiting actions are equal. This follows from Theorem 3.1.1.

**Theorem 3.2.1.** In the continuous utility model

$$A^i_{\infty} = P[S = 1 | H^i_{\infty}]$$

and furthermore

$$A^j_{\infty} = A^i_{\infty}$$

for all $i, j \in V$.

Note again that this holds also for private signals that are not conditionally i.i.d.

**Proof.** As was mentioned above, since the actions $A^i_t$ are equal to the beliefs $B^i_t$, they are a bounded martingale and therefore converge. Hence $A^i_{\infty} = B^i_{\infty}$ and, by Eq. 17,

$$A^i_{\infty} = P[S = 1 | H^i_{\infty}] .$$

Assume $(i, j) \in E$. By Theorem 3.1.1 we have that

$$E[u(S, A^i_{\infty}) | H^i_{\infty}] = E[u(S, A^j_{\infty}) | H^i_{\infty}] .$$

It hence follows from Claim 3.0.13 that both $A^i_{\infty}$ and $A^j_{\infty}$ maximize $E[u(S, \cdot) | H^i_{\infty}]$. But the unique maximizer is $P[S = 1 | H^i_{\infty}]$, and so $A^i_{\infty} = A^j_{\infty}$. For general $i$ and $j$, the claim now follows from the fact that the graph is strongly connected. □

### 3.3. Bounds on number of rounds in finite probability spaces

In this section we consider the case of a finite probability space. Let $S$ be binary, and let the private signals $\overline{W} = (W_1, \ldots, W_{|V|})$ be chosen from an arbitrary (not necessarily conditionally independent) distribution over a finite joint probability space of size $M$. Consider general utility functions $U^i_t = u(S, A^i_t)$.

The following theorem is a strengthening of a theorem by Geanakoplos [18], using ideas from [32].

**Theorem 3.3.1 (Geanakoplos).** Let $d$ be the diameter of the graph $G$. Then the actions of each agent converge after at most $M \cdot |V|$ time periods:

$$A^i_t = A^i_{t'}$$
for all \( i \in V \) and all \( t, t' \geq M \cdot |V| \). Furthermore, the number of time periods \( t \) such that \( A_{i_{t+1}} \neq A_{i_t} \) is at most \( M \).

The key observation is that each sigma-algebra \( \sigma(H_i^t) \) is generated by some subset of the set of random variables \( \{1(W = m)\}_{m \in \{1, \ldots, M\}} \).

**Proof.** By Eq. 10, if \( \sigma(H_i^t) = \sigma(H_i^{t'}) \) then \( A_i^t = A_i^{t'} \). It remains to show, then, that \( \sigma(H_i^t) = \sigma(H_i^{t'}) \) for all \( t, t' \geq M \cdot |V| \), and that \( \sigma(H_i^t) \neq \sigma(H_i^{t+1}) \) at most \( M \) times.

Now, every sub-sigma-algebra of \( \sigma(W) \) (such as \( \sigma(H_i^t) \)) is simply a partition of the finite space \( \{1, \ldots, M\} \). Furthermore, for every \( i \), the sequence \( \sigma(H_i^t) \) is a filtration, so that each \( \sigma(H_i^{t+1}) \) is a refinement of \( \sigma(H_i^t) \). A simple combinatorial argument shows that any such sequence has at most \( M \) unique partitions, and so \( \sigma(H_i^t) \neq \sigma(H_i^{t+1}) \) at most \( M \) times.

Finally, note that if \( \sigma(H_i^t) = \sigma(H_i^{t+1}) \) for all \( i \in V \) at some time \( t \), then this is also the case for all later time periods. Hence, as long as the process hasn’t ended, it must be that \( \sigma(H_i^t) \neq \sigma(H_i^{t+1}) \) for some agent \( i \). It follows that the process ends after at most \( M \cdot |V| \) time periods.

\[\Box\]

### 3.4. From agreement to learning

This section is adapted from Mossel, Sly and Tamuz [30].

In this section we prove two very general results that relate agreement and learning in Bayesian models. As in our general framework, we consider a binary state of the world \( S \in \{0, 1\} \) chosen from the uniform distribution, with conditionally i.i.d. private signals. We do not define actions, but only study what can be said when, at the end of the process (whatever it may be) the agents reach agreement.

Formally, consider a finite set of agents of size \( n \), or an countably infinite set of agents, each with a private signal \( W_i \). Let \( \mathcal{F}_i \) be the sigma-algebra that represents what is known by agent \( i \). We require that \( W_i \) is \( \mathcal{F}_i \) measurable (i.e., each agent knows its own private signal), and that each \( \mathcal{F}_i \) is a sub-sigma-algebra of \( \sigma(W_1, \ldots, W_n) \). Let agent \( i \)'s belief be

\[ B_i = \mathbb{P}[S = 1|\mathcal{F}_i] \]

and let agent \( i \)'s action be

\[ A_i = \text{argmax}_{s \in \{0, 1\}} \mathbb{P}[S = s|\mathcal{F}_i] \]

We let \( A_i = \{0, 1\} \) when both maximize \( \mathbb{P}[S = s|\mathcal{F}_i] \).
We say that agents agree on beliefs when there exists a random variable $B$ such that almost surely $B_i = B$ for all agents $i$. Likewise, we say that agents agree on actions when there exists a random variable $A$ such that almost surely $A_i = A$ for all agents $i$. Such agreement arises often as a result of repeated interaction of Bayesian agents.

We show below that agreement on beliefs is a sufficient condition for learning, and in fact implies the strongest possible type of learning. We also show that when private signals are unbounded beliefs then agreement on actions is also a condition for learning.

3.4.1. Agreement on beliefs. The following theorem and its proof is taken from Mossel, Sly and Tamuz [30]. This theorem also admits a proof as a corollary of some well known results on rational expectation equilibria (see, e.g., [14, 33]), but we will not delve into this topic.

**Theorem 3.4.1.** Let the private signals $(W_1, \ldots, W_n)$ be independent conditioned on $S$, and let the agents agree on beliefs. Then

$$B = \mathbb{P}[S = 1|W_1, \ldots, W_n].$$

That is, if the agents have exchanged enough information to agree on beliefs, they have exchanged all the relevant information, in the sense that they have the same belief that they would have had they shared all the information.

**Proof.** Denote agent $i$’s private log-likelihood ratio by

$$Z_i = \log \frac{d\mu_i}{d\mu_0}(W_i).$$

Since $\mathbb{P}[S = 1] = \mathbb{P}[S = 0] = 1/2$ it follows that

$$Z_i = \log \frac{\mathbb{P}[S = 1|W_i]}{\mathbb{P}[S = 0|W_i]}.$$

Denote $Z = \sum_{i \in [n]} Z_i$. Then, since the private signals are conditionally independent, it follows by Bayes’ rule that

$$\mathbb{P}[S = 1|W_1, \ldots, W_n] = \text{logit} (Z),$$

where $\text{logit}(z) = e^z/(e^z + e^{-z})$.

Since

$$B = \mathbb{P}[S = 1|B] = \mathbb{E}[\mathbb{P}[S = 1|B, W_1, \ldots, W_n]|B]$$

then

$$B = \mathbb{E}[\text{logit}(Z)|B],$$
since, given the private signals \((W_1, \ldots, W_n)\), further conditioning on \(B\) (which is a function of the private signals) does not change the probability of the event \(S = 1\).

Our goal is to show that \(B = \mathbb{P}[S = 1|B] = \mathbb{P}[S = 1|Z(B)] = \mathbb{P}[S = 1|W_1, \ldots, W_n]\).

By the law of total expectation we have that
\[
\mathbb{E}[Z_i \cdot \logit(Z)|B] = \mathbb{E}[\mathbb{E}[Z_i \cdot \logit(Z)|B, Z_i]|B].
\]

Note that \(\mathbb{E}[Z_i \cdot \logit(Z)|B, Z_i] = Z_i \mathbb{E}[\logit(Z)|B, Z_i]\) and so we can write
\[
\mathbb{E}[Z_i \cdot \logit(Z)|B] = \mathbb{E}[Z_i \cdot \mathbb{E}[\logit(Z)|B, Z_i]|B].
\]

Since \(Z_i\) is \(\mathcal{F}_i\) measurable, and since, by Eq. 19, \(B = \mathbb{E}[\logit(Z)|\mathcal{F}_i] = \mathbb{E}[\logit(Z)|B]\), then \(B = \mathbb{E}[\logit(Z)|B, Z_i]\) and so it follows that
\[
\mathbb{E}[Z_i \cdot \logit(Z)|B] = \mathbb{E}[Z_i B|B] = B \cdot \mathbb{E}[Z_i|B] = \mathbb{E}[\logit(Z)|B] \cdot \mathbb{E}[Z_i|B].
\]

where the last equality is another substitution of Eq. 19. Summing this equation over \(i \in [n]\) we get that
\[
\mathbb{E}[Z \cdot \logit(Z)|B] = \mathbb{E}[\logit(Z)|B] \mathbb{E}[Z|B].\tag{21}
\]

Now, since \(\logit(Z)\) is a monotone function of \(Z\), by Chebyshev’s sum inequality we have that
\[
\mathbb{E}[Z \cdot \logit(Z)|B] \geq \mathbb{E}[\logit(Z)|B] \mathbb{E}[Z|B]\tag{22}
\]

with equality only if \(Z\) (or, equivalently \(\logit(Z)\)) is constant. Hence \(Z\) is constant conditioned on \(B\) and the proof is concluded.

3.4.2. Agreement on actions. In this section we consider the case that the agents agree on actions, rather than beliefs. The boundedness of private beliefs plays an important role in the case of agreement on actions. When private beliefs are bounded then agreement on actions does not imply learning, as shown by the following example, which is reminiscent of Bala and Goyal’s royal family. However, when private beliefs are unbounded then learning does occur with high probability, as we show below.
Example 3.4.2. Let there be \( n > 100 \) agents, and call the first hundred “the Senate”. The private signals are bits that are independently equal to \( S \) with probability \( \frac{2}{3} \). Let

\[
A_S = \arg\max_a \mathbb{P}[S = a|W_1, \ldots, W_{100}],
\]

and let \( \mathcal{F}_i = \sigma(W_i, A_S) \).

This example describes the case in which the information available to each agent is the decision of the senate - which aggregates the senators’ private information optimally - and its own private signal. It is easy to convince oneself that \( A_i = A_S \) for all \( i \in [n] \), and so actions are indeed agreed upon. However, the probability that \( A_S \neq S \) - i.e., the Senate makes a mistake - is constant and does not depend on the number of agents \( n \). Hence the probability that the agents choose the wrong action does not tend to zero as \( n \) tends to infinity. This cannot be the case when private beliefs are unbounded, as Mossel, Sly and Tamuz [30] show.

Theorem 3.4.3 (Mossel, Sly and Tamuz). Let the private signals \((W_1, \ldots, W_n)\) be i.i.d. conditioned on \( S \), and have unbounded beliefs. Let the agents agree on actions. Then there exists a sequence \( q(n) = q(n, \mu_0, \mu_1) \), depending only on the conditional private signal distributions \( \mu_1 \) and \( \mu_0 \), such that \( q(n) \to 1 \) as \( n \to \infty \), and

\[
\mathbb{P}[A = S] \geq q(n).
\]

In particular,

\[
q(n) \leq \min_{\epsilon > 0} \max \left\{ \frac{2\epsilon}{1 - \epsilon}, \frac{4}{n \mathbb{P}[B_i < \epsilon | S = 0]} \right\}.
\]

For the case of a countably infinite set of agents, we prove (using an essentially identical technique) the following similar statement.

Theorem 3.4.4. Identify the set of agents with \( \mathbb{N} \), let the private signals \((W_1, W_2, \ldots)\) be i.i.d. conditioned on \( S \), and have unbounded beliefs. Let all but a vanishing fraction of the agents agree on actions. That is, let there exist a random variable \( A \) such that almost surely

\[
\limsup_n \frac{1}{n} |\{i \in \mathbb{N} : A_i \neq A\}| = 0.
\]

Then \( \mathbb{P}[A = S] = 1 \).

Recall that \( B^i_0 \) denoted the probability of \( S = 1 \) given agent \( i \)'s private signal:

\[
B^i_0 = \mathbb{P}[S = 1|W_i].
\]
The condition of unbounded beliefs can be equivalently formulated to be that for any $\epsilon > 0$ it holds that $\mathbb{P}[B_0^i < \epsilon] > 0$ and $\mathbb{P}[B_0^i > 1 - \epsilon] > 0$.

We shall need two standard lemmas to prove this theorem.

**Lemma 3.4.5.** $\mathbb{P}[S = 0|B_0^i < \epsilon] > 1 - \epsilon$.

**Proof.** Since $B_0^i$ is a function of $W_i$ then $\mathbb{P}[S = 1|B_0^i = b_i] = \mathbb{E}[\mathbb{P}[S = 1|F_i]|B_0^i = b_i] = \mathbb{E}[B_i^i|B_0^i = b_i] = b_i$, and so $\mathbb{P}[S = 0|B_0^i < \epsilon] > 1 - \epsilon$.

Lemma 3.4.6 below is a version of Chebyshev’s inequality, quantifying the idea that the expectation of a random variable $Z$, conditioned on some event $A$, cannot be much lower than its unconditional expectation when $A$ has high probability.

**Lemma 3.4.6.** Let $Z$ be a real valued random variable with finite variance, and let $A$ be an event. Then

$$\mathbb{E}[Z] - \sqrt{\frac{\text{Var}[Z]}{\mathbb{P}[A]}} \leq \mathbb{E}[Z|A] \leq \mathbb{E}[Z] + \sqrt{\frac{\text{Var}[Z]}{\mathbb{P}[A]}}$$

**Proof.** By Cauchy-Schwarz

$$|\mathbb{E}[Z1(A)] - \mathbb{E}[Z] \mathbb{P}[A]| = |\mathbb{E}[(Z - \mathbb{E}[Z]) \cdot 1(A)]| \leq \sqrt{\text{Var}[Z] \mathbb{P}[A]}.$$  

Dividing by $\mathbb{P}[A]$ and noting that $\mathbb{E}[Z1(A)]/\mathbb{P}[A] = \mathbb{E}[Z|A]$ we obtain the statement of the lemma.

We are now ready to prove Theorem 3.4.4.

**Proof of Theorem 3.4.4.** Consider a set of agents $N$ who agree (except for a vanishing fraction) on the action. Assume by contradiction that $q = \mathbb{P}[A \neq 0|S = 0] > 0$.

Recall that $B_i = \mathbb{P}[S = 1|F_i]$. Since $\mathbb{P}[S = 1|B_0^i] = B_0^i$,

$$\mathbb{E}[B_i|B_0^i] = \mathbb{E}[\mathbb{P}[S = 1|F_i]|B_0^i] = \mathbb{P}[S = 1|B_0^i] = B_0^i.$$  

Applying Markov’s inequality to $B_i$ we have that $\mathbb{P}[B_i \geq \frac{1}{2}|B_0^i < \epsilon] < 2\epsilon$, and in particular

$$\mathbb{P}[A_i \neq 0, S = 0|B_0^i < \epsilon] = \mathbb{P}[B_i \geq \frac{1}{2}, S = 0|B_0^i < \epsilon] < 2\epsilon$$

so

$$\mathbb{P}[A_i \neq 0, S = 0, B_0^i < \epsilon] \leq 2\epsilon \mathbb{P}[B_0^i < \epsilon]$$
Denote
\begin{equation}
K(n) = \frac{1}{n} \sum_{i \in [n]} \mathbb{1}(B^i_0 < \epsilon) = \frac{1}{n} \sum_{i \in [n]} \mathbb{1}(B^i_0 < \epsilon, A_i = 0) + \frac{1}{n} \sum_{i \in [n]} \mathbb{1}(B^i_0 < \epsilon, A_i \neq 0)
\end{equation}

Let $K_1(n)$ denote the first sum and $K_2(n)$ denote the second sum. From our assumption that a vanishing fraction of agents disagree it follows that a.s.
\begin{align*}
\limsup_n \mathbb{E}[K_1(n)|A \neq 0, S = 0] &\leq \frac{1}{q} \limsup_n \mathbb{E}[K_1(n)|A \neq 0] \\
&\leq \frac{1}{q} \limsup_n \mathbb{E} \left[ \frac{1}{n} \sum_{i \in [n]} \mathbb{1}(A_i = 0) \Big| A \neq 0 \right] = 0.
\end{align*}

It also follows that for all $n$
\begin{align*}
\mathbb{E}[K_2(n)|A \neq 0, S = 0] &\leq \frac{1}{q} \mathbb{E}[K_2(n), A \neq 0, S = 0] \leq \frac{2\epsilon \mathbb{P}[B^i_0 < \epsilon]}{q}.
\end{align*}

Thus
\begin{equation}
\limsup_n \mathbb{E}[K(n)|A \neq 0, S = 0] \leq \frac{2\epsilon \mathbb{P}[B^i_0 < \epsilon]}{q}.
\end{equation}

We hence bound $\mathbb{E}[K|A \neq 0, S = 0]$ from above. We will now bound it from above to obtain a contradiction.

Applying lemma 3.4.6 to $K$ and the event “$A \neq 0$” (under the conditional measure $S = 0$) yields that
\begin{align*}
\mathbb{E}[K(n)|A \neq 0, S = 0] \geq \mathbb{E}[K(n)|S = 0] - \sqrt{\frac{\text{Var}[K(n)|S = 0]}{q}}.
\end{align*}

Since the agents’ private signals (and hence their private beliefs) are independent conditioned on $S = 0$, $K$ (conditioned on $S$) is the average of $n$ i.i.d. variables. Hence $\text{Var}[K(n)|S = 0] = n^{-1} \text{Var}[\mathbb{1}(B^i_0 < \epsilon)|S = 0]$ and $\mathbb{E}[K(n)|S = 0] = \mathbb{P}[B^i_0 < \epsilon|S = 0]$. Thus we have that
\begin{align*}
\mathbb{E}[K(n)|A \neq 0, S = 0] \geq \mathbb{P}[B^i_0 < \epsilon|S = 0] - n^{-1/2} \sqrt{\frac{\text{Var}[\mathbb{1}(B^i_0 < \epsilon)|S = 0]}{q}}.
\end{align*}

and so
\begin{align*}
\liminf_n \mathbb{E}[K(n)|A_i \neq 0, S = 0] \geq \mathbb{P}[B^i_0 < \epsilon|S = 0]
\end{align*}
Joining the lower bound with the upper bound we obtain that
\[ P[B_0^i < \epsilon | S = 0] \leq \frac{2\epsilon P[B_0^i < \epsilon]}{q}, \]
and applying Bayes rule we obtain
\[ q \leq \frac{\epsilon}{P[S = 0 | B_0^i < \epsilon]}. \]
Since by Lemma 3.4.5 above we know that \( P[S = 0 | B_0^i < \epsilon] > 1 - \epsilon \), then
\[ q \leq \frac{\epsilon}{1 - \epsilon}. \]
Since this holds for all \( \epsilon \), we have shown that \( q = 0 \), which is a contradiction. \( \square \)

### 3.5. Sequential Models

In this section we consider a classical class of learning models called **sequential models**. We retain a binary state of the world \( S \) and conditionally i.i.d. private signals, but relax two assumptions.

- We no longer assume that the graph \( G \) is strongly connected. In fact, we consider the particular case that the set of agents is countably infinite, identify it with \( \mathbb{N} \), let and \((i,j) \in E \) iff \( j < i \). That is, the agents are ordered, and each agent observes the actions of its predecessors.
- We assume that each agent acts once, after observing the actions of its predecessors. That is, agent \( i \) acts only once, at time \( i \).

In this section, we denote agent \( i \)'s (single) action by \( A_i \). Hence agent \( i \)'s information when taking its action, which we denote by \( H_i \), is
\[ H_i = \{ W_i, A_j : j < i \}. \]
We likewise denote agent \( i \)'s belief at time \( i \) by \( B_i = P[S = 1 | H_i] \). We assume discrete utilities, so that
\[ A_i = \arg\max_{s \in \{0,1\}} P[S = s | H_i], \]
and let \( A_i = 1 \) when \( P[S = 1 | H_i] = 1/2 \).

Since each agent acts only once, we explore a different notion of learning in this section. The question we consider is the following: when is it the case that \( \lim_{i \to \infty} A_i = S \) with probability one? Since the graph is fixed, the answer to this question depends only on the private signal distributions \( \mu_0 \) and \( \mu_1 \).
This model (in a slightly different form) was introduced independently by Bikhchandani, Hirshleifer and Welch [8], and Banerjee [4]. A significant later contribution was that of Smith and Sørensen [39].

An interesting phenomenon that arises in this model is that of an information cascade. An information cascade is said to occur if, given an agent $i$’s predecessor’s actions, $i$’s action does not depend on its private signal. This happens if the previous agents’ actions present such compelling evidence towards the event that (say) $S = 1$, that any realization of the private signal would not change this conclusion. Once this occurs - that is, once one agent’s action does not depend on its private signal - then this will also hold for all the agents who act later.

3.5.1. The external observer at infinity. An important tool in the analysis of this model is the introduction of an external observer $x$ that observes all the agents’ actions but none of their private signals. We denote by $H^t_x = \{A_j : j < i\}$ the information available to $x$ at time $i$, and denote by

$$B^t_x = \mathbb{P} [S = 1 | H^t_x]$$

and

$$B^\infty_x = \lim_{i \to \infty} B^t_x = \mathbb{P} [S = 1 | H^\infty_x]$$

the beliefs of $x$ at times $t$ and infinity respectively, where, as before, $H^\infty_x = \bigcup_i H^t_x$. The same martingale argument used above can also be used here to show that the limit $B^\infty_x$ indeed exists and satisfies the equality above.

A more subtle argument reveals that the likelihood ratio

$$L^t_x = \frac{1 - B^t_x}{B^t_x}$$

is also a martingale, conditioned on $S = 1$. This fact won’t be used below. See Smith and Sørensen [39] for a proof.

The martingale $\{B^t_x\}$ converges almost surely to $B^\infty_x$ in $[0, 1]$, and conditioned on $S = 1$, $B^\infty_x$ has support $\subseteq (0, 1]$. The reason that $B^\infty_x \neq 0$ when conditioning on $S = 1$, is the fact that $\mathbb{P} [S = 1 | B^\infty_x] = B^\infty_x$, and so $\mathbb{P} [S = 1 | B^\infty_x = 0] = 0$.

We also define actions for $x$, given by

$$A^t_x = \arg\max_{s \in \{0, 1\}} \mathbb{P} [S = s | H^t_x] = \text{round} (B^t_x) .$$

We again assume that in cases of indifference, the action 1 is chosen.

Claim 3.5.1. $A^t_{i+1} = A_i$
That is, the external observer simply copies, at time $t+1$, the action of agent $t$. This follows immediately from the fact that $A_t$ is $\sigma(H_t)$-measurable, and so $H_{t+1}^x \subseteq H_t$. It follows that $\lim_t A_t = \lim_t A_t^x$, and so we have learning - in the sense we defined above for this section by $\lim_t A_t = S$ - iff the external observer learns in the usual sense of $\lim_t A_t^x = S$.

3.5.2. The agents’ calculation. We write out each agent’s calculation of its belief $B_i$, from which follows its action $A_i$. This is more easily done by calculating the likelihood ratio

$$L_i = \frac{1 - B_i}{B_i}.$$ 

By Bayes’ law, since $\mathbb{P}[S = 1] = \mathbb{P}[S = 0] = \frac{1}{2}$, and since $H_i = (H_i^x, W_i)$

$$L_i = \frac{\mathbb{P}[S = 0|H_i]}{\mathbb{P}[S = 1|H_i]} = \frac{\mathbb{P}[H_i|S = 0]}{\mathbb{P}[H_i|S = 1]} = \frac{\mathbb{P}[H_i^x, W_i|S = 0]}{\mathbb{P}[H_i^x, W_i|S = 1]}.$$ 

Since the private signals are conditionally i.i.d., $W_i$ is conditionally independent of $H_i^x$, and so

$$L_i = \frac{\mathbb{P}[H_i^x|S = 0]}{\mathbb{P}[H_i^x|S = 1]} \cdot \frac{\mathbb{P}[W_i|S = 0]}{\mathbb{P}[W_i|S = 1]}.$$ 

We denote by $P_i$ the private likelihood ratio $\mathbb{P}[W_i|S = 0]/\mathbb{P}[W_i|S = 1]$, so that

(25) $$L_i = L_i^x \cdot P_i.$$ 

3.5.3. The Markov chain and the martingale. Another useful observation is that $\{B_i^x\}_{i \in \mathbb{N}}$ is not only a martingale, but also a Markov chain. We denote this Markov chain on $[0, 1]$ by $\mathcal{M}$. To see this, note that conditioned on $S$, the private likelihood ratio $P_i$ is independent of $B_j^x$, $j < i$, and so its distribution conditioned on $B_i^x = \mathbb{P}[S = 1|H_i^x]$ is the same as its distribution conditioned on $(B_0^x, \ldots, B_i^x)$, which are $\sigma(H_i^x)$-measurable.

3.5.4. Information cascades, convergence and learning. An information cascade is the event that, for some $i$, conditioned on $H_i^x$, $A_i$ is independent of $W_i$. That is, an information cascade is the event that the observer at infinity knows, at time $i$, which action agent $i$ is going to take, even though it only knows the actions of $i$’s predecessors and does not know $i$’s private signal. Equivalently, an information cascade occurs when $A_i$ is $\sigma(H_i^x)$-measurable. It is easy to see that it follows that $A_j$ will also be $\sigma(H_j^x)$-measurable, for all $j \geq i$. 
Claim 3.5.2. An information cascade is the event that $B_i^x$ is a fixed point of $M$.

Proof. If $A_i$ is $\sigma(H_i^x)$ measurable then $\sigma(H_i^x) = \sigma(H_i^x, A_i) = \sigma(H_{i+1}^x)$. It follows that

$$B_i^x = \mathbb{P}[S = 1|H_i^x] = \mathbb{P}[S = 1|H_{i+1}^x] = B_{i+1}^x.$$  

Conversely, if $B_i^x = B_{i+1}^x$ w.p. one, then $A_i^x = A_{i+1}^x$ with probability one, and it follows that $A_i = A_{i+1}$ is $\sigma(H_i^x)$-measurable. □

Theorem 3.5.3. The limit $\lim_{i} A_i$ exists almost surely.

Proof. As noted above, $A_i = A_{i+1}^x$. Assume by contradiction that $A_{i+1}^x$ takes both values infinitely often. Since $A_i^x = 1(B_i^x \geq \frac{1}{2})$, and since $B_i^x$ converges to $B_{\infty}^x$, it follows that $B_{\infty}^x = \frac{1}{2}$.

Note that by the Markov chain nature of $\{B_i^x\}$,

$$B_{i+1}^x = f(B_i^x, A_i)$$

for $f : [0,1] \times \{0,1\} \rightarrow [0,1]$ independent of $i$ and given by

$$f(b,a) = \mathbb{E}[B_i|B_i^x = b, A_i = a].$$

Since $A_i = 1(B_i \geq \frac{1}{2})$, it follows that $B_i = |B_i - \frac{1}{2}|(2A_i - 1) + \frac{1}{2}$, and so

$$f(b,a) = \mathbb{E}[|B_i - \frac{1}{2}|B_i^x = b, A_i = a](2a - 1) + 1/2.$$  

Hence $f$ is continuous at $(1/2,1)$ and $(1/2,0)$, even if $B_i = \frac{1}{2}$ with positive probability. It follows by taking the limit of Eq. 26 that if $\lim B_i^x = 1/2$ then $f(1/2,1) = f(1/2,0)$. But then $B_i^x$ would equal $f(1/2, \cdot)$ for all $i$, since $B_0^x = 1/2$, and $A_i^x = 1$ for all $i$, which is a contradiction. □

Since $\lim_i A_i$ exists almost surely we can define

$$A = \lim_i A_i.$$  

Since $A_i \neq A$ for only a finite number of agents, we can directly apply Theorem 3.4.4 to arrive at the following result.

Theorem 3.5.4. When private signals are unbounded then $A = S$ w.p. one.

When private signals are bounded then information cascades occur with probability one, and $A$ is no longer almost surely equal to $S$.

Theorem 3.5.5. When private signals are bounded then $\mathbb{P}[A = S] < 1$.  


3.6. Learning from discrete actions

Proof. When private signals are bounded then the convex closure of the support of $P_i$ is equal to $[\epsilon, M]$ for some $\epsilon, M > 0$. It follows then from Eq. 25 that if $L^x_i \leq 1/M$ then a.s. $L_i \leq 1$, and so $A_i = 1$. Likewise, if $L^x_i > 1/\epsilon$ then a.s. $A_i = 0$. Hence $[0, 1/M]$ and $(1/\epsilon, \infty)$ are all fixed points of $\mathcal{M}$.

Note that $P[A^x_i = S | H^x_i] = \max\{B^x_i, 1 - B^x_i\}$. Hence we can prove the claim by showing that $B^x_\infty = \lim_i B^x_i$ is in $(0, 1)$, since then it would follow that $\lim_i P[A^x_i = S] < 1$, and in particular $\lim_i A_i = S < 1$.

Indeed, condition on $S = 1$, and assume by contradiction that $\lim_i B^x_i = 1$. Then $L^x_i$ will equal some $\delta \in (0, 1/M)$ for $i$ large enough. But $\delta$ is a fixed point of $\mathcal{M}$, and so $L^x_j$ will equal $\delta$ hence and $B^x_i$ will not converge to one. The same argument applies if we condition on $S = 0$ and argue that $L^x_i$ will equal some $N \in (1/\epsilon, \infty)$ for $i$ large enough. □

3.6. Learning from discrete actions

This section is adapted from Mossel, Sly and Tamuz [31].

We consider agents who maximize, at each time $t$, the utility function (Eq. 11)

$$U^i_t = 1(A^i_t = S).$$

Hence they choose actions (Eq. 14)

$$A^i_t = \arg\max_{s \in \{0, 1\}} P[S = s | H^i_t].$$

We assume that the social network is undirected, and consider both the finite and the infinite case.

We ask the following questions:

1. **Agreement.** Do the agents reach agreement? In this model we say that $i$ and $j$ agree if $A^i_\infty = A^j_\infty$. We show that this happens under a weak condition on the private signals.

2. **Learning.** When the agents do agree on some limit action $A_\infty$, does this action equal $S$? We show that the answer to this question depends on the graph, and that for undirected graphs indeed $A_\infty = S$ with high probability (for large finite graphs) or with probability one (for infinite graphs).

The condition on private signal that implies agreement on limit actions is the following. By the definition of beliefs, $B^0_i = P[S = 1 | W_i]$. We say that the private signals induce non-atomic beliefs when the distribution of $B^0_i$ is non-atomic.

The rational behind this definition is that it precludes the possibility of indifference or ties. As we show below, indifference is the only cause
of disagreement, in the sense that agreement follows once indifference is done away with.

**3.6.1. Main results.** In our first theorem we show that when initial private beliefs are non-atomic, then at the limit $t \to \infty$ the limit action sets of the players are identical.

**Theorem 3.6.1 (Mossel, Sly and Tamuz).** Let $(\mu_0, \mu_1)$ induce non-atomic beliefs. Then there exists a random variable $A_\infty$ such that almost surely $A^i_\infty = A_\infty$ for all $i$.

I.e., when initial private beliefs are non-atomic then agents, at the limit, agree on the optimal action. The following theorem states that when such agreement is guaranteed then the agents learn the state of the world with high probability, when the number of agents is large. This phenomenon is known as asymptotic learning. This theorem is our main result.

**Theorem 3.6.2 (Mossel, Sly and Tamuz).** Let $\mu_0, \mu_1$ be such that for every connected, undirected graph $G$ there exists a random variable $A_\infty$ such that almost surely $A^u_\infty = A_\infty$ for all $u \in V$. Then there exists a sequence $q(n) = q(n, \mu_0, \mu_1)$ such that $q(n) \to 1$ as $n \to \infty$, and $P[A_\infty = S] \geq q(n)$, for any choice of undirected, connected graph $G$ with $n$ agents.

Informally, when agents agree on limit action sets then they necessarily learn the correct state of the world, with probability that approaches one as the number of agents grows. This holds uniformly over all possible connected and undirected social network graphs.

The following theorem is a direct consequence of the two theorems above, since the property proved by Theorem 3.6.1 is the condition required by Theorem 3.6.2.

**Theorem 3.6.3.** Let $\mu_0$ and $\mu_1$ induce non-atomic beliefs. Then there exists a sequence $q(n) = q(n, \mu_0, \mu_1)$ such that $q(n) \to 1$ as $n \to \infty$, and $P[A^i_\infty = S] \geq q(n)$, for all agents $i$ and for any choice of undirected, connected $G$ with $n$ agents.

Before delving into the proofs of Theorems 3.6.1 and 3.6.2 we introduce additional definitions in subsection 3.6.2 and prove some general lemmas in subsections 3.6.3, 3.6.4 and 3.6.5. Note that Lemma 3.6.6, which is the main technical insight in the proof of Theorem 3.6.2, may be of independent interest. We prove Theorem 3.6.2 in subsection 3.6.6 and Theorem 3.6.1 in subsection 3.6.7.
3.6. LEARNING FROM DISCRETE ACTIONS

3.6.2. Additional general notation. We denote the log-likelihood ratio of agent $i$’s belief at time $t$ by

$$Z^i_t = \log \frac{B^i_t}{1 - B^i_t},$$

and let

$$Z^i_\infty = \lim_{t \to \infty} Z^i_t.$$ 

Note that

$$Z^i_t = \log \frac{P[S = 1 | H^i_t]}{P[S = 0 | H^i_t]},$$

and that

$$Z^i_0 = \log \frac{d\mu_1}{d\mu_0}(W_i).$$

Note also that $Z^i_t$ converges almost surely since $B^i_t$ does.

We denote the set of actions of agent $i$ up to time $t$ by

$$A^i_{[0,t)} = (A^i_0, \ldots, A^i_{t-1}).$$

The set of all actions of $i$ is similarly denoted by

$$A^i_{[0,\infty)} = (A^i_0, A^i_1, \ldots).$$

We denote the actions of the neighbors of $i$ up to time $t$ by

$$I^i_t = \{A^j_{[0,t)} : j \in \partial i\} = \{A^j_{t'} : j \in \partial i, t' < t\},$$

and let $I^i_t$ denote all the actions of $i$’s neighbors:

$$I^i_\infty = \{A^j_{[0,\infty)} : j \in \partial i\} = \{A^j_{t'} : j \in \partial i, t' \geq 0\}.$$ 

Note that using this notation we have that $H^i_t = \{W_i, I^i_t\}$ and $\mathcal{F}^i_t = \{W_i, I^i_t\}$.

We denote the probability that $i$ chooses the correct action at time $t$ by

$$p^i_t = P[A^i_t = S],$$

and accordingly

$$p^i_\infty = \lim_{t \to \infty} p^i_t.$$ 

For a set of vertices $U \subseteq V$ we denote by $W(U)$ the private signals of the agents in $U$. 
3.6.3. Sequences of rooted graphs and their limits. In this section we define a topology on rooted graphs. We call convergence in this topology convergence to local limits, and use it repeatedly in the proof of Theorem 3.6.2. The core of the proof of Theorem 3.6.2 is the topological Lemma 3.6.6 which we prove here. This lemma is a claim related to local graph properties, which we also introduce here.

Let \( G = (V, E) \) be a finite or countably infinite graph, and let \( i \in V \) be a vertex in \( G \) such that for every vertex \( j \in V \) there exists a (directed) path in \( G \) from \( i \) to \( j \). We denote by \( (G, i) \) the rooted graph \( G \) with root \( i \).

Note that the requirement that there exist paths from the root to all other vertices is non-standard in the definition of rooted graphs. However, we will eventually only consider strongly connected graphs, and so this will always hold.

Let \( G = (V, E) \) and \( G' = (V', E') \) be graphs. \( h : V \to V' \) is a graph isomorphism between \( G \) and \( G' \) if \( (i, j) \in E \iff (h(i), h(j)) \in E' \).

Let \( (G, i) \) and \( (G', i') \) be rooted graphs. Then \( h : V \to V' \) is a rooted graph isomorphism between \( (G, i) \) and \( (G', i') \) if \( h \) is a graph isomorphism and \( h(u) = u' \).

We write \( (G, i) \cong (G', i') \) whenever there exists a rooted graph isomorphism between the two rooted graphs.

Given a (perhaps directed) graph \( G = (V, E) \) and two vertices \( i, j \in V \), the graph distance \( d(i, j) \) is equal to the length in edges of a shortest (directed) path between \( i \) and \( j \). We denote by \( B_r(G, i) \) the ball of radius \( r \) around the vertex \( i \) in the graph \( G = (V, E) \): Let \( V' \) be the set of vertices \( j \) such that \( d(i, j) \) is at most \( r \). Let \( E' = \{(i, j) \in E : i, j \in V'\} \). Then \( B_r(G, i) \) is the rooted graph with vertices \( V' \), edges \( E' \) and root \( i' \).

Note that the requirement that rooted graphs have paths from the root to all other vertices is equivalent to having \( B_\infty(G, i) \cong (G, i) \).

We next define a topology on strongly connected rooted graphs (or rather on their isomorphism classes; we shall simply refer to these classes as graphs). A natural metric between strongly connected rooted graphs is the following (see Benjamini and Schramm [6], Aldous and Steele [1]). Given \( (G, i) \) and \( (G', i') \), let

\[
D((G, i), (G', i')) = 2^{-R},
\]

where

\[
R = \sup\{r : B_r(G, i) \cong B_r(G', i')\}.
\]

This is indeed a metric: the triangle inequality follows immediately, and a standard diagonalization argument is needed to show that if
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\[ D((G, i), (G', i')) = 0 \] then \( B_\infty(G, i) \cong B_\infty(G', i') \) and so \( (G, i) \cong (G', i') \).

This metric induces a topology that will be useful to us. As usual, the basis of this topology is the set of balls of the metric; the ball of radius \( 2^{-R} \) around the graph \( (G, i) \) is the set of graphs \( (G', i') \) such that \( B_R(G, i) \cong B_R(G', i') \). We refer to convergence in this topology as convergence to a local limit, and provide the following equivalent definition for it.

Let \( \{(G_r, i_r)\}_{r=1}^\infty \) be a sequence of strongly connected rooted graphs. We say that the sequence converges if there exists a strongly connected rooted graph \( (G', i') \) such that \( B_r(G', i') \cong B_r(G_r, i_r) \), for all \( r \geq 1 \). We then write

\[ (G', i') = \lim_{r \to \infty} (G_r, i_r), \]

and call \( (G', i') \) the local limit of the sequence \( \{(G_r, i_r)\}_{r=1}^\infty \).

Let \( \mathcal{G}_d \) be the set of strongly connected rooted graphs with degree at most \( d \). Another standard diagonalization argument shows that \( \mathcal{G}_d \) is compact (see again [6, 1]). Then, since the space is metric, every sequence in \( \mathcal{G}_d \) has a converging subsequence:

**Lemma 3.6.4.** Let \( \{(G_r, i_r)\}_{r=1}^\infty \) be a sequence of rooted graphs in \( \mathcal{G}_d \). Then there exists a subsequence \( \{(G_{r_n}, i_{r_n})\}_{n=1}^\infty \) with \( r_{n+1} > r_n \) for all \( n \), such that \( \lim_{n \to \infty} (G_{r_n}, u_{r_n}) \) exists.

We next define local properties of rooted graphs. Let \( P \) be property of rooted graphs or a Boolean predicate on rooted graphs. We write \( (G, i) \in P \) if \( (G, i) \) has the property, and \( (G, i) \notin P \) otherwise.

We say that \( P \) is a local property if, for every \( (G, i) \in P \) there exists an \( r > 0 \) such that if \( B_r(G, i) \cong B_r(G', i') \), then \( (G', i') \in P \). Let \( r \) be such that \( B_r(G, i) \cong B_r(G', i') \Rightarrow (G', i') \in P \). Then we say that \( (G, i) \) has property \( P \) with radius \( r \), and denote \( (G, i) \in P^{(r)} \).

That is, if \( (G, i) \) has a local property \( P \) then there is some \( r \) such that knowing the ball of radius \( r \) around \( i \) in \( G \) is sufficient to decide that \( (G, i) \) has the property \( P \).

An alternative name for a local property would therefore be a locally decidable property. In our topology, local properties are nothing but open sets: the definition above states that if \( (G, i) \in P \) then there exists an element of the basis of the topology that includes \( (G, i) \) and is also in \( P \). This is a necessary and sufficient condition for \( P \) to be open.
We use this fact to prove the following lemma. Let \( \mathcal{B}_d \) be the set of infinite, connected, undirected graphs of degree at most \( d \), and let \( \mathcal{B}_d^r \) be the set of \( \mathcal{B}_d \)-rooted graphs

\[
\mathcal{B}_d^r = \{(G, i) : G \in \mathcal{B}_d, i \in G\}.
\]

**Lemma 3.6.5.** \( \mathcal{B}_d^r \) is compact.

**Proof.** Lemma 3.6.4 states that \( \mathcal{G}_d \), the set of strongly connected rooted graphs of degree at most \( d \), is compact. Since \( \mathcal{B}_d^r \) is a subset of \( \mathcal{G}_d \), it remains to show that \( \mathcal{B}_d^r \) is closed in \( \mathcal{G}_d \).

The complement of \( \mathcal{B}_d^r \) in \( \mathcal{G}_d \) is the set of graphs in \( \mathcal{G}_d \) that are either finite or directed. These are both local properties: if \((G, i)\) is finite (or directed), then there exists a radius \( r \) such that examining \( B_r(G, i) \) is enough to determine that it is finite (or directed). Hence the sets of finite graphs and directed graphs in \( \mathcal{G}_d \) are open in \( \mathcal{G}_d \), their intersection is open in \( \mathcal{G}_d \), and their complement, \( \mathcal{B}_d^r \), is closed in \( \mathcal{G}_d \). \(\square\)

We now state and prove the main lemma of this subsection. Note that the set of graphs \( \mathcal{B}_d \) satisfies the conditions of this lemma.

**Lemma 3.6.6.** Let \( \mathcal{A} \) be a set of infinite, strongly connected graphs, let \( \mathcal{A}^r \) be the set of \( \mathcal{A} \)-rooted graphs

\[
\mathcal{A}^r = \{(G, i) : G \in \mathcal{A}, i \in G\},
\]

and assume that \( \mathcal{A} \) is such that \( \mathcal{A}^r \) is compact.

Let \( P \) be a local property such that for each \( G \in \mathcal{A} \) there exists a vertex \( j \in G \) such that \( (G, j) \in P \). Then for each \( G \in \mathcal{A} \) there exist an \( r_0 \) and infinitely many distinct vertices \( \{j_n\}_{n=1}^{\infty} \) such that \( (G, j_n) \in P(r_0) \) for all \( n \).

**Proof.** Let \( G \) be an arbitrary graph in \( \mathcal{A} \). Consider a sequence \( \{k_r\}_{r=1}^{\infty} \) of vertices in \( G \) such that for all \( r, s \in \mathbb{N} \) the balls \( B_r(G, k_r) \) and \( B_s(G, k_s) \) are disjoint.

Since \( \mathcal{A}^r \) is compact, the sequence \( \{(G, k_r)\}_{r=1}^{\infty} \) has a converging subsequence \( \{(G, k_{r_n})\}_{n=1}^{\infty} \) with \( r_{n+1} > r_n \). Write \( i_r = k_{r_n} \), and let

\[
(G', i') = \lim_{r \to \infty} (G, i_r).
\]

Note that since \( \mathcal{A}^r \) is compact, \( (G', i') \in \mathcal{A}^r \) and in particular \( G' \in \mathcal{A} \) is an infinite, strongly connected graph. Note also that since \( r_{n+1} > r_n \), it also holds that the balls \( B_r(G, i_r) \) and \( B_s(G, i_s) \) are disjoint for all \( r, s \in \mathbb{N} \).

Since \( G' \in \mathcal{A} \), there exists a vertex \( j' \in G' \) such that \( (G', j') \in P \). Since \( P \) is a local property, \( (G', j') \in P(r_0) \) for some \( r_0 \), so that if \( B_{r_0}(G', j') \cong B_{r_0}(G, j) \) then \( (G, j) \in P \).
Let $R = d(i', j') + r_0$, so that $B_{r_0}(G', j') \subseteq B_R(G', i')$. Then, since the sequence $(G, i_r)$ converges to $(G', i')$, for all $r \geq R$ it holds that $B_R(G, i_r) \cong B_R(G', i')$. Therefore, for all $r > R$ there exists a vertex $j_r \in B_R(G, j_r)$ such that $B_{r_0}(G, j_r) \cong B_{r_0}(G', j')$. Hence $(G, j_r) \in P^{(r_0)}$ for all $r > R$ (see Fig. 1). Furthermore, for $r, s > R$, the balls $B_R(G, i_r)$ and $B_R(G, i_s)$ are disjoint, and so $j_r \neq j_s$.

We have therefore shown that the vertices $\{j_r\}_{r>R}$ are an infinite set of distinct vertices such that $(G, j_r) \in P^{(r_0)}$, as required. 

3.6.4. Coupling isomorphic balls. This section includes three claims that we will use repeatedly later. Their spirit is that everything that happens to an agent up to time $t$ depends only on the state of the world and a ball of radius $t$ around it."
Recall that $H_i^t$, the information available to agent $i$ at time $t$, includes $W_i$ and $A^j_t$ for all $j$ neighbors of $i$ and $t' < t$. Recall that $I^i_t$ denotes this exact set of actions:

$$I^i_t = \{A^j_{[0,t]} : j \in \partial i\} = \{A^j_{t'} : j \in \partial i, t' < t\}.$$  

**Claim 3.6.7.** For all agents $i$ and times $t$, $I^i_t$ a deterministic function of $W(B_t(G,i))$.

Recall that $W(B_t(G,i))$ are the private signals of the agents in $B_t(G,i)$, the ball of radius $t$ around $i$.

**Proof.** We prove by induction on $t$. $I^i_0$ is empty, and so the claim holds for $t = 1$.

Assume the claim holds up to time $t$. By definition, $A^i_{t+1}$ is a function of $W_i$ and of $I^i_{t+1}$, which includes $\{A^j_{t'} : w \in \partial i, t' \leq t\}$. $A^j_{t'}$ is a function of $W^j_t$ and $I^i_{t'}$, and hence by the inductive assumption it is a function of $W(B_{t'}(G,w))$. Since $t' < t + 1$ and the distance between $i$ and $j$ is one, $W(B_{t'}(G,j)) \subseteq W(B_{t+1}(G,i))$, for all $j \in \partial i$ and $t' \leq t$. Hence $I^i_{t+1}$ is a function of $W(B_{t+1}(G,i))$, the private signals in $B_{t+1}(G,i)$. \qed

The following lemma follows from Claim 3.6.7 above:

**Lemma 3.6.8.** Consider two processes with identical private signal distributions $(\mu_0, \mu_1)$, on different graphs $G = (V, E)$ and $G' = (V', E')$.

Let $t \geq 1$, $i \in V$ and $i' \in V'$ be such that there exists a rooted graph isomorphism $h : B_t(G,i) \rightarrow B_t(G',i')$.

Let $M$ be a random variable that is measurable in $\sigma(H^i_t)$. Then there exists an $M'$ that is measurable in $H^i_t$ such that the distribution of $(M,S)$ is identical to the distribution of $(M',S')$.

Recall that a graph isomorphism between $G = (V, E)$ and $G' = (V', E')$ is a bijective function $h : V \rightarrow V$ such that $(u,v) \in E$ iff $(h(u), h(v)) \in E'$.

**Proof.** Couple the two processes by setting $S = S'$, and letting $W_j = W_{j'}$ when $h(j) = j'$. Note that it follows that $W_i = W_{i'}$. By Claim 3.6.7 we have that $I^i_t = I^i_t$, when using $h$ to identify vertices in $V$ with vertices in $V'$.

Since $M$ is measurable in $\sigma(H^i_t)$, it must, by the definition of $H^i_t$, be a function of $I^i_t$ and $W_i$. Denote then $M = f(I^i_t, W_i)$. Since we showed that $I^i_t = I^i_t$, if we let $M' = f(I^i_t, W_{i'})$ then the distribution of $(M, S)$ and $(M', S')$ will be identical. \qed
In particular, we use this lemma in the case where $M$ is an estimator of $S$. Then this lemma implies that the probability that $M = S$ is equal to the probability that $M' = S'$.

Recall that $p_i = \mathbb{P}[A_i = S] = \max_{A \in \sigma(H_i)} \mathbb{P}[A = S]$. Hence we can apply this lemma above to $A_i$ and $A_i'$:

**Corollary 3.6.9.** If $B_i(G, i)$ and $B_i(G', i')$ are isomorphic then $p_i = p_i'(t)$.

3.6.5. **$\delta$-independence.** To prove that agents learn $S$ we will show that the agents must, over the duration of this process, gain access to a large number of measurements of $S$ that are almost independent. To formalize the notion of almost-independence we define $\delta$-independence and prove some easy results about it. The proofs in this subsection are relatively straightforward.

Let $\mu$ and $\nu$ be two measures defined on the same space. We denote the total variation distance between them by $d_{TV}((\cdot, \mu), \nu)$. Let $A$ and $B$ be two random variables with joint distribution $\mu_{(A,B)}$. Then we denote by $\mu_A$ the marginal distribution of $A$, $\mu_B$ the marginal distribution of $B$, and $\mu_A \times \mu_B$ the product distribution of the marginal distributions.

Let $(X_1, X_2, \ldots, X_k)$ be random variables. We refer to them as $\delta$-independent if their joint distribution $\mu_{(X_1, \ldots, X_k)}$ has total variation distance of at most $\delta$ from the product of their marginal distributions $\mu_{X_1} \times \cdots \times \mu_{X_k}$:

$$d_{TV}((\cdot, \mu_{(X_1, \ldots, X_k)}), \mu_{X_1} \times \cdots \times \mu_{X_k}) \leq \delta.$$ 

Likewise, $(X_1, \ldots, X_l)$ are $\delta$-dependent if the distance between the distributions is more than $\delta$.

We remind the reader that a coupling $\nu$, between two random variables $A_1$ and $A_2$ distributed $\nu_1$ and $\nu_2$, is a distribution on the product of the spaces $\nu_1$, $\nu_2$ such that the marginal of $A_i$ is $\nu_i$. The total variation distance between $A_1$ and $A_2$ is equal to the minimum, over all such couplings $\nu$, of $\nu(A_1 \neq A_2)$.

Hence to prove that $X, Y$ are $\delta$-independent it is sufficient to show that there exists a coupling $\nu$ between $\nu_1$, the joint distribution of $(X, Y)$ and $\nu_2$, the products of the marginal distributions of $X$ and $Y$, such that $\nu((X_1, Y_1) \neq (X_2, Y_2)) \leq \delta$.

Alternatively, to prove that $(A, B)$ are $\delta$-independent, one could directly bound the total variation distance between $\mu_{(A,B)}$ and $\mu_A \times \mu_B$ by $\delta$. This is often done below using the fact that the total variation distance satisfies the triangle inequality $d_{TV}((\cdot, \mu), \nu) \leq d_{TV}((\cdot, \mu), \gamma) + d_{TV}((\cdot, \gamma), \nu)$. 


We state and prove some straightforward claims regarding $\delta$-independence.

**Claim 3.6.10.** Let $A$, $B$ and $C$ be random variables such that $\mathbb{P}[A \neq B] \leq \delta$ and $(B, C)$ are $\delta'$-independent. Then $(A, C)$ are $2\delta + \delta'$-independent.

**Proof.** Let $\mu_{(A,B,C)}$ be a joint distribution of $A$, $B$ and $C$ such that $\mathbb{P}[A \neq B] \leq \delta$.

Since $\mathbb{P}[A \neq B] \leq \delta$, $\mathbb{P}[(A, C) \neq (B, C)] \leq \delta$, in both cases that $A, B, C$ are picked from either $\mu_{(A,B,C)}$ or $\mu_{(A,B)} \times \mu_C$. Hence

$$d_{\text{TV}}((\cdot, \mu_{(A,C)}), \mu_{(B,C)}) \leq \delta$$

and

$$d_{\text{TV}}((\cdot, \mu_A \times \mu_C), \mu_B \times \mu_C) \leq \delta.$$ 

Since $(B, C)$ are $\delta'$-independent,

$$d_{\text{TV}}((\cdot, \mu_B \times \mu_C), \mu_{(B,C)}) \leq \delta'.$$

The claim follows from the triangle inequality

$$d_{\text{TV}}((\cdot, \mu_{(A,C)}), \mu_A \times \mu_C) \leq d_{\text{TV}}((\cdot, \mu_{(A,C)}), \mu_{(B,C)}) + d_{\text{TV}}((\cdot, \mu_{(B,C)}), \mu_B \times \mu_C) + d_{\text{TV}}((\cdot, \mu_B \times \mu_C), \mu_A \times \mu_C) \leq 2\delta + \delta'.$$

**Claim 3.6.11.** Let $(X, Y)$ be $\delta$-independent, and let $Z = f(Y, B)$ for some function $f$ and $B$ that is independent of both $X$ and $Y$. Then $(X, Z)$ are also $\delta$-independent.

**Proof.** Let $\mu_{(X,Y)}$ be a joint distribution of $X$ and $Y$ satisfying the conditions of the claim. Then since $(X, Y)$ are $\delta$-independent,

$$d_{\text{TV}}((\cdot, \mu_{(X,Y)}), \mu_X \times \mu_Y) \leq \delta.$$ 

Since $B$ is independent of both $X$ and $Y,$

$$d_{\text{TV}}((\cdot, \mu_{(X,Y)}), \mu_B \times \mu_X \times \mu_Y \times \mu_B) \leq \delta$$

and $(X, Y, B)$ are $\delta$-independent. Therefore there exists a coupling between $(X_1, Y_1, B_1) \sim \mu_{(X,Y)} \times \mu_B$ and $(X_2, Y_2, B_2) \sim \mu_X \times \mu_Y \times \mu_B$ such that $\mathbb{P}[(X_1, Y_1, B_1) \neq (X_2, Y_2, B_2)] \leq \delta.$ Then

$$\mathbb{P}[(X_1, f(Y_1, B_1)) \neq (X_2, f(Y_2, B_2))] \leq \delta$$

and the proof follows.

**Claim 3.6.12.** Let $A = (A_1, \ldots, A_k)$, and $X$ be random variables. Let $(A_1, \ldots, A_k)$ be $\delta_1$-independent and let $(A, X)$ be $\delta_2$-independent. Then $(A_1, \ldots, A_k, X)$ are $(\delta_1 + \delta_2)$-independent.
Proof. Let $\mu_{(A_1,\ldots,A_k,X)}$ be the joint distribution of $A = (A_1,\ldots,A_k)$ and $X$. Then since $(A_1,\ldots,A_k)$ are $\delta_1$-independent,

$$d_{TV}((\mu_A,\mu_{A_1}\times\cdots\times\mu_{A_k}) \leq \delta_1.$$  

Hence

$$d_{TV}((\mu_A \times \mu_X,\mu_{A_1}\times\cdots\times\mu_{A_k} \times \mu_X) \leq \delta_1.$$  

Since $(A,X)$ are $\delta_2$-independent,

$$d_{TV}((\mu_{(A,X)},\mu_A \times \mu_X) \leq \delta_2.$$  

The claim then follows from the triangle inequality

$$d_{TV}((\mu_{(A,X)},\mu_{A_1}\times\cdots\times\mu_{A_k} \times \mu_X) \leq d_{TV}((\mu_{(A,X)},\mu_A \times \mu_X) + d_{TV}((\mu_A \times \mu_X,\mu_{A_1}\times\cdots\times\mu_{A_k} \times \mu_X)).$$

\[ \square \]

**Lemma 3.6.13.** For every $1/2 < p < 1$ there exist $\delta = \delta(p) > 0$ and $\eta = \eta(p) > 0$ such that if $S$ and $(X_1,X_2,X_3)$ are binary random variables with $P[S = 1] = 1/2$, $1/2 < p - \eta \leq P[X_i = S] < 1$, and $(X_1,X_2,X_3)$ are $\delta$-independent conditioned on $S$ then $P[a(X_1,X_2,X_3) = S] > p$, where $a$ is the MAP estimator of $S$ given $(X_1,X_2,X_3)$.  

In other words, one’s odds of guessing $S$ using three conditionally almost-independent bits are greater than using a single bit.

Proof. We apply Lemma 3.6.14 below to three conditionally independent bits which are each equal to $S$ w.p. at least $p - \eta$. Then

$$P[a(X_1,X_2,X_3) = S] \geq p - \eta + \epsilon_{p-\eta}$$

where $\epsilon_q = \frac{1}{100}(2q - 1)(3q^2 - 2q^3 - q)$.

Since $\epsilon_q$ is continuous in $q$ and positive for $1/2 < q < 1$, it follows that for $\eta$ small enough $p - \eta + \epsilon_{p-\eta} > p$. Now, take $\delta < \epsilon_{p-\eta} - \eta$. Then, since we can couple $\delta$-independent bits to independent bits so that they differ with probability at most $\delta$, the claim follows.  

\[ \square \]

**Lemma 3.6.14.** Let $S$ and $(X_1,X_2,X_3)$ be binary random variables such that $P[S = 1] = 1/2$. Let $1/2 < p \leq P[X_i = S] < 1$. Let $a(X_1,X_2,X_3)$ be the MAP estimator of $S$ given $(X_1,X_2,X_3)$. Then there exists an $\epsilon_p > 0$ that depends only on $p$ such that if $(X_1,X_2,X_3)$ are independent conditioned on $S$ then $P[a(X_1,X_2,X_3) = S] \geq p + \epsilon_p$.

In particular the statement holds with

$$\epsilon_p = \frac{1}{100}(2p - 1)(3p^2 - 2p^3 - p).$$


3. BAYESIAN MODELS

PROOF. Denote $X = (X_1, X_2, X_3)$. Assume first that $\mathbb{P} [X_i = S] = p$ for all $i$. Let $\delta_1, \delta_2, \delta_3$ be such that $p + \delta_1 = \mathbb{P} [X_i = 1 | S = 1]$ and $p - \delta_i = \mathbb{P} [X_i = 0 | S = 0]$.

To show that $\mathbb{P} [a(X) = S] \geq p + \epsilon_p$ it is enough to show that $\mathbb{P} [b(X) = S] \geq p + \epsilon_p$ for some estimator $b$, by the definition of a MAP estimator. We separate into three cases.

1. If $\delta_1 = \delta_2 = \delta_3 = 0$ then the events $X_i = S$ are independent and the majority of the $X_i$’s is equal to $S$ with probability $p' = p^3 + 3p^2(1 - p)$, which is greater than $p$ for $\frac{1}{2} < p < 1$. Denote $\eta_p = p' - p$. Then $\mathbb{P} [a(X) = S] \geq p + \eta_p$.

2. Otherwise if $|\delta_i| \leq \eta_p/6$ for all $i$ then we can couple $X$ to three bits $Y = (Y_1, Y_2, Y_3)$ which satisfy the conditions of case 1 above, and so that $\mathbb{P} [X \neq Y] \leq \eta_p/2$. Then $\mathbb{P} [a(X) = S] \geq p + \eta_p/2$.

3. Otherwise we claim that there exist $i$ and $j$ such that $|\delta_i + \delta_j| > \eta_p/12$.

Indeed assume w.l.o.g. that $\delta_1 \geq \eta_p/6$. Then if it doesn’t hold that $\delta_1 + \delta_2 \geq \eta_p/12$ and it doesn’t hold that $\delta_1 + \delta_3 \geq \eta_p/12$ then $\delta_2 \leq -\eta_p/12$ and $\delta_3 \leq -\eta_p/12$ and therefore $\delta_2 + \delta_3 \leq -\eta_p/12$.

Now that this claim is proved, assume w.l.o.g. that $\delta_1 + \delta_2 \geq \eta_p/12$. Recall that $X_i \in \{0, 1\}$, and so the product $X_1X_2$ is also an element of $\{0, 1\}$. Then

$$\mathbb{P} [X_1X_2 = S] = \frac{1}{2} \mathbb{P} [X_1X_2 = 1 | S = 1] + \frac{1}{2} \mathbb{P} [X_1X_2 = 0 | S = 0]$$

$$= \frac{1}{2} \left( (p + \delta_1)(p + \delta_2) + (p - \delta_1)(p - \delta_2) + (p - \delta_1)(1 - p + \delta_2) + (1 - p + \delta_1)(p - \delta_2) \right)$$

$$= p + \frac{1}{2} (2p - 1)(\delta_1 + \delta_2) \geq p + \eta_p/12,$$

and so $\mathbb{P} [a(X) = S] \geq p + (2p - 1)\eta_p/12$.

Finally, we need to consider the case that $\mathbb{P} [X_i = S] = p_i^\infty > p$ for some $i$. We again consider two cases. Denote $\epsilon_p = (2p - 1)\eta_p/100$. If there exists an $i$ such that $p_i^\infty > \epsilon_p$ then this bit is by itself an estimator that equals $S$ with probability at least $p + \epsilon_p$, and therefore the MAP estimator equals $S$ with probability at least $p + \epsilon_p$.

Otherwise $p \leq p_i^\infty \leq p_i^\infty + \epsilon_p$ for all $i$. We will construct a coupling between the distributions of $X = (X_1, X_2, X_3)$ and $Y = (Y_1, Y_2, Y_3)$.
(Y_1, Y_2, Y_3) such that the Y_i's are conditionally independent given S and $P[Y_i = S] = p$ for all i, and furthermore $P[Y \neq X] \leq 3\epsilon_p$. By what we've proved so far the MAP estimator of S given Y equals S with probability at least $p + (2p - 1)\eta_p/12 \geq p + 8\epsilon_p$. Hence by the coupling, the same estimator applied to X is equal to S with probability at least $p + 8\epsilon_p - 3\epsilon_p > p + \epsilon_p$.

To couple X and Y let $Z_t$ be a real i.i.d. random variables uniform on $[0, 1]$. When $S = 1$ let $X_i = Y_i = S$ if $Z_i > p^\infty + \delta_i$, let $X_i = S$ and $Y_i = 1 - S$ if $Z_i \in [p + \delta_i, p^\infty + \delta_i]$, and otherwise $X_i = Y_i = 1 - S$. The construction for $S = 0$ is similar. It is clear that X and Y have the required distribution, and that furthermore $P[X_i \neq Y_i] = p^i - p \leq \epsilon_p$. Hence $P[X \neq Y] \leq 3\epsilon_p$, as needed.

\[\square\]

3.6.6. Asymptotic learning. In this section we prove Theorem 3.6.2

THEOREM (3.6.2). Let $\mu_0, \mu_1$ be such that for every connected, undirected graph G there exists a random variable $A^i_{\infty}$ such that almost surely $A^i_{\infty} = A_{\infty}$ for all $i \in V$. Then there exists a sequence $q(n) = q(n, \mu_0, \mu_1)$ such that $q(n) \to 1$ as $n \to \infty$, and $P[A_{\infty} = S] \geq q(n)$, for any choice of undirected, connected graph G with n agents.

To prove this theorem we will need a number of intermediate results, which are given over the next few subsections.

3.6.6.1. Estimating the limiting optimal action set $A_{\infty}$. We would like to show that although the agents have a common optimal action set $A_{\infty}$ only at the limit $t \to \infty$, they can estimate this set well at a large enough time $t$.

The action $A^i_t$ is agent i's MAP estimator of S at time t. We likewise define $K^i_t$ to be agent i's MAP estimator of $A_{\infty}$, at time t:

\[K^i_t = \arg\max_{K \in \{0, 1\}} P[A_{\infty} = K|H^i_t].\] (27)

We show that the sequence of random variables $K^i_t$ converges to $A_{\infty}$ for every $i$, or that alternatively $K^i_t = A_{\infty}$ for each agent $i$ and $t$ large enough:

LEMMA 3.6.15. $P[\lim_{t \to \infty} K^i_t = A_{\infty}] = 1$ for all $i \in V$.

This lemma follows by direct application of the more general Lemma 3.6.16 which we prove below. Note that a consequence is that $\lim_{t \to \infty} P[K^i_t = A_{\infty}] = 1$.

LEMMA 3.6.16. Let $K_1 \subseteq K_2, \ldots$ be a filtration of $\sigma$-algebras, and let $K_{\infty} = \cup_t K_t$. Let $K$ be a random variable that takes a finite number of
values and is measurable in $\mathcal{K}_\infty$. Let $M(t) = \arg\max_k \mathbb{P} [K = k|\mathcal{K}(t)]$ be the MAP estimator of $K$ given $\mathcal{K}_t$. Then

$$\mathbb{P} \left[ \lim_{t \to \infty} M(t) = K \right] = 1.$$  

**Proof.** For each $k$ in the support of $K$, $\mathbb{P} [K = k|\mathcal{K}_t]$ is a bounded martingale which converges almost surely to $\mathbb{P} [K = k|\mathcal{K}_\infty]$, which is equal to $1 (K = k)$, since $K$ is measurable in $G_\infty$. Therefore $M(t) = \arg\max_k \mathbb{P} [K = k|\mathcal{K}_t]$ converges almost surely to $\arg\max_k \mathbb{P} [K = k|\mathcal{K}_\infty] = K$. □

We would like at this point to provide the reader with some more intuition on $A_i$, $K_i$ and the difference between them. Assuming that $A_\infty = 1$ then by definition, from some time $t_0$ on, $A_i^t = 1$, and from Lemma 3.6.15 $K_i^t = 1$. The same applies when $A_\infty = 0$. However, when $A_\infty = \{0, 1\}$ then $A_i^t$ takes both values 0 and 1 infinitely often, but $K_i^t$ will eventually equal $\{0, 1\}$. That is, agent $i$ will realize at some point that, although it thinks at the moment that 1 is preferable to 0 (for example), it is in fact the most likely outcome that its belief will converge to 1/2. In this case, although it is not optimal, a uniformly random guess of which is the best action may not be so bad. Our next definition is based on this observation.

Based on $K_i^t$, we define a second “action” $C_i^t$. Let $C_i^t$ be picked uniformly from $K_i^t$: if $K_i^t = 1$ then $C_i^t = 1$, if $K_i^t = 0$ then $C_i^t = 0$, and if $K_i^t = \{0, 1\}$ then $C_i^t$ is picked independently from the uniform distribution over $\{0, 1\}$.

Note that we here extend our probability space by including in $I_i^t$ (the observations of agent $i$ up to time $t$) an extra uniform bit that is independent of all else and $S$ in particular. Hence this does not increase $i$’s ability to estimate $S$, and if we can show that in this setting $i$ learns $S$ then $i$ can also learn $S$ without this bit. In fact, we show that asymptotically it is as good an estimate for $S$ as the best estimate $A_i^t$:

**Claim 3.6.17.** $\lim_{t \to \infty} \mathbb{P} [C_i^t = S] = \lim_{t \to \infty} \mathbb{P} [A_i^t = S] = p$ for all $i$.

**Proof.** We prove the claim by showing that it holds both when conditioning on the event $A_\infty = \{0, 1\}$ and when conditioning on its complement.

When $A_\infty \neq \{0, 1\}$ then for $t$ large enough $A_\infty = \{A_i^t\}$. Since (by Lemma 3.6.15) $\lim K_i^t = A_\infty$ with probability 1, in this case $C_i^t = A_i^t$.
for $t$ large enough, and
\[
\lim_{t \to \infty} \Pr \left[ C^i_t = S \middle| A^i_\infty \neq \{0, 1\} \right] = \Pr \left[ A^i_\infty = S \middle| A^i_\infty \neq \{0, 1\} \right] = \lim_{t \to \infty} \Pr \left[ A^i_t = S \middle| A^i_\infty \neq \{0, 1\} \right].
\]

When $A^i_\infty = \{0, 1\}$ then $\lim B^i_t = \lim \Pr \left[ A^i_t = S \middle| H^i_t \right] = 1/2$ and so $\lim \Pr \left[ A^i_t = S \right] = 1/2$. This is again also true for $C^i_t$, since in this case it is picked at random for $t$ large enough, and so
\[
\lim_{t \to \infty} \Pr \left[ C^i_t = S \middle| A^i_\infty = \{0, 1\} \right] = 1/2 = \lim_{t \to \infty} \Pr \left[ A^i_t = S \middle| A^i_\infty = \{0, 1\} \right].
\]

3.6.6.2. The probability of getting it right. Recall that $p^i_0 = \Pr \left[ A^i_0 = S \right]$ and $p^i_\infty = \lim_{t \to \infty} p^i_t$ (i.e., $p^i_t$ is the probability that agent $i$ takes the right action at time $t$). We state here a few easy related claims that will later be useful to us. The next claim is a rephrasing of the first part of Claim 3.1.2.

**Claim 3.6.18.** $p^i_{t+1} \geq p^i_t$.

The following claim is a rephrasing of Corollary 3.1.3.

**Claim 3.6.19.** There exists a $p \in [0, 1]$ such that $p^i_\infty = p$ for all $i$.

We make the following definition in the spirit of these claims:

$$p = \lim_{t \to \infty} \Pr \left[ A^i_t = S \right].$$

In the context of a specific social network graph $G$ we may denote this quantity as $p(G)$.

For time $t = 1$ the next standard claim follows from the fact that the agents’ signals are informative.

**Claim 3.6.20.** $p^i_1 > 1/2$ for all $i$ and $t$.

**Proof.** Note that
\[
\Pr \left[ A^i_0 = S \middle| W_i \right] = \max \{ B^i_0, 1 - B^i_0 \} = \max \{ \Pr \left[ S = 0 \middle| W_i \right], \Pr \left[ S = 1 \middle| W_i \right] \}.
\]

Recall that $p^i_0 = \Pr \left[ A^i_0 = S \right]$. Hence
\[
p^i_0 = \mathbb{E} \left[ \Pr \left[ A^i_0 = S \middle| W_i \right] \right]
= \mathbb{E} \left[ \max \{ \Pr \left[ S = 0 \middle| W_i \right], \Pr \left[ S = 1 \middle| W_i \right] \} \right].
\]
Since \( \max\{a, b\} = \frac{1}{2}(a+b)+\frac{1}{2}|a-b| \), and since \( P[S = 0|W_i] + P[S = 1|W_i] = 1 \), it follows that
\[
p'_\infty(1) = \frac{1}{2} + \frac{1}{2}E\left[|P[S = 0|W_i] - P[S = 1|W_i]|\right]
= \frac{1}{2} + \frac{1}{2}D_{TV}(\mu_0, \mu_1),
\]
where the last equality follows by Bayes’ rule. Since \( \mu_0 \neq \mu_1 \), the total variation distance \( D_{TV}(\mu_0, \mu_1) > 0 \) and \( p'_0 > \frac{1}{2} \). For \( t > 1 \) the claim follows from Claim \ref{claim:Bayes} above. \( \square \)

Recall that \( |\partial i| \) is the out-degree of \( i \), or the number of neighbors that \( i \) observes. The next lemma states that an agent with many neighbors will have a good estimate of \( S \) already at the second round, after observing the first action of its neighbors.

**Lemma 3.6.21.** There exist constants \( C_1 = C_1(\mu_0, \mu_1) \) and \( C_2 = C_2(\mu_0, \mu_1) \) such that for any agent \( i \) it holds that
\[
p'_i \geq 1 - C_1e^{-C_2|\partial i|}.
\]

**Proof.** Conditioned on \( S \), private signals are independent and identically distributed. Since \( A^j_0 \) is a deterministic function of \( W_j \), the initial actions \( A^j_0 \) are also identically distributed, conditioned on \( S \). Hence there exists a \( q \) such that \( p'_0 = P[A^j_0 = S] = q \) for all agents \( j \). By Lemma \ref{claim:Bayes} above, \( q > 1/2 \). Therefore
\[
P[A^j_0 = 1|S = 1] \neq P[A^j_0 = 1|S = 0],
\]
and the distribution of \( A^j_0 \) is different when conditioned on \( S = 0 \) or \( S = 1 \).

Fix an agent \( i \), and let \( n = |\partial i| \) be the out-degree of \( i \), or the number of neighbors that it observes. Let \( \{j_1, \ldots, j_{|\partial i|}\} \) be the set of \( i \)'s neighbors. Recall that \( A^i_1 \) is the MAP estimator of \( S \) given \( (A^j_0, \ldots, A^n_0) \), and given \( i \)'s private signal.

By standard asymptotic statistics of hypothesis testing (cf. \cite{12}), testing an hypothesis (in our case, say, \( S = 1 \) vs. \( S = 0 \)) given \( n \) informative, conditionally i.i.d. signals, succeeds except with probability that is exponentially low in \( n \). It follows that \( P[A^i_1 \neq S] \) is exponentially small in \( n \), so that there exist \( C_1 \) and \( C_2 \) such that
\[
p'_i = P[A^i_1 = S] \geq 1 - C_1e^{-C_2|\partial i|}.
\]

\( \square \)

The following claim is a direct consequence of the previous lemmas of this section.
Claim 3.6.22. Let \( d(G) = \sup \{|\partial i|\} \) be the out-degree of the graph \( G \); note that for infinite graphs it may be that \( d(G) = \infty \). Then there exist constants \( C_1 = C_1(\mu_0, \mu_1) \) and \( C_2 = C_2(\mu_0, \mu_1) \) such that
\[
p(G) \geq 1 - C_1 e^{-C_2 \cdot d(G)}
\]
for all agents \( i \).

Proof. Let \( i \) be an arbitrary vertex in \( G \). Then by Lemma 3.6.21 it holds that
\[
p^i_1 \geq 1 - C_1 e^{-C_2 \cdot \partial i},
\]
for some constants \( C_1 \) and \( C_2 \). By Lemma 3.6.18 we have that \( p^i_{t+1} \geq p^i_t \), and therefore
\[
p^i_\infty = \lim_{n \to \infty} p^i_n \geq 1 - C_1 e^{-C_2 \cdot \partial i}.
\]
Finally, \( p(G) = p^i_\infty \) by Lemma 3.6.19 and so
\[
p^i_\infty \geq 1 - C_1 e^{-C_2 \cdot \partial i}.
\]
Since this holds for an arbitrary vertex \( i \), the claim follows.

3.6.6.3. Local limits and pessimal graphs. We now turn to apply local limits to our process. We consider here and henceforth the same model as applied, with the same private signals, to different graphs. We write \( p(G) \) for the value of \( p \) on the process on \( G \), \( A_\infty(G) \) for the value of \( A_\infty \) on \( G \), etc.

Lemma 3.6.23. Let \((G, i) = \lim_{r \to \infty} (G_r, i_r)\). Then \( p(G) \leq \lim \inf_r p(G_r) \).

Proof. Since \( B_r(G_r, i_r) \simeq B_r(G, i) \), by Lemma 3.6.9 we have that \( p^r_r = p^r_{i_r} \). By Claim 3.6.18 \( p^r_{i_r} \leq p(G_r) \), and therefore \( p^r_i \leq p(G_r) \). The claim follows by taking the \( \lim \inf \) of both sides.

A particularly interesting case in the one the different \( G_r \)'s are all the same graph:

Corollary 3.6.24. Let \( G \) be a (perhaps infinite) graph, and let \( \{i_r\} \) be a sequence of vertices. Then if the local limit \( (H, u) = \lim_{r \to \infty} (G, i_r) \) exists then \( p(H) \leq p(G) \).

Recall that \( B_d \) denotes the set of infinite, connected, undirected graphs of degree at most \( d \). Let
\[
B = \bigcup_d B_d.
\]
Let
\[
p^* = p^*(\mu_0, \mu_1) = \inf_{G \in B} p(G)
\]
be the probability of learning in the pessimal graph.

Note that by Claim 3.6.20 we have that $p^* > 1/2$. We show that this infimum is in fact attained by some graph:

**Lemma 3.6.25.** There exists a graph $H \in \mathcal{B}$ such that $p(H) = p^*$.

**Proof.** Let $\{G_r = (V_r, E_r)\}_{r=1}^\infty$ be a series of graphs in $\mathcal{B}$ such that $\lim_{r \to \infty} p(G_r) = p^*$. Note that $\{G_r\}$ must all be in $\mathcal{B}_d$ for some $d$ (i.e., have uniformly bounded degrees), since otherwise the sequence $p(G_r)$ would have values arbitrarily close to 1 and its limit could not be $p^*$ (unless indeed $p^* = 1$, in which case our main Theorem 3.6.2 is proved). This follows from Lemma 3.6.21.

We now arbitrarily mark a vertex $i_r$ in each graph, so that $i_r \in V_r$, and let $(H, i)$ be the limit of some subsequence of $\{G_r, i_r\}_{r=1}^\infty$. Since $\mathcal{B}_d$ is compact (Lemma 3.6.5), $(H, i)$ is guaranteed to exist, and $H \in \mathcal{B}_d$.

By Lemma 3.6.23 we have that $p(H) \leq \liminf_r p(G_r) = p^*$. But since $H \in \mathcal{B}$, $p(H)$ cannot be less than $p^*$, and the claim is proved. □

**3.6.6.4. Independent bits.** We now show that on infinite graphs, the private signals in the neighborhood of agents that are “far enough away” are (conditioned on $S$) almost independent of $A_\infty$ (the final consensus estimate of $S$).

**Lemma 3.6.26.** Let $G$ be an infinite graph. Fix a vertex $i_0$ in $G$. Then for every $\delta > 0$ there exists an $r_\delta$ such that for every $r \geq r_\delta$ and every vertex $i$ with $d(i_0, i) > 2r$ it holds that $W(B_r(G, i))$, the private signals in $B_r(G, i)$, are $\delta$-independent of $A_\infty$, conditioned on $S$. Here we denote graph distance by $d(\cdot, \cdot)$.

**Proof.** Fix $i_0$, and let $i$ be such that $d(i_0, u) > 2r$. Then $B_r(G, i_0)$ and $B_r(G, i)$ are disjoint, and hence independent conditioned on $S$. Hence $K^{i_0}_r$ is independent of $W(B_r(G, i))$, conditioned on $S$.

Lemma 3.6.15 states that $\mathbb{P} [\lim_{r \to \infty} K^{i_0}_r = A_\infty] = 1$, and so there exists an $r_\delta$ such that for every $r \geq r_\delta$ it holds that $\mathbb{P} [K^{i_0}_r = A_\infty] > 1 - \frac{1}{2}\delta$.

Recall Claim 3.6.10 for any $A, B, C$, if $\mathbb{P} [A = B] = 1 - \frac{1}{2}\delta$ and $B$ is independent of $C$, then $(A, C)$ are $\delta$-independent.

Applying Claim 3.6.10 to $A_\infty$, $K^{i_0}_r$ and $W(B_r(G, i))$ we get that for any $r$ greater than $r_\delta$ it holds that $W(B_r(G, i))$ is $\delta$-independent of $A_\infty$, conditioned on $S$. □

We will now show, in the lemmas below, that in infinite graphs each agent has access to any number of “good estimators”: $\delta$-independent measurements of $S$ that are each almost as likely to equal $S$ as $p^*$, the minimal probability of estimating $S$ on any infinite graph.
We say that agent \( i \in G \) has \( k (\delta, \epsilon) \)-good estimators if there exists a time \( t \) and estimators \( M_1, \ldots, M_k \) such that \( (M_1, \ldots, M_k) \in H^i_t \) and

1. \( \mathbb{P}[M_i = S] > p^* - \epsilon \) for \( 1 \leq i \leq k \).
2. \( (M_1, \ldots, M_k) \) are \( \delta \)-independent, conditioned on \( S \).

**Claim 3.6.27.** Let \( P \) denote the property of having \( k (\delta, \epsilon) \)-good estimators. Then \( P \) is a local property of the rooted graph \((G, i)\). Furthermore, if \( u \in G \) has \( k (\delta, \epsilon) \)-good estimators measurable in \( H^i_t \) then \((G, i) \in P(t)\), i.e., \((G, i)\) has property \( P \) with radius \( t \).

**Proof.** If \((G, i) \in P\) then by definition there exists a time \( t \) such that \((M_1, \ldots, M_k) \in H^i_t \). Hence by Lemma 3.6.28, if \( B_t(G, i) \equiv B_t(G', i') \) then \( i' \in G' \) also has \( k (\delta, \epsilon) \)-good estimators \((M'_1, \ldots, M'_k) \subseteq \sigma(H^i_t)\) and \((G', i') \in P\). In particular, \((G, i) \in P(t)\), i.e., \((G, i)\) has property \( P \) with radius \( t \). \(\square\)

We are now ready to prove the main lemma of this subsection:

**Lemma 3.6.28.** For every \( d \geq 2 \), \( G \in B_d \), \( \epsilon, \delta > 0 \) and \( k \geq 0 \) there exists a vertex \( i \), such that \( i \) has \( k (\delta, \epsilon) \)-good estimators.

Informally, this lemma states that if \( G \) is an infinite graph with bounded degrees, then there exists an agent that eventually has \( k \) almost-independent estimates of \( S \) with quality close to \( p^* \), the minimal probability of learning.

**Proof.** In this proof we use the term “independent” to mean “independent conditioned on \( S \”).

We choose an arbitrary \( d \) and prove by induction on \( k \). The basis \( k = 0 \) is trivial. Assume the claim holds for \( k \), any \( G \in B_d \) and all \( \epsilon, \delta > 0 \). We shall show that it holds for \( k + 1 \), any \( G \in B_d \) and any \( \delta', \epsilon > 0 \).

By the inductive hypothesis for every \( G \in B_d \) there exists a vertex in \( G \) that has \( k (\delta/100, \epsilon) \)-good estimators \((M_1, \ldots, M_k)\).

Now, having \( k (\delta/100, \epsilon) \)-good estimators is a local property (Claim 3.6.27). We now therefore apply Lemma 3.6.28 since every graph \( G \in B_d \) has a vertex with \( k (\delta/100, \epsilon) \)-good estimators, any graph \( G \in B_d \) has a time \( t_k \) for which infinitely many distinct vertices \( \{j_r\} \) have \( k (\delta/100, \epsilon) \)-good estimators measurable at time \( t_k \).

In particular, if we fix an arbitrary \( i_0 \in G \) then for every \( r \) there exists a vertex \( j \in G \) that has \( k (\delta/100, \epsilon) \)-good estimators and whose distance \( d(i_0, j) \) from \( i_0 \) is larger than \( r \).

We shall prove the lemma by showing that for a vertex \( j \) that is far enough from \( i_0 \) which has \( (\delta/100, \epsilon) \)-good estimators \((M_1, \ldots, M_k)\), it
holds that for a time $t_{k+1}$ large enough $(M_1, \ldots, M_k, C_{t_{k+1}}^j)$ are $(\delta, \epsilon)$-good estimators.

By Lemma 3.6.26 there exists an $r_\delta$ such that if $r > r_\delta$ and $d(i_0, j) > 2r$ then $W(B_r(G, j))$ is $\delta/100$-independent of $A_\infty$. Let $r^* = \max\{r_\delta, t_k\}$, where $t_k$ is such that there are infinitely many vertices in $G$ with $k$ good estimators measurable at time $t_k$.

Let $j$ be a vertex with $k$ $(\delta/100, \epsilon)$-good estimators $(M_1, \ldots, M_k)$ at time $t_k$, such that $d(i_0, j) > 2r^*$. Denote $\bar{M} = (M_1, \ldots, M_k)$.

Since $d(i_0, j) > 2r_\delta$, $W(B_{r^*}(G, j))$ is $\delta/100$-independent of $A_\infty$, and since $B_{t_k}(G, j) \subseteq B_{r^*}(G, j)$, $W(B_{t_k}(G, j))$ is $\delta/100$-independent of $A_\infty$.

Finally, since $\bar{M} \in \sigma(H_{t_k})$, $\bar{M}$ is a function of $W(B_{t_k}(G, j))$, and so by Claim 3.6.11 we have that $\bar{M}$ is also $\delta/100$-independent of $A_\infty$.

For $t_{k+1}$ large enough it holds that

- $K_{t_{k+1}}^j$ is equal to $A_\infty$ with probability at least $1 - \delta/100$, since
  $$\lim_{t \to \infty} \mathbb{P}[K_t^j = A_\infty] = 1,$$
  by Claim 3.6.15
- Additionally, $\mathbb{P}[C_{t_{k+1}}^j = S] > p^* - \epsilon$, since
  $$\lim_{t \to \infty} \mathbb{P}[C_t^j = S] = p \geq p^*,$$
  by Claim 3.6.17

We have then that $(\bar{M}, A_\infty)$ are $\delta/100$-independent and $\mathbb{P}[K_{t_{k+1}}^j \neq A_\infty] \leq \delta/100$. Claim 3.6.10 states that if $(A, B)$ are $\delta$-independent $\mathbb{P}[B \neq C] \leq \delta'$ then $(A, C)$ are $\delta + 2\delta'$-independent. Applying this here we get that $(\bar{M}, K_{t_{k+1}}^j)$ are $\delta/25$-independent.

It follows by application of Claim 3.6.12 that $(M_1, \ldots, M_k, K_{t_{k+1}}^j)$ are $\delta$-independent. Since $C_{t_{k+1}}^j$ is a function of $K_{t_{k+1}}^j$ and an independent bit, it follows by another application of Claim 3.6.11 that $(M_1, \ldots, M_k, C_{t_{k+1}}^j)$ are also $\delta$-independent.

Finally, since $\mathbb{P}[C_{t_{k+1}}^j = S] > p^* - \epsilon$, $j$ has the $k+1$ $(\delta, \epsilon)$-good estimators $(M_1, \ldots, C_{t_{k+1}}^j)$ and the proof is concluded.

3.6.6.5. Asymptotic learning. As a tool in the analysis of finite graphs, we would like to prove that in infinite graphs the agents learn the correct state of the world almost surely.
**Theorem 3.6.29.** Let \( G = (V, E) \) be an infinite, connected undirected graph with bounded degrees (i.e., \( G \) is a general graph in \( B \)). Then \( p(G) = 1 \).

Note that an alternative phrasing of this theorem is that \( p^* = 1 \).

**Proof.** Assume the contrary, i.e. \( p^* < 1 \). Let \( H \) be an infinite, connected graph with bounded degrees such that \( p(H) = p^* \), such as we’ve shown exists in Lemma 3.6.25.

By Lemma 3.6.28 there exists for arbitrarily small \( \epsilon, \delta > 0 \) a vertex \( w \in H \) that has access at some time \( T \) to three \( \delta \)-independent estimators (conditioned on \( S \)), each of which is equal to \( S \) with probability at least \( p^* - \epsilon \). By Claims 3.6.13 and 3.6.20, the MAP estimator of \( S \) using these estimators equals \( S \) with probability higher than \( p^* \), for the appropriate choice of low enough \( \epsilon, \delta \). Therefore, since \( j \)’s action \( A_j^t \) is the MAP estimator of \( S \), its probability of equaling \( S \) is \( p^* \) as well, and so \( p(H) > p^* \) - contradiction. \( \square \)

Using Theorem 3.6.29 we prove Theorem 3.6.2, which is the corresponding theorem for finite graphs:

**Theorem (3.6.2).** Let \( \mu_0, \mu_1 \) be such that for every connected, undirected graph \( G \) there exists a random variable \( A_\infty^i \) such that almost surely \( A_\infty^i = A_\infty \) for all \( u \in V \). Then there exists a sequence \( q(n) = q(n, \mu_0, \mu_1) \) such that \( q(n) \to 1 \) as \( n \to \infty \), and \( \mathbb{P}[A_\infty = S] \geq q(n) \), for any choice of undirected, connected graph \( G \) with \( n \) agents.

**Proof.** Assume the contrary. Then there exists a series of graphs \( \{G_r\} \) with \( r \) agents such that \( \lim_{r \to \infty} \mathbb{P}[A_\infty^r(G_r) = S] < 1 \), and so also \( \lim_{r \to \infty} p(G_r) < 1 \).

By the same argument of Theorem 3.6.29 these graphs must all be in \( B_d \) for some \( d \), since otherwise, by Lemma 3.6.22 there would exist a subsequence of graphs \( \{G_{r_d}\} \) with degree at least \( d \) and \( \lim_{d \to \infty} p(G_{r_d}) = 1 \). Since \( B_d \) is compact (Lemma 3.6.5), there exists a graph \((G, i) \in B_d\) that is the limit of a subsequence of \( \{(G_r, i_r)\}_{r = 1}^\infty \).

Since \( G \) is infinite and of bounded degree, it follows by Theorem 3.6.29 that \( p(G) = 1 \), and in particular \( \lim_{r \to \infty} p^r_\infty(r) = 1 \). As before, \( p_{i_r}(r) = p^r_\infty(r) \), and therefore \( \lim_{r \to \infty} p_{i_r}(r) = 1 \). Since \( p(G_r) \geq p_{i_r}(r), \lim_{r \to \infty} p(G_r) = 1 \), which is a contradiction. \( \square \)

**3.6.7. Convergence to identical optimal action sets.** In this section we prove Theorem 3.6.1.

**Theorem (3.6.1).** Let \( (\mu_0, \mu_1) \) induce non-atomic beliefs. Then there exists a random variable \( A_\infty \) such that almost surely \( A_\infty^i = A_\infty \) for all \( i \).
In this section we shall assume henceforth that the distribution of initial private beliefs is non-atomic.

Given two agents $i$ and $j$, let $E_i^0$ denote the event that $A_i^t$ equals 0 infinitely often, $E_j^1$ and the event that $A_j^t$ equals 1 infinitely often. Then a rephrasing of Theorem 3.1.1 is

**Theorem 3.6.30.** If agent $i$ observes agent $j$’s actions then

$$ \mathbb{P} \left[ E_i^0, E_j^1 \right] = \mathbb{P} \left[ B^i_{\infty} = 1/2, E_i^0, E_j^1 \right]. $$

I.e., if agent $i$ takes action 0 infinitely often, agent $j$ takes action 1 infinitely, and $i$ observes $j$ then $i$’s belief is 1/2 at the limit, almost surely.

**Corollary 3.6.31.** If agent $i$ observes agent $j$’s actions, and $j$ takes both actions infinitely often then $B^i_{\infty} = 1/2$.

**Proof.** Assume by contradiction that $B^i_{\infty} < 1/2$. Then $i$ takes action 0 infinitely often. Therefore Theorem 3.6.30 implies that $B^i_{\infty} = 1/2$ - contradiction.

The case where $B^i_{\infty} > 1/2$ is treated similarly. $\square$

3.6.7.1. **Limit log-likelihood ratios.** Denote

$$ Y^i_t = \log \frac{\mathbb{P} \left[ I_i^t \mid S = 1, A_i^t \right]}{\mathbb{P} \left[ I_i^t \mid S = 0, A_i^t \right]}. $$

In the next claim we show that $Z^i_t$, the log-likelihood ratio inspired by $i$’s observations up to time $t$, can be written as the sum of two terms: $Z^i_t = \sum_{t=0}^{\infty} (W_i)$, which is the log-likelihood ratio inspired by $i$’s private signal $W_i$, and $Y^i_t$, which depends only on the actions of $i$ and its neighbors, and does not depend directly on $W_i$.

**Claim 3.6.32.**

$$ Z^i_t = Z^i_0 + Y^i_t. $$

**Proof.** By definition we have that

$$ Z^i_t = \log \frac{\mathbb{P} \left[ S = 1 \mid I_i^t \right]}{\mathbb{P} \left[ S = 0 \mid I_i^t \right]} = \log \frac{\mathbb{P} \left[ S = 1 \mid I_i^t, W_i \right]}{\mathbb{P} \left[ S = 0 \mid I_i^t, W_i \right]}.$$

and by the law of conditional probabilities

$$ Z^i_t = \log \frac{\mathbb{P} \left[ I_i^t \mid S = 1, W_i \right] \mathbb{P} \left[ W_i \mid S = 1 \right]}{\mathbb{P} \left[ I_i^t \mid S = 0, W_i \right] \mathbb{P} \left[ W_i \mid S = 0 \right]} = \log \frac{\mathbb{P} \left[ I_i^t \mid S = 1, W_i \right]}{\mathbb{P} \left[ I_i^t \mid S = 0, W_i \right]} + Z^i_0. $$
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Now \( I^i_t \), the actions of the neighbors of \( i \) up to time \( t \), are a deterministic function of \( W(B_t(G,i)) \), the private signals in the ball of radius \( t \) around \( i \), by Claim 3.6.7. Conditioned on \( S \) these are all independent, and so, from the definition of actions, these actions depend on \( i \)'s private signal \( W_i \) only in as much as it affects the actions of \( i \). Hence

\[
P[I^i_t | S = s, W_i] = P[I^i_t | S = s, A^i_{[0,t]}],
\]

and therefore

\[
Z^i_t = \log \frac{P[I^i_t | S = 1, A^i_{[0,t]}]}{P[I^i_t | S = 0, A^i_{[0,t]}]} + Z^i_0
\]

\[
= Z^i_0 + Y^i_t.
\]

Note that \( Y^i_t \) is a deterministic function of \( I^i_t \) and \( A^i_{[0,t]} \).

Following our notation convention, we define \( Y^i_\infty = \lim_{t \to \infty} Y^i_t \). Note that this limit exists almost surely since the limit of \( Z^i_t \) exists almost surely. The following claim follows directly from the definitions:

CLAIM 3.6.33. \( Y^i_\infty \) is measurable in \((A^i_{[0,\infty)}, I_i)\), the actions of \( i \) and its neighbors.

3.6.7.2. Convergence of actions. The event that an agent takes both actions infinitely often is (almost surely) a sufficient condition for convergence to belief \( 1/2 \). This follows from the fact that these actions imply that its belief takes values both above and below \( 1/2 \) infinitely many times. We show that it is also (almost surely) a necessary condition. Denote by \( E^a_i \) the event that \( i \) takes action \( a \) infinitely often.

THEOREM 3.6.34.

\[
P[E^0_i \cap E^1_i, B^i_\infty = 1/2] = P[B^i_\infty = 1/2].
\]

I.e., it a.s. holds that \( B^i_\infty = 1/2 \) iff \( i \) takes both actions infinitely often.

PROOF. We’ll prove the claim by showing that \( P[\neg(E^0_i \cap E^1_i), B^i_\infty = 1/2] = 0 \), or equivalently that \( P[\neg(E^0_i \cap E^1_i), Z^i_\infty = 0] = 0 \) (recall that \( Z^i_\infty = \log B^i_\infty/(1 - B^i_\infty) \) and so \( B^i_\infty = 1/2 \Leftrightarrow Z^i_\infty = 0 \)).

Let \( \bar{a} = (a(1), a(2), \ldots) \) be a sequence of actions, and denote by \( W_{-i} \) the private signals of all agents except \( i \). Conditioning on \( W_{-i} \) and \( S \) we can write:

\[
P[A^i_{[0,\infty)} = \bar{a}, Z^i_\infty = 0] = E[P[A^i_{[0,\infty)} = \bar{a}, Z^i_\infty = 0 | W_{-i}, S]]
\]

\[
= E[P[A^i_{[0,\infty)} = \bar{a}, Z^i_0 = -Y^i_\infty | W_{-i}, S]]
\]
where the second equality follows from Claim 3.6.32. Note that by Claim 3.6.33, \( Y^i_\infty \) is fully determined by \( A^i_{[0,\infty)} \) and \( W_{-i} \). We can therefore write

\[
P\left[ A^i_{[0,\infty)} = \bar{a}, Z^i_\infty = 0 \right] = \mathbb{E} \left[ \mathbb{P} \left[ A^i_{[0,\infty)} = \bar{a}, Z^i_0 = -Y^i_\infty(W_{-i}, \bar{a}) \mid W_{-i}, S \right] \right] \leq \mathbb{E} \left[ \mathbb{P} \left[ Z^i_0 = -Y^i_\infty(W_{-i}, \bar{a}) \mid W_{-i}, S \right] \right]
\]

Now, conditioned on \( S \), the private signal \( W_i \) is distributed \( \mu_S \) and is independent of \( W_{-i} \). Hence its distribution when further conditioned on \( W_{-i} \) is still \( \mu_S \). Since \( Z^i_0 = \log \frac{\mu_S(W_i)}{\mu_S(0)} \), its distribution is also unaffected, and in particular is still non-atomic. It therefore equals \(-Y^i_\infty(W_{-i}, \bar{a})\) with probability zero, and so

\[
P\left[ A^i_{[0,\infty)} = \bar{a}, Z^i_\infty = 0 \right] = 0.
\]

Since this holds for all sequences of actions \( \bar{a} \), it holds in particular for all sequences which converge. Since there are only countably many such sequences, the probability that the action converges (i.e., \(- (E^i_0 \cap E^i_1)\)) and \( Z^i_\infty = 0 \) is zero, or

\[
P\left[ - (E^i_0 \cap E^i_1), Z^i_\infty = 0 \right] = 0.
\]

Hence it impossible for an agent’s belief to converge to 1/2 and for the agent to only take one action infinitely often. A direct consequence of this, together with Thm. 3.6.30, is the following corollary:

**Corollary 3.6.35.** The union of the following three events occurs with probability one:

1. \( \forall u \in V : \lim_{t \to \infty} A^i_t = S \). Equivalently, all agents converge to the correct action.
2. \( \forall u \in V : \lim_{t \to \infty} A^i_t = 1 - S \). Equivalently, all agents converge to the wrong action.
3. \( \forall u \in V : B^i_\infty = 1/2 \), and in this case all agents take both actions infinitely often and hence don’t converge at all.

**Proof.** Consider first the case that there exists a vertex \( i \) such that \( i \) takes both actions infinitely often. Let \( j \) be a vertex that observes \( i \). Then by Corollary 3.6.31 we have that \( B^j_\infty = 1/2 \), and by Theorem 3.6.34 \( j \) also takes both actions infinitely often. Continuing by induction and using the fact that the graph is strongly connected we obtain the third case that none of the agents converge and \( B^i_\infty = 1/2 \) for all \( i \).

It remains to consider the case that all agents’ actions converge to either 0 or 1. Using strong connectivity, to prove the theorem it
suffices to show that it cannot be the case that \( j \) observes \( i \) and they converge to different actions. In this case, by Corollary 3.6.31 we have that \( B^j_\infty = 1/2 \), and then by Theorem 3.6.34 agent \( j \)'s actions do not converge - contradiction.

\[ \square \]

Theorem 3.6.1 is an easy consequence of this theorem.

**Theorem (3.6.1).** Let \((\mu_0, \mu_1)\) induce non-atomic beliefs. Then there exists a random variable \( A_\infty \) such that almost surely \( A^i_\infty = A_\infty \) for all \( i \).

**Proof.** Fix an agent \( j \). When \( B^j_\infty < 1/2 \) (resp. \( B^j_\infty > 1/2 \)) then the first (resp. second) case of corollary 3.6.35 occurs and \( A_\infty = 0 \) (resp. \( A_\infty = 1 \)). Likewise when \( B^j_\infty = 1/2 \) then the third case occurs, \( B^i_\infty = 1/2 \) for all \( i \in V \) and \( A^i_\infty = \{0, 1\} \) for all \( i \in V \).

\[ \square \]

### 3.6.8. Extension to \( L \)-locally connected graphs.

The main result of this article, Theorem 3.6.2, is a statement about undirected graphs. We can extend the proof to a larger family of graphs, namely, \( L \)-locally connected graphs.

Let \( G = (V, E) \) be a directed graph. \( G \) is \( L \)-locally strongly connected if, for each \((i, j) \in E\) there exists a path in \( G \) of length at most \( L \) from \( j \) to \( i \).

Theorem 3.6.2 can be extended as follows.

**Theorem 3.6.36.** Fix \( L \), a positive integer. Let \( \mu_0, \mu_1 \) be such that for every strongly connected, directed graph \( G \) there exists a random variable \( A_\infty \) such that almost surely \( A^u_\infty = A_\infty \) for all \( u \in V \). Then there exists a sequence \( q(n) = q(n, \mu_0, \mu_1) \) such that \( q(n) \to 1 \) as \( n \to \infty \), and \( \mathbb{P}[A_\infty = S] \geq q(n) \), for any choice of \( L \)-locally strongly connected graph \( G \) with \( n \) agents.

The proof of Theorem 3.6.36 is essentially identical to the proof of Theorem 3.6.2. The latter is a consequence of Theorem 3.6.29, which shows learning in bounded degree infinite graphs, and of Lemma 3.6.22, which implies asymptotic learning for sequences of graphs with diverging maximal degree.

Note first that the set of \( L \)-locally strongly connected rooted graphs with degrees bounded by \( d \) is compact. Hence the proof of Theorem 3.6.29 can be used as is in the \( L \)-locally strongly connected setup.

In order to apply Lemma 3.6.22 in this setup, we need to show that when in-degrees diverge then so do out-degrees. For this note that if \((i, j)\) is a directed edge then \( i \) is in the (directed) ball of radius \( L \) around \( j \). Hence, if there exists a vertex \( j \) with in-degree \( D \) then in the ball of radius \( L \) around it there are at least \( D \) vertices. On the other hand,
if the out-degree is bounded by $d$, then the number of vertices in this ball is at most $L \cdot d^L$. Therefore, $d \to \infty$ as $D \to \infty$.

3.6.9. Example of Non-atomic private beliefs leading to non-learning. We sketch an example in which private beliefs are atomic and asymptotic learning does not occur.

Example 3.6.37. Let the graph $G$ be the undirected chain of length $n$, so that $V = \{1, \ldots, n\}$ and $(i, j)$ is an edge if $|i - j| = 1$. Let the private signals be bits that are each independently equal to $S$ with probability $2/3$. We choose here the tie breaking rule under which agents defer to their original signal\footnote{We conjecture that changing the tie-breaking rule does not produce asymptotic learning, even for randomized tie-breaking.}.

We leave the following claim as an exercise to the reader.

Claim 3.6.38. If an agent $i$ has at least one neighbor with the same private signal (i.e., $W_i = W_j$ for $j$ a neighbor of $i$) then $i$ will always take the same action $A_i = W_i$.

Since this happens with probability that is independent of $n$, with probability bounded away from zero an agent will always take the wrong action, and so asymptotic learning does not occur. It is also clear that optimal action sets do not become common knowledge, and these fact are indeed related.
Bibliography


