STRONG AMENABILITY AND THE INFINITE CONJUGACY CLASS PROPERTY

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Abstract. A group is said to be strongly amenable if each of its proximal topological actions has a fixed point. We show that a countable discrete group is strongly amenable if and only if none of its quotients have the infinite conjugacy class property.

1. Introduction

Let $G \curvearrowright X$ be a continuous action of a countable discrete group on a compact Hausdorff space. This action is said to be *proximal* if for any $x, y \in X$ there exists a sequence (g_n) in G such that $\lim_n g_n x = \lim_n g_n y$. G is said to be *strongly amenable* if every such proximal action of G has a fixed point. Glasner introduced these notions in [5], and proved a number of results: he showed that every virtually nilpotent group is strongly amenable, and that non-amenable groups are not strongly amenable. He also gave some examples of amenable groups that are not strongly amenable. Since then, a handful of papers have studied strong amenability [1, 4, 6, 7], but none have made significant progress on relating it to other group properties.

Our main result is a characterization of strongly amenable groups. Recall that a group has the infinite conjugacy class property (ICC) if each of its non-trivial elements has an infinite conjugacy class.

Theorem 1. *A countable discrete group is strongly amenable if and only if it has no ICC quotients.*

For example, this implies that the lamplighter $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ is not strongly amenable, as is the group S_∞ of finite permutations on $\mathbb{N}$. Likewise, the alternating subgroup of S_∞ is not strongly amenable, as is every infinite simple group. Groups that have no ICC quotients—which by Theorem 1 are exactly the strongly amenable groups—are also known as *hyper-FC-central* [8] or *hyper-FC* [2]. This implies that

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subgroups of strongly amenable countable discrete groups are again strongly amenable.

The case of groups with no ICC quotients is a straightforward consequence of Glasner's work. To prove that groups with ICC quotients are not strongly amenable, we consider an ICC group G, and a certain class of symbolic dynamical systems for G. Using a topological genericity argument, we show that in this class there is a proximal action without a fixed point. We furthermore, in Proposition 3.13, show that there exists such an action which is faithful.

In light of Theorem 1, it is natural to ask whether, in some larger class of topological groups, a group is strongly amenable if and only if each of its quotients has a non-trivial element with a compact (or perhaps precompact) conjugacy class.

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2. Overview of the proof of Theorem 1

That a group with no ICC quotients is strongly amenable follows immediately from from the following proposition.

Proposition 2.1. *Let G be a countable discrete group that acts faithfully, minimally and proximally on a compact Hausdorff space X. Then each non-trivial element of G has an infinite conjugacy class.*

Proof. Let g be a non-trivial element of G. Assume by contradiction that g has a finite conjugacy class. Let H be the centralizer of g, so that H has finite index in G. By [5, Lemma 3.2] H acts proximally and minimally on X. Since g is in the center of H, it acts trivially on X, by [5, Lemma 3.3]. This contradicts the assumption that the action in faithful. □

Thus, to prove Theorem 1, we consider any G that is ICC, and prove that it has a proximal action that does not have a fixed point. This is without loss of generality, since if G has a proximal action without a fixed point, then so does any group that has G as a quotient.

Our general strategy for the proof of Theorem 1 is to consider a certain space $\mathcal{S}$ of actions of G. We show that this space includes a proximal action without a fixed point by showing that, in fact, a *generic* action in this space is proximal, and an open set of actions has no fixed point. Genericity here is in the Baire category sense.

To define the space $\mathcal{S}$, let A be a finite alphabet of size at least 2. The *full shift* A^G, equipped with the product topology, is a space on which G acts continuously by left translations. Enumerate elements of $G = \{g_1, g_2, \cdots,\}$ and endow A^G with the metric $d(\cdot,\cdot)$ given by $d(s,t) = 1/k$ where $k = \inf\{n : s(g_n) \neq t(g_n)\}$. An element of A^G is called a *configuration*.

The closed, G-invariant subsets of A^G are called *shifts*. The space of shifts is endowed with the subspace topology of the Hausdorff topology (or Fell topology) on the closed subsets of A^G. This topology is also metrizable: take, for example, the metric that assigns to a pair of shifts $S, T \subseteq A^G$ the distance $1/(n+1)$, where n is the largest index such that S and T agree on $\{g_1, \cdots, g_n\}$; by agreement on a finite $X \subseteq G$ we mean that the restriction of the configurations in S to X is equal to the restrictions of the configurations in T to X. Note that for any shift $S \subseteq A^G$, the sets of the form $\{T \subseteq A^G \mid T \text{ agrees with } S \text{ on } X\}$ for different finite subsets $X \subseteq G$ form a basis of the neighborhoods for S.

We define the space $\mathcal{S}$ to be the closure, in the space of shifts, of the strongly irreducible shifts, which are defined as follows:

Definition 2.2. A shift $S \subseteq A^G$ is said to be strongly irreducible if there exist a finite $X \subseteq G$ including the identity such that for any two subsets $E_1, E_2 \subseteq G$ with $E_1X \cap E_2X = \emptyset$ and any two configurations $s_1, s_2 \in S$ there is a configuration $s \in S$ such that s restricted to E_1 equals s_1 restricted to E_1, and s restricted to E_2 equals s_2 restricted to E_2.

To show that the proximal actions are generic in $\mathcal{S}$, we define ε-proximal actions; proximal actions will be the actions which are ε-proximal for each $\varepsilon > 0$.

Definition 2.3. An action $G \curvearrowright X$ on a compact metric space with metric $d(\cdot,\cdot)$ is ε-proximal if for all $x, y \in X$ there exists a $g \in G$ such that $d(gx, gy) < \varepsilon$.

A subset of a topological space is generic (in the Baire category sense) if it contains a dense G_δ. To prove our main result, we show that the proximal actions are a dense G_δ in $\mathcal{S}$. The proof of density is the main challenge of this paper, while proving that this subset is a G_δ is straightforward.

Claim 2.4. *The set of ε-proximal shifts is an open set in $\mathcal{S}$. Thus the set of proximal shifts is a G_δ set in $\mathcal{S}$.*

The Baire Category Theorem guarantees that for well behaved spaces (e.g., compact, as is our space $\mathcal{S}$), a countable intersection of dense open sets is dense. Thus, to prove that the proximal shifts are dense in the closure of the strongly proximal shifts, it suffices to show that the ε-proximal shifts are dense in $\mathcal{S}$ for each ε. That is, fixing ε, we must show that for each strongly irreducible subshift $S \subseteq A^G$ and each finite subset $X \subseteq G$ there exists a strongly irreducible subshift S' that agrees with S on X, and is ε-proximal.

To this end, we construct a class of subshifts of $\{0,1\}^G$ (which we denote by 2^G) which are ε-proximal. Furthermore, for these shifts ε-proximality is witnessed by a particular configuration around the origin: one having a 1 at the origin, and zeros close to it. For a finite symmetric subset $X \subset G$ and $g, h \in G$ we say that g and h are *X-apart* if $g^{-1}h \notin X$.

Definition 2.5. Let X be a finite symmetric subset of G. A shift $S \subset 2^G$ is an *X-witness* shift if

(1) For each $s \in S$, $s(a) = 1$ and $s(b) = 1$ implies that a and b are X-apart.
(2) For each $s, t \in S$ there exists an $a \in G$ such that $s(a) = t(a) = 1$.

The first step in the construction of X-witness shifts is to construct a single configuration which is an X-witness in a large finite ball.

Proposition 2.6. *Let G be an ICC group. For each finite symmetric $X \subset G$ there exists an $s \in 2^G$ and a finite symmetric $Y \supset X$ such that*

(1) For every $a, b \in G$, if $s(a) = s(b) = 1$ then a and b are X-apart.
(2) For every $a, b \in Y^{100}$ there exists some $g \in Y$ such that $s(ag) = s(bg) = 1$.

The proof of this proposition—along with Proposition 2.7 below—contains the main technical effort of this paper. The construction is a probabilistic one: we choose the configuration s at random, and then show that it has the desired properties with positive probability. This stage crucially uses the assumption that the group is ICC, which translates to independence of some events that arise in the analysis of this random choice. This is the only step in the proof of Theorem 1 in which we use the ICC property of G.

We use the configuration constructed in Proposition 2.6 to construct X-witness shifts. These shifts will additionally (and importantly) be strongly irreducible.

Proposition 2.7. *Let G be a group for which, for each finite symmetric $X \subset G$, there exists a configuration that satisfies the conditions of Proposition 2.6. Then for each such X there also exists a strongly irreducible X-witness shift.*

The combination of Propositions 2.6 and 2.7 immediately yields the following.

Proposition 2.8. *Let G be an ICC group. Then for each finite symmetric $X \subset G$ there exists a strongly irreducible X-witness shift $S \subset 2^G$.*

Finally, we use these strongly irreducible X-witness shifts to construct ε-proximal approximations to a given strongly irreducible shift S.

Proposition 2.9. *Let G be a group for which there exists, for each finite symmetric $X \subset G$, a strongly irreducible X-witness shift. Let $T \subseteq A^G$ be a strongly irreducible shift. Then for each ε and finite $X \subset G$ there exists a strongly irreducible shift $T' \subseteq 2^G$ that is ε-proximal and agrees with T on X.*

An immediate consequence of Proposition 2.9 and Claim 2.4 is the following.

Proposition 2.10. *Let G be an ICC group. Then the proximal shifts are a dense G_δ in $\mathcal{S}$.*

Given all this, the proof of our main theorem follows easily.

Proof of Theorem 1. That groups with no ICC quotients are strongly amenable follows immediately from Proposition 2.1. Let G be ICC. By Proposition 2.10 the proximal shifts are a dense G_δ in $\mathcal{S}$. Now, it is easy to see that the set of shifts without fixed points is open, since, by a compactness argument, this is witnessed by the restriction of the shift to some finite subset of G. Thus it remains to be shown that there exists a single strongly irreducible shift without fixed points. One is given, for example, in the proof of Claim 3.9. Thus there exists a proximal shift in $\mathcal{S}$ without a fixed point, and so G is not strongly amenable. □

3. Proofs

3.1. **Proof of Proposition 2.6.** Let G be an ICC group, and let X be a finite, symmetric subset of G. We choose a random configuration $u \in 2^G$ as follows. Assign to each element of G an independent uniform random variable in $[0, 1]$. Let V_a be the random variable corresponding to $a \in G$. For each $a \in G$, let $u(a) = 1$ iff $V_a > V_{ax}$ for all $x \in X \setminus \{e\}$. That is, let $u(a) = 1$ if $V_a > V_b$ whenever $a^{-1}b \in X$ and $b \neq a$. The following claim is an immediate consequence of the definition of u.

Claim 3.1. *If $a_1, \cdots, a_n$ are X^2-apart*[1] *for $a_i \in G$, then $\{u(a_i) = 1\}$ are independent events.*

Clearly, for all values of the random configuration, $u(a) = u(b) = 1$ implies that $a^{-1}b \notin X$ for all $a, b \in G$, which means that a and b are X-apart. So the random configuration u almost surely satisfies the first part of the proposition. It thus remains to find a finite symmetric subset $Y \supset X$ such that for each $a, b \in Y^{100}$ there exists some $g \in Y$ such that $u(ag) = u(bg) = 1$.

The next lemma claims that there exists a subset Y with certain useful properties. We use this lemma to prove our proposition, and then prove the lemma.

Lemma 3.2. *There exists a $Y \supset X$ with the following properties.*

(1)
$$|Y|^{200}(1 - |X|^{-2})^{|Y|/(10|X^2|+5)} < 1.$$

(2) For each $a, b \in G$ there exists a subset $Y_{a,b} \subseteq Y$ with the following properties.
  (a) $|Y_{a,b}| \geq |Y|/(10|X^2| + 5)$.
  (b) For $y_1, y_2 \in Y_{a,b}$, ay_1 and by_2 are X^2-apart.

For $c, d, y \in G$, let E_c be the event that $u(c) = 1$, and let $E^y_{c,d} = E_{cy} \cap E_{dy}$. Now fix $a, b \in G$.

(1) By the second property of $Y_{a,b}$, ay and by are X^2-apart for any $y \in Y_{a,b}$. Hence E_{ay} and E_{by} are independent, by Claim 3.1.
(2) $\mathbb{P}[E_c] = 1/|X|$ for all $c \in G$.
(3) Combining the previous two results: $\mathbb{P}\left[E^y_{a,b}\right] = |X|^{-2}$ for all $y \in Y_{a,b}$. So $\mathbb{P}\left[\neg E^y_{a,b}\right] = 1 - |X|^{-2}$.
(4) $E^y_{a,b}$ are independent events for different values of $y \in Y_{a,b}$. This is because ay_1 and by_2 are X^2-apart for any $y_1, y_2 \in Y_{a,b}$, which means $\{E_{ay}, E_{by} \mid y \in Y_{a,b}\}$ are independent events. And finally, since $E^y_{a,b} = E_{ay} \cap E_{by}$, we get that $E^y_{a,b}$ are independent events for $y \in Y_{a,b}$.
(5) We say that the pair (a, b) *fails* if $E^y_{a,b}$ does not happen for any $y \in Y_{a,b}$. So, by the previous two results,
$$\begin{aligned}\mathbb{P}[(a,b) \text{ fails}] &= \mathbb{P}\left[E^y_{a,b} \text{ for no } y \in Y_{a,b}\right] \\ &= (1 - |X|^{-2})^{|Y_{a,b}|} \\ &\leq (1 - |X|^{-2})^{|Y|/(10|X^2|+5)},\end{aligned}$$
where the last inequality follows from the first property of $Y_{a,b}$.

[1]Recall that given a finite symmetric subset $X \subset G$ and $g, h \in G$, we say that g and h are X-apart if $g^{-1}h \notin X$.

By the last inequality, union bound, and the first property of Y:

$$\begin{aligned}\mathbb{P}\left[(a,b) \text{ fails for some } a,b \in Y^{100}\right] &\leq |Y^{100}|^2(1-|X|^{-2})^{|Y|/(10|X^2|+5)} \\ &\leq |Y|^{200}(1-|X|^{-2})^{|Y|/(10|X^2|+5)} < 1.\end{aligned}$$

So, there is at least one configuration, say s, for which no (a,b) fails for $a,b \in Y^{100}$. Therefore, for all $a,b \in Y^{100}$ there is a $y \in Y$ such that $s(ay) = s(by) = 1$. So this s satisfies the second part of the proposition, which concludes the proof of Proposition 2.6, except the proof of Lemma 3.2, to which we turn now.

Proof of Lemma 3.2. We call an element $g \in G$ *switching* if for all non-identity $x \in X^2$ we have $g^{-1}xg \notin X^2$.

Claim 3.3. *There exists at least one switching element $g_s \in G$.*

Proof. Let C_x be the centralizer of x for each $x \in X^2$. Then there are finitely many cosets of C_x, say $g_1^x C_x, \cdots, g_{n_x}^x C_x$, such that $g^{-1}xg \in X^2$ only if $g \in g_i^x C_x$ for some $i \in \{1, \cdots, n_x\}$. So, non-switching elements are in the union of finitely many cosets of subgroups with infinite index, i.e. g is non-switching only if $g \in g_i^x C_x$ for some $x \in X^2$ and some $i \in \{1, \cdots, n_x\}$. Since G is ICC, each C_x has infinite index in G. By [9, Lemma 4.1] a finite collection of cosets of infinite index does not cover the whole group G, so, there is at least one switching element in G. □

Let g_s be a switching element. We can choose an arbitrarily large finite subset $Y_1 \subseteq G$ which includes the identity and such that $Y_1 \cap Y_1 g_s = \emptyset$. Choose such a Y_1 that is large enough so that

$$(5|Y_1|)^{200}(1-|X|^{-2})^{2|Y_1|/(10|X^2|+5)} < 1 \quad \text{and} \quad |Y_1| \geq |X|$$

and let $Y = (Y_1 \cup Y_1 g_s) \cup (Y_1 \cup Y_1 g_s)^{-1} \cup X$. Note that Y is symmetric and $5|Y_1| \geq |Y| \geq 2|Y_1|$ which implies

$$|Y|^{200}(1-|X|^{-2})^{|Y|/(10|X^2|+5)} < 1.$$

This establishes the first property of Y.

Fix $a,b \in G$ with $a \neq b$. We say $g \in G$ is *distancing* for the pair (a,b) if ag and bg are X^2-apart.

Claim 3.4. *If $h \in G$ is not distancing for (a,b) then hg_s is distancing for (a,b).*

Proof. Since h is not distancing for (a,b), $(ah)^{-1}(bh) = h^{-1}a^{-1}bh \in X^2$. By the definition of a switching element $g_s^{-1}[(ah)^{-1}(bh)]g_s = (ahg_s)^{-1}(bhg_s) \notin X^2$, which means that hg_s is distancing for (a,b). □

By this observation, if $y_1 \in Y_1$ is not distancing for (a,b) then $y_1 g_s \in Y_1 g_S$ is distancing for (a,b). So at least half of the elements in $Y_1 \cup Y_1 g_s$ are distancing for (a,b) and thus at least one fifth of the elements in Y are distancing for (a,b). Let $Y'_{a,b}$ be the collection of elements in Y that are distancing for (a,b). We just saw that $|Y'_{a,b}| \geq |Y|/5$.

Now define a graph on $Y'_{a,b}$ by connecting $y_1 \neq y_2 \in Y'_{a,b}$ if either ay_1 and by_2 are not X^2-apart or ay_2 and by_1 are not X^2-apart. Call this graph $G'_{a,b}$. Note that the degree of each $y \in Y'_{a,b}$ in $G'_{a,b}$ is at most $2|X^2|$. So, we can find an independent set of size at least $|Y'_{a,b}|/(2|X^2|+1) \geq |Y|/(10|X^2|+5)$ in $G'_{a,b}$. Call this independent set $Y_{a,b}$.

Claim 3.5. $|Y_{a,b}| \geq |Y|/(10|X^2|+5)$ *and for* $y_1, y_2 \in Y_{a,b}$, *we have that* ay_1 *and* by_2 *are* X^2*-apart.*

Proof. The bound on the size of Y is established in the previous paragraph. To see the second property, note that in the case that $y_1 \neq y_2$, this follows from independence of $Y_{a,b}$ in $G'_{a,b}$. In the case $y_1 = y_2$, it follows from the fact that all elements of $Y'_{a,b}$ are distancing for (a,b) and that $Y_{a,b} \subseteq Y'_{a,b}$. □

This establishes the two properties of $Y_{a,b}$, and thus concludes the proof of the lemma. □

3.2. **Proof of Proposition 2.7.**

Definition 3.6. Let $Z_1, \cdots, Z_n$ be finite subsets of G. A $\{Z_1, \cdots, Z_n\}$-*packing* is a $p \in \{Z_1, \cdots, Z_n, \emptyset\}^G$ with $h\, p(h) \cap g\, p(g) = \emptyset$ for all $g, h \in G$; note that $h\, p(h)$ and $g\, p(g)$ are each a translate, by h and g respectively, of some element of $\{Z_1, \cdots, Z_n, \emptyset\}$. When $p(g) \neq \emptyset$, we call the translate $g\, p(g)$ a *block*.

By an abuse of notation, we use the term Z-packing instead of $\{Z\}$-packing when we have only one subset.

Definition 3.7. Let $Z_1, \cdots, Z_n$ be finite subsets of G. We say that a $\{Z_1, \cdots, Z_n\}$-packing p is *saturated* if there is no $\{Z_1, \cdots, Z_n\}$-packing $p' \neq p$ such that $p(g) \neq \emptyset$ implies $p'(g) = p(g)$.

Saturated Z-packings are packings to which one cannot add any blocks (i.e., Z-translates) without removing at least one (note, however, that it may be possible to add more Z translates by first removing some).

The proof of the following claim is standard.

Claim 3.8. *Let* Z *be a finite subset of* G. *The set of all saturated* Z*-packings is a shift. We denote this shift by* P_Z.

The first part of following claim will be useful in the construction of strongly irreducible shifts, while the second is used in the proof of Theorem 1 to show that there exists a proximal shift without a fixed point.

Claim 3.9. *P_Z is strongly irreducible. If Z is of size at least 2 then P_Z does not have a fixed point.*

A similar claim with a similar proof appears in [3, Lemma 2.2].

Proof. Let $X = Z \cup Z^{-1}$. Let $F_1, F_2 \subset G$ be any two subsets of G that are X^4-apart. To prove the claim, it suffices to show that for any $p_1, p_2 \in P_Z$ there is a $p \in P_Z$ that agrees with p_1 on F_1 and with p_2 on F_2. Let $F_1' = F_1X$ and $F_2' = F_2X$. Then F_1' and F_2' are disjoint, and furthermore, if $a_1 \in F_1'$ and $a_2 \in F_2'$ then $p_1(a_1)$ and $p_2(a_2)$ are disjoint. Thus there is a packing that is equal to p_1 on F_1' and to p_2 on F_2'. Consider the set of all such saturated packings. This set is non-empty by Zorn's Lemma. Furthermore, it is easy to see that each packing in this set agrees with p_1 on F_1, agrees with p_2 on F_2, and is saturated, i.e. is an element of P_Z. Thus P_Z is strongly irreducible.

Note that the only fixed elements of $\{Z, \emptyset\}^G$ are the two constant configurations. When $|Z| \geq 2$ then the constant configuration Z is not a packing, since the blocks overlap, and the constant configuration $\emptyset$ is not saturated. Thus, in this case, P_Z has no fixed points. □

We can now start the proof of proposition 2.7. Assume that X, a finite symmetric subset of G, is given. We now seek to construct a strongly irreducible X-witness shift T. Since G satisfies proposition 2.6, we can let Y and s be a finite symmetric subset of G and a configuration on G that satisfy the statement of proposition 2.6 for $X \subseteq G$.

Define $\phi : P_{Y^{100}X} \times P_{YX} \to \{Y^{100}X, YX, \emptyset\}^G$ as follows. For $p_1 \in P_{Y^{100}X}$ and $p_2 \in P_{YX}$ let

$$\phi(p_1,p_2)(g) = \begin{cases} p_1(g) & \text{if } p_1(g) \neq \emptyset \\ p_2(g) & \text{if } p_2(g) \neq \emptyset \text{ and} \\ & \quad g\,p_2(g) \cap h\,p_1(h) = \emptyset \text{ for all } h \in G \\ \emptyset & \text{otherwise} \end{cases}$$

So ϕ receives a $Y^{100}X$-packing p_1 and a YX-packing p_2, and produces a $\{Y^{100}X, YX, \emptyset\}$-packing $\phi(p_1,p_2)$, which can be informally described as follows. Add all the blocks in p_1 to $\phi(p_1,p_2)$. For any block in p_2, if it does not intersect with any of the blocks in p_1, add it to $\phi(p_1,p_2)$, and otherwise ignore it. In other words, $\phi(p_1,p_2)$ consists of all the blocks in p_1, and all the blocks in p_2 that are disjoint from the blocks in p_1.

It is easy to verify that ϕ is continuous and equivariant, and that $P = \phi(P_{Y^{100}X} \times P_{YX})$ is a saturated $\{Y^{100}X, YX\}$-packing. Since P is a factor of a product of strongly irreducible shifts it is also strongly irreducible.

Define $\psi : P \to \{0,1\}^G$ by

$$[\psi(p)](g) = \begin{cases} s(h^{-1}g) & \text{if } g \in h\,Y^{100} \text{ for some } h \in G \\ & \quad \text{with } p(h) = Y^{100}X \\ s(h^{-1}g) & \text{if } g \in h\,Y \text{ for some } h \in G \\ & \quad \text{with } p(h) = YX \\ 0 & \quad \text{otherwise} \end{cases}$$

What ψ does is produce a configuration which is 0 outside of the X-interior[2] of blocks, and is equal to translates of $s|_{Y^{100}}$ and $s|_Y$ inside the interior of blocks.

It is again easy to see that ψ is continuous and equivariant, so $T = \psi(P)$ is a strongly irreducible shift. The following claim completes the proof of proposition 2.7.

Claim 3.10. *T is an X-witness shift.*

The claim follows immediately from the following two lemmas. The first of the lemmas is straightforward from our construction, while the second is less immediate.

Lemma 3.11. *For all $t \in T$, the 1's in t are X-apart.*

Proof. Let $t \in T$ and $a, b \in G$ with $t(a) = t(b) = 1$. Since $t \in T$, there is a $p \in P$ with $\psi(p) = t$. By the definition of ψ, since $[\psi(p)](a) = [\psi(p)](b) = 1$, we get that $a \in hp(h)$ and $b \in gp(g)$ for some $h, g \in G$. If $g = h$, i.e. a and b are in the same block of p, then $s(h^{-1}a) = t(a) = 1$ and $s(h^{-1}b) = t(b) = 1$. But, since s satisfies proposition 2.6 (in particular, the 1's in s are X-apart), $h^{-1}a$ and $h^{-1}b$ are X-apart, which implies a and b are X-apart. If $g \neq h$, then $hp(h)$ and $gp(g)$ are disjoint, so the X-interior of $h\,p(h)$ and the X-interior of $g\,p(g)$ are X-apart. We also know that a is in the X-interior of $h\,p(h)$ and b is in the X-interior of $g\,p(g)$. Therefore, a and b are X-apart. □

Lemma 3.12. *For any $t_1, t_2 \in T$ there is a $g \in G$ with $t_1(g) = t_2(g) = 1$.*

[2]The X-interiors of $Y^{100}X$ and YX are Y^{100} and Y.

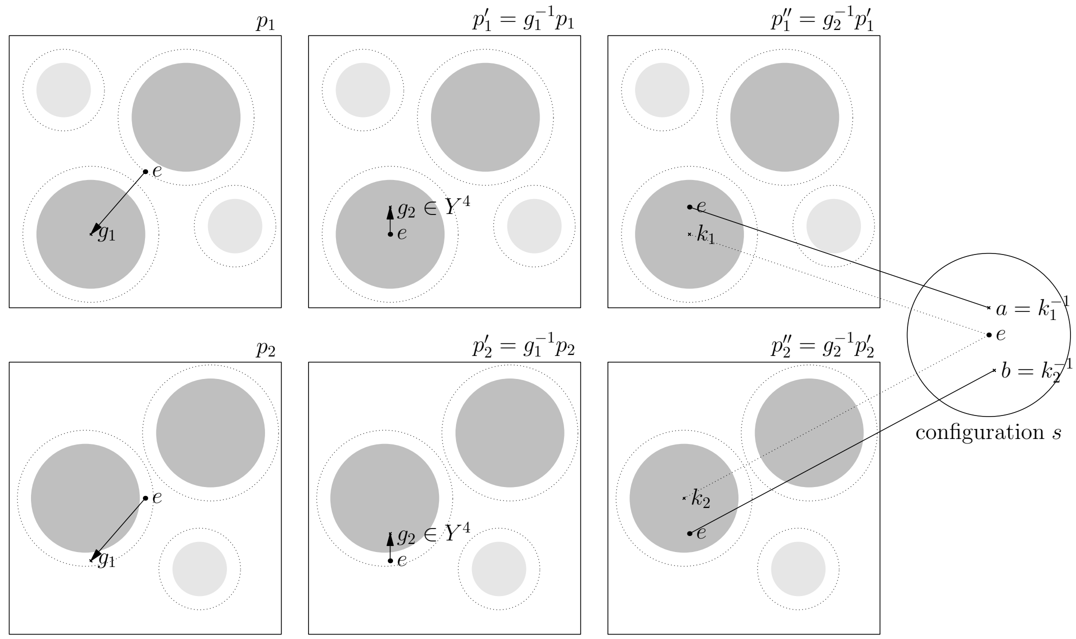

FIGURE 1. Case (I).

Proof. We essentially prove this lemma by a series of reductions.

Let $t_1, t_2 \in T$. So there are $p_1, p_2 \in P$ with $\psi(p_1) = t_1$ and $\psi(p_2) = t_2$. Pick a $g_1 \in G$ with $p_1(g_1) = Y^{100}X$. This means that $g_1^{-1}p_1$ has a block of shape $Y^{100}X$ centered at the identity. Let $p'_1 = g_1^{-1}p_1$, $p'_2 = g_1^{-1}p_2$, and let $t'_1 = \psi(p'_1)$, $t'_2 = \psi(p'_2)$. So, p'_1 has a block of shape $Y^{100}X$ centered at the identity.

Since p'_2 is saturated, we know there is a $g_2 \in Y^4$ such that either (I) g_2 is in the YX-interior of a block of shape $Y^{100}X$ in p'_2, or (II) g_2 is the center of a block of shape YX in p'_2. Let $p''_1 = g_2^{-1}p'_1$, $p''_2 = g_2^{-1}p'_2$, and let $t''_1 = \psi(p''_1)$, $t''_2 = \psi(p''_2)$. Observe that in p''_1 the identity is in the YX-interior of a block of shape $Y^{100}X$, say $e \in k_1Y^{99}$ for some $k_1 \in G$ with $p''_1(k_1) = Y^{100}X$. Moreover, in p''_2 the identity is either (I) in the YX-interior of a block of shape $Y^{100}X$, or (II) in the center of a block of shape YX.

In case (I), since in p''_2 the identity is in the YX-interior of a block of shape $Y^{100}X$, $e \in k_2Y^{99}$ for some $k_2 \in G$ with $p''_2(k_2) = Y^{100}X$. So $k_2^{-1} \in Y^{99}$. By the second part of proposition 2.6 applied to $a = k_1^{-1}$ and $b = k_2^{-1}$, we know that there is a $g_3 \in Y$ such that $s(k_1^{-1}g_3) = s(k_2^{-1}g_3) = 1$. So, by the definition of ψ, the fact that $k_1^{-1}g_3 \in Y^{100}$, and the fact that $p''_1(k_1) = Y^{100}X$, we get $t''_1(g_3) = s(k_1^{-1}g_3) = 1$, and similarly, we get $t''_2(g_3) = s(k_2^{-1}g_3) = 1$. Therefore, $t_1(g_1g_2g_3) =$

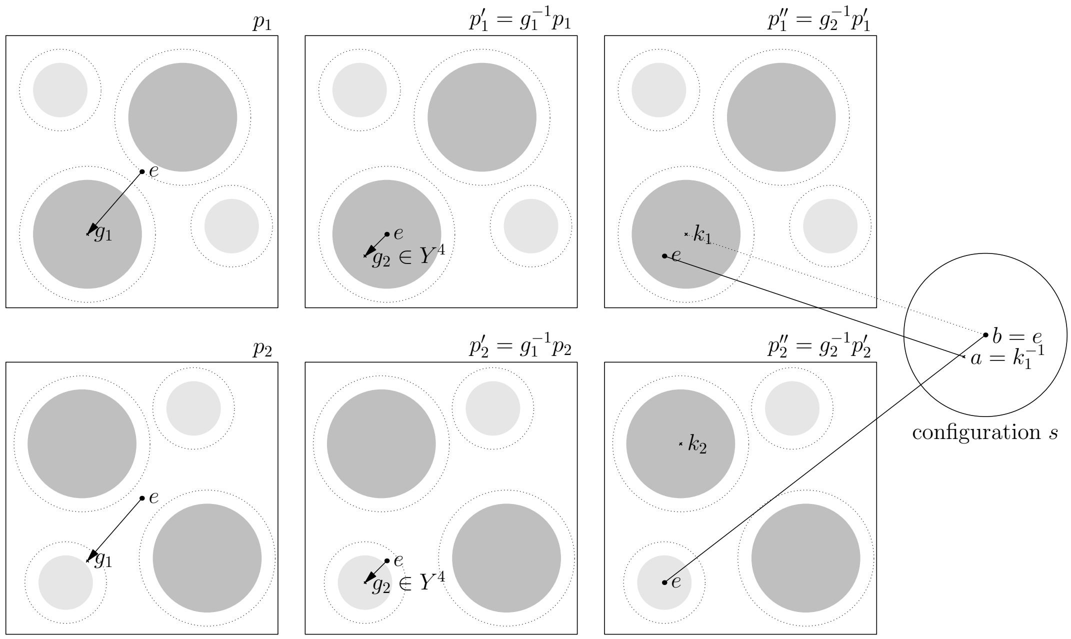

FIGURE 2. Case (II).

$t''_1(g_3) = 1$ and $t_2(g_1g_2g_3) = t''_2(g_3) = 1$. Case (I) is schematically depicted in Figure 1.

In case (II), $p''_2(e) = YX$. Again, if we apply the second part of proposition 2.6 to $a = k_1^{-1}$ and $b = e$, we get that there is a $g_3 \in Y$ such that $s(k_1^{-1}g_3) = s(g_3) = 1$. So, by the definition of ψ, the fact that $k_1^{-1}g_3 \in Y^{100}$, and the fact that $p''_1(k_1) = Y^{100}X$, we get $t''_1(g_3) = s(k_1^{-1}g_3) = 1$. Also, by the definition of ψ, the fact that $g_3 \in Y$, and the fact that $p''_2(e) = YX$, we get $t''_2(g_3) = s(g_3) = 1$. Therefore, $t_1(g_1g_2g_3) = t''_1(g_3) = 1$ and $t_2(g_1g_2g_3) = t''_2(g_3) = 1$. Case (II) is schematically depicted in Figure 2.

In both cases we showed there is a $g \in G$ with $t_1(g) = t_2(g) = 1$. This completes the proof. □

3.3. **Proof of Proposition 2.9.** Fix $\epsilon > 0$ and X a finite symmetric subset of G. Without loss of generality, we may assume that X includes the identity.

Since $T \subseteq A^G$ is a strongly irreducible shift, there is a finite symmetric $U \subseteq G$ including the identity such that for any two subsets $E_1, E_2 \subseteq G$ with $E_1U \cap E_2U = \emptyset$ and any two configurations $t_1, t_2 \in T$ there is a configuration $t \in T$ such that t restricted to E_1 equals t_1 restricted to E_1, and t restricted to E_2 equals t_2 restricted to E_2

Given a shift $S \subseteq A^G$ and a finite $Y \subset G$, we call a map $p\colon Y \to A$ a *Y-pattern of S* if it is equal to $s|_Y$, the restriction of some $s \in S$ to Y. In this case we say that s contains the X-pattern p.

By strong irreducibility of T, we can find a $u \in T$ whose orbit $\{gu : g \in G\}$ contains all the X-patterns of T. Furthermore, since there are only finitely many X-patterns in S, there must be a finite $V \subset G$ (which we assume w.l.o.g. to be symmetric and contain the identity) such that $\{gu : g \in$ the X-interior of $V\}$ contains all the X-patterns of T. By making V even larger, we can assume that $d(t, t') < \varepsilon$ for any two configurations $t, t' \in T$ that agree on V, where $d(\cdot, \cdot)$ is the metric on T.

Let $Z = (VU^2X)(VU^2X)^{-1}$. By the assumption in the statement, there is a strongly irreducible Z-witness shift for G. Call this shift S.

Now, define a continuous equivariant function $\phi : S \times T \to A^G$. Let $s \in S, t \in T$. Let $t' = \phi(s, t)$ be defined as follows, in the following cases:

(1) $g = kh$ for some $k \in G$ with $s(k) = 1$ and some $h \in V$:
In this case, let $t'(g) = u(h)$.

(2) $g = kh$ for some $k \in G$ with $s(k) = 1$ and some $h \in VU^2 \setminus V$:
In this case let $E_1 = kV$ and $E_2 =$ complement of kVU^2. Since $E_1U \cap E_2U = \emptyset$, there is a $v \in T$ with $v|_{E_1} = (ku)|_{E_1}$ and $v|_{E_2} = t|_{E_2}$. If there are multiple choices for v, choose the lexicographically least configuration for a fixed ordering of G and a fixed ordering of A. Let $t'(g) = v(kh)$.

(3) $g \neq kh$ for $s(k) = 1$ and $h \in VU^2$:
In this case, let $t'(g) = t(g)$.

Since the 1's in S are Z-apart, this leads to a well-defined definition for t'. Informally, t' is constructed from t as follows: the configuration t' mostly agrees with t. The first exceptions are the V-neighborhoods of any $k \in G$ such that $s(k) = 1$, where we set t' to equal the pattern that appears around the origin in u. The second exceptions are the borders of these V-neighborhoods, where some adjustments need to be made so that—as we explain below—t' and t agree on any translate of X. This construction is schematically depicted in Figure 3.

The following hold:

- ϕ is continuous and equivariant. So $T' = \phi(S \times T)$ is a subshift.
- Since strong irreducibility is closed under taking products and factors, we see that T' is strongly irreducible.
- Let $t'_1 = \phi(s_1, t_1), t'_2 = \phi(s_2, t_2) \in T'$. Since S is a $(VU^2X)(VU^2X)^{-1}$-witness shift, there is a $g \in G$ with $s_1(g) = s_2(g) = 1$. So, $t'_1|_{gV}$ and $t'_2|_{gV}$ are both translates of $u|_V$, which means $(g^{-1}t'_1)|_V =$

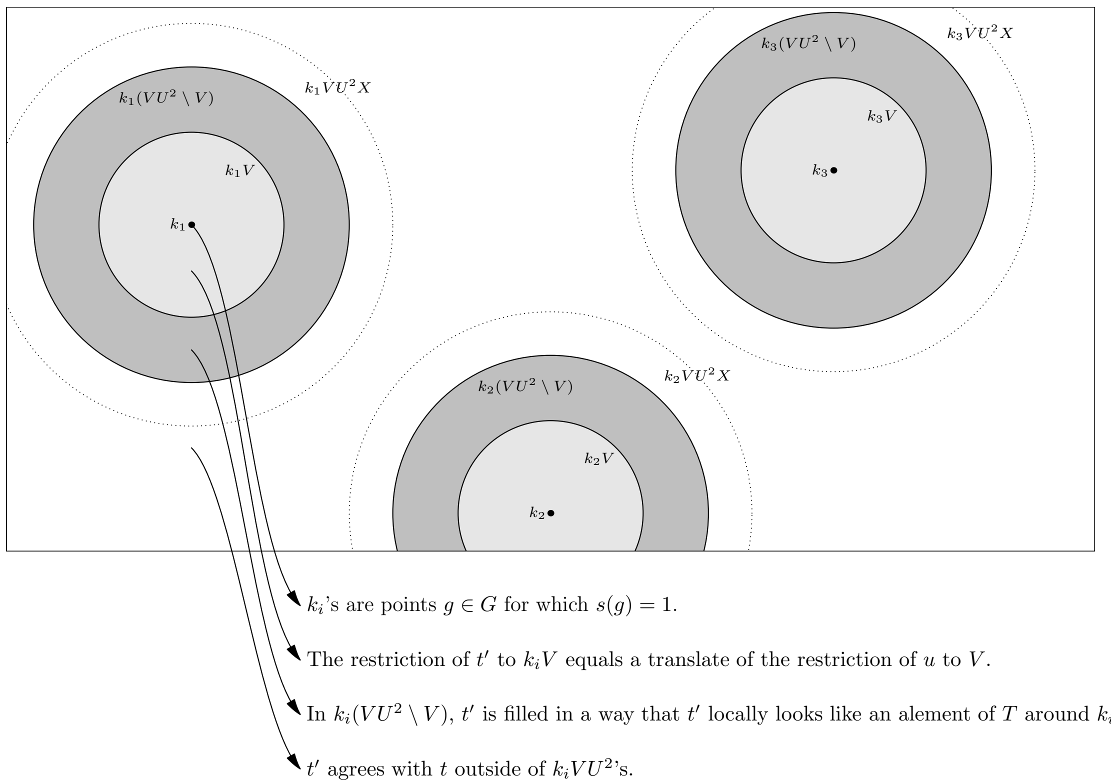

FIGURE 3. $t' = \phi(s,t)$ for $s \in S$ and $t \in T$.

$(g^{-1}t_2')|_V$. So from the definition of V, we get $d(g^{-1}t_1', g^{-1}t_2') < \epsilon$. Hence T' is ϵ-proximal.

- Now we claim that the set of X-patterns of T' and T are equal.

  First notice that since $u|_V$ has all the X-patterns in T, and $u|_V$ appears in T', we get that all the X-patterns of T appear in T'.

  Now let $t' = \phi(s,t) \in T'$ and fix a X-pattern in t', located at gV. If gV does not meet any $k(VU^2X)$ for $s(k) = 1$, then $t'|_{gV} = t|_{gV}$ and so the pattern appears in T. If, on the other hand, gV intersects $k(VU^2X)$ for some k with $s(k) = 1$, by the definition of t' around k, we again see that the pattern in gV appears in T.

This concludes the proof of Proposition 2.9.

3.4. **Proof of Claim 2.4.** Given a shift S, an $\varepsilon > 0$ and a $g \in G$, let $P_g \subset S \times S$ be the set of pairs of configurations s_1, s_2 such that $d(gs_1, gs_2) < \varepsilon$. Since P_g is the preimage of an open set under a continuous map P_g is open. Thus, whenever S is ε-proximal the collection

$\{P_g : g \in G\}$ forms an open cover of $S \times S$ and thus, by compactness, whenever a shift is ε-proximal there is a finite subset $X \subset G$ which suffice to demonstrate this. For each $X \subset G$, whether X demonstrates ε-proximality is determined by the restriction of S to a finite set of elements of G. But this is exactly the definition of a clopen set in the topology on the space of shifts. Thus the set of ε-proximal shifts is the union of a collection of clopen sets and is therefore open.

3.5. **Faithfulness.** Denote by $\mathcal{S}'$ the set $\mathcal{S}$, with the $|A|$ trivial shifts (i.e., singleton) shifts removed.

Proposition 3.13. *Let G be an ICC group. Then there is a dense G_δ set in $\mathcal{S}'$ for which the action $G \curvearrowright S$ is faithful and proximal.*

In the proof of Theorem 1 we show that there is a non-empty open set in $\mathcal{S}$ of shifts without a fixed point. Thus Proposition 3.13 implies that there exist proximal shifts without fixed points which are faithful.

Proof. By Proposition 2.10 the proximal shifts are a dense G_δ in $\mathcal{S}'$. It thus remains to be shown that faithfulness is also generic. Given an element $g \in G$ call a shift *g-faithful* if g acts non-trivially on the shift. It is easy to see that g-faithfulness is an open condition, and so the intersection over all non-trivial $g \in G$, which is faithfulness of the action of G, is a G_δ set. It remains to show that it is dense. To do this we show that each non-trivial $g \in G$ acts non-trivially on every non-trivial strongly irreducible shift. Suppose g is not the identity and acts trivially on a shift S. Then all conjugates of g also act trivially on S, so that $hs = s$ for every h a conjugate of g and $s \in S$. In particular, $s(h^{-1})$ must be the same for every h a conjugate of g and every $s \in S$. Since g has an infinite conjugacy class this holds for infinitely many such h. But if S is strongly irreducibile and non-trivial, then there is some finite $X \subset G$ such that, if $h \notin X$ then there is an $s \in S$ such that $s(h) \neq s(e)$. Thus g must act non-trivially on every non-trivial strongly irreducible shift, and so we have proved the claim. ☐

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