CHOQUET-DENY GROUPS AND THE INFINITE CONJUGACY CLASS PROPERTY

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Abstract. A countable discrete group G is said to be Choquet-Deny if it has a trivial Poisson boundary for every generating probability measure. We show that a finitely generated group G is Choquet-Deny if and only if it is virtually nilpotent. Moreover, when G is not virtually nilpotent, then the Poisson boundary is non-trivial for a generating measure that is symmetric and has finite entropy. For general countable discrete groups, we show that G is Choquet-Deny if and only if none of its quotients have the infinite conjugacy class property.

1. Introduction

Let G be a countable discrete group. A probability measure μ on G is *generating* if its support generates G as a semigroup. A function $f\colon G \to \mathbb{R}$ is *μ-harmonic* if $f(k) = \sum_{g\in G} \mu(g)f(kg)$ for all $k \in G$. We say that the *measured group* (G, μ) is *Liouville* if all the bounded μ-harmonic functions are constant; this is equivalent to the triviality of the Poisson boundary $\Pi(G, \mu)$ [12–14] (also called the Furstenberg-Poisson boundary; for formal definitions see also, e.g., Furstenberg and Glasner [11], Bader and Shalom [1], or a survey by Furman [10]).

When G is non-amenable, (G, μ) is not Liouville for every generating μ [14]. Conversely, when G is amenable, then there exists some generating μ such that (G, μ) is Liouville, as has been shown by Kaimanovich and Vershik [21] and Rosenblatt [23]. It is natural to ask for which groups G it holds that (G, μ) is Liouville for *every* generating μ. Such groups are known in the literature as *Choquet-Deny* groups (see, e.g., [15–17]).

The classical Choquet-Deny Theorem states that abelian groups are Choquet-Deny [3, 4], and the same holds for virtually nilpotent groups [6]. There are many examples of amenable groups that are not Choquet-Deny (see, e.g., [21]), Erschler shows that even some groups of intermediate growth are not Choquet-Deny [7], and that solvable

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groups that are not virtually nilpotent are not Choquet-Deny [8]. See Bartholdi and Erschler [2] for additional related results and further references and discussion.

Our main result is a characterization of the countable discrete Choquet-Deny groups. Recall that G has the *infinite conjugacy class* property (ICC) if each of its non-trivial elements has an infinite conjugacy class.

Theorem 1. *A countable discrete group G is Choquet-Deny if and only if it has no ICC quotients. Moreover, when G does have an ICC quotient, then there exists a generating, symmetric, finite entropy probability measure μ on G such that (G, μ) is not Liouville. In particular, if G is finitely generated, then there exists such a μ if and only if G is not virtually nilpotent.*

The case of a group with no ICC quotients is straightforward and has been previously observed by Jaworski [18] and Jaworski and Raja [19]. We provide a proof for completeness, in §3. Our contribution is therefore in the proof of the other direction, which appears in §2. The implication for finitely generated groups is a consequence of the fact that in this class, virtually nilpotent groups are precisely those with no ICC quotients [22, Theorem 2].

A very recent result by three of the authors of this paper shows that a group is strongly amenable if and only it has no ICC quotients [9]. This implies that a countable discrete G is strongly amenable if and only if (G, μ) is Liouville for every generating μ, paralleling the above mentioned characterization of amenability as equivalent to the existence of such a μ. While the proofs of these two similar results are different, it is natural to ask whether there is some deeper connection between strong amenability and the Choquet-Deny property.

The remainder of the paper consists of §2, in which we prove Proposition 2.1, and §3 in which we prove Proposition 3.1. Each proposition shows one of the directions of Theorem 1, with the former establishing the theorem for groups with ICC quotients, and the latter for groups with no ICC quotients. For the convenience of the reader, we informally explain in §2.1 the main ideas behind the proof of Proposition 2.1.

2. Groups with ICC quotients

In this section we prove the following proposition. Recall that a probability measure μ on G is symmetric if $\mu(g) = \mu(g^{-1})$ for all $g \in G$. Its Shannon entropy (or just entropy) is $H(\mu) = -\sum_{g \in G} \mu(g) \log \mu(g)$.

Proposition 2.1. *Let G be a countable discrete ICC group with an ICC quotient. Then there exists a generating, symmetric, finite entropy probability measure μ on G such that $\Pi(G,\mu)$ is non-trivial.*

The main technical effort in the proof of Proposition 2.1 is in the proof of the following proposition.

Proposition 2.2. *Let G be an amenable countable discrete ICC group. There exists a generating, symmetric, finite entropy probability measure μ on G and an element of the group $h \in G$, such that*

$$\liminf_{M\to\infty} \left\| \frac{1}{M}\sum_{m=1}^{M} h\mu^{*m} - \frac{1}{M}\sum_{m=1}^{M} \mu^{*m} \right\| > 0. \tag{2.1}$$

Here μ^{*m} is the m-fold convolution $\mu * \cdots * \mu$. We will prove this Proposition later, and now turn to the proof of Proposition 2.1.

Proof of Proposition 2.1. The case of non-amenable G is known, so assume that G is amenable and has an ICC quotient Q. Applying Proposition 2.2 to Q yields a finite entropy, symmetric measure $\bar{\mu}$ on Q that is generating, and satisfies (2.1) for some $h \in Q$. By [20, Theorem 2.8] this implies that $(Q,\bar{\mu})$ has a non-trivial Poisson boundary. Let μ be any symmetric, finite entropy generating probability measure on G that is projected to $\bar{\mu}$; the existence of such a μ is straightforward. Then (G,μ) has a non-trivial Poisson boundary. □

Before we turn to the proof of Proposition 2.2, we would like to give some of the intuition behind it.

2.1. Ideas behind the proof of Proposition 2.2.

In this section we provide a simplified description of the main components of the construction in the proof of Proposition 2.2. Note that some of the statements in this description are imprecise, but still give the correct intuition.

To build our measure μ we fix in a probability measure p on the natural numbers, construct inductively a sequence $g_1, g_2, \ldots$ of elements of G, and set $\mu(g_n) = p(n)$. With foresight, we choose $p(n) \sim n^{-5/4}$, which will guarantee some desirable properties, as we explain below.

For simplicity assume that G is finitely generated by $\{a_1, \ldots, a_k\}$. To construct our sequence $g_1, g_2, \ldots$, begin by setting $g_1 = a_1, \ldots, g_k = a_k$. This guarantees that μ is generating. To ensure that μ^{*m} is distant from $h\mu^{*m}$ for some h and large m, we now explain how we choose h and $g_{k+1}, g_{k+2}, \ldots$ so that, after removing a low probability set, we are left with disjoint supports for μ^{*m} and $h\mu^{*m}$.

Assume that we already picked $g_1, \ldots, g_{n-1}$. Choose $s_1, \ldots, s_m$ i.i.d. from p, and assume that m is such that, with high probability, all the

s_i's are at most n. Then, a random choice from μ^{*m} will be a product $g_{s_1}\cdots g_{s_m}$ of elements in $\{g_1,\ldots,g_n\}$. Our choice of p guarantees that with high probability, if g_n appears in this product, then it appears once. In §2.3 we show that p has the properties that we will require.

Assume first, that when picking m i.i.d. draws from μ, there will be a unique one which is g_n and all the rest will have strictly smaller indices. Namely, two typical draws from μ^{*m} are of the form $g_{s_1}\cdots g_n\cdots g_{s_{m-1}}$ and $g_{t_1}\cdots g_n\cdots g_{t_m}$ where all the $s_i,t_i<n$.

For the supports of μ^{*m} and $h\mu^{*m}$ to be disjoint (after removing a small probability set), we want to ensure that

$$h\underbrace{g_{s_1}\cdots}_{b_1}g_n\underbrace{\cdots g_{s_{m-1}}}_{b_2}\neq\underbrace{g_{t_1}\cdots}_{c_1}g_n\underbrace{\cdots g_{t_m}}_{c_2}$$

which is equivalent to

$$hb_1g_nb_2\neq c_1g_nc_2.$$

After moving elements around we need to have $g_n^{-1}c_1^{-1}hb_1g_n\neq c_2b_2^{-1}$. Let $B_n=\{g_1,\ldots,g_n\}$, let $X=(B_{n-1})^{-m}h(B_{n-1})^m$ and let $Y=(B_{n-1})^m(B_{n-1})^{-m}$. Then we want g_n to satisfy that $g_n^{-1}Xg_n\cap Y=\emptyset$. This is impossible in general since the trivial element might belong to X. However, using induction on n, we show that with high probability $c_1^{-1}hb_1\neq e$, for an appropriate initial choice of h.

We call g_n a *switching element for* (X,Y) if $g_n^{-1}(X\setminus\{e\})g_n\cap Y=\emptyset$, and prove in §2.2 that for any finite sets X and Y in an amenable ICC group, there are infinitely many switching elements.

The case that g_n does not appear in both draws but only in one of them requires additionally that g_n is outside some finite set. Since we know that there are infinitely many switching elements, this is always possible. So we will always have a possible choice for g_n.

To ensure that the measure μ is symmetric, we in fact pick g_n or g_n^{-1} with equal probability. This imposes additional constraints on the choice of g_n. Most importantly, we will need to verify not only that $g_n^{-1}(X\setminus\{e\})g_n\cap Y=\emptyset$, but that all the four equations $g_n^{\pm1}(X\setminus\{e\})g_n^{\pm1}\cap Y=\emptyset$ are satisfied. We call such elements g_n *super-switching elements for* (X,Y). In §2.2 we prove that like regular switching elements, there are infinitely many super-switching elements in any amenable ICC group. Hence we will be able to find our g_n, avoiding any finite set.

This completes the description of the basic idea behind our construction. In the formal proof we need to address some additional technical challenges. One problem is to show that the difference between the Cesàro average of the measure's convolutions and its h-translation is large. Therefore, we need to consider the case that the number of element (m above) is different in the two sides of the equations. This adds some cases to verify but is not a deep complication.

Another issue is that for non-finitely generated groups, there is no guarantee that the sequence $g_1, g_2, \ldots$ generates the whole group. For that we enumerate all the elements in G by $\{a_1, a_2, \ldots\}$ and choose a_n instead of g_n with very small probability.

Now we turn to the formal proof of these ideas.

2.2. Switching Elements.

Here we introduce two notions: switching elements and super-switching elements. We will use these notions in the proof of Proposition 2.2.

Definition 2.3. Let G be a countable group and X, Y be finite symmetric subsets of G.

- We call $g \in G$ a *switching element for* (X, Y) if for all non-identity $x \in X$ we have $g^{-1}xg \notin Y$.
- We call $g \in G$ a *super-switching element for* (X, Y) if for all non-identity $x \in X$ we have $g^{\pm 1}xg^{\pm 1} \notin Y$. That is, if $g^{w_1}xg^{w_2} \notin Y$ for all non-trivial $x \in X$ and $w_1, w_2 \in \{-1, +1\}$.

Note that since X and Y are symmetric, for $g \in G$ to be super-switching for (X, Y), we just need to have that g and g^{-1} are switching for (X, Y), and that $gxg \notin Y$ for all non-identity $x \in X$.

Proposition 2.4. *Let G be a countable discrete amenable ICC group and let X, Y be finite symmetric subsets of G. The set of super-switching elements for (X, Y) is infinite.*

Proof of Proposition 2.4. Fix an invariant finitely additive probability measure d on G.[1] For $A \subseteq G$, we call $d(A)$ the density of A. We will need the fact that infinite index subgroups have zero density.

Let $C_G(x)$ be the centralizer of a non-identity $x \in X$. Then, since Y is finite, there is a finite set of cosets of $C_G(x)$ that includes all $g \in G$ such that $g^{-1}xg \in Y$. So, non-switching elements for (X, Y) are in the union of finitely many cosets of subgroups with infinite index, since G is ICC. This means that the set of non-switching elements for (X, Y) has zero density, and so the set of switching elements for (X, Y) has density one, as does the set S of elements $g \in G$ such that both g and g^{-1} are switching for (X, Y).

[1]The argument in this proof can be reproduced using finitary Følner sequence arguments that do not use the axiom of choice.

Let T be the set of all super-switching elements for (X, Y). Let $A \subseteq G$ be the set of involutions $\{g \in G \mid g^2 = e\}$. If $d(A) > 0$, then $d(A \cap S) > 0$. On the other hand, for any $g \in A \cap S$, since g is switching for (X, Y) and $g^{-1} = g$, g is super-switching for (X, Y). Hence $A \cap S \subseteq T$. This shows that if $d(A) > 0$, then $d(T) \geq d(A \cap S) > 0$, and so we are done.

So, we can assume that $d(A) = 0$. For any $x \in X \setminus \{e\}$ and $y \in Y$, let $S_{x,y} = \{g \in S \mid gxg = y\}$. Note that

$$T = S \setminus \bigcup_{\substack{x \in X \setminus \{e\} \\ y \in Y}} S_{x,y}.$$

It is thus enough to be shown that each $S_{x,y}$ has zero density. So assume for the sake of contradiction that $d(S_{x,y}) > 0$. Fix $g \in S_{x,y}$. We have the following for all $h \in g^{-1}S_{x,y}$.

$$\begin{aligned}
gxg = y = ghxgh &\implies (xg) = h(xg)h \\
&\implies (xg)^{-1}h^{-1}(xg) = h \\
&\implies h = (xg)^{-1}h^{-1}(xg) \\
&\qquad = (xg)^{-1}[(xg)^{-1}h^{-1}(xg)]^{-1}(xg) \\
&\qquad = (xg)^{-2}h(xg)^2 \\
&\implies h \text{ is in the centralizer of } (xg)^2.
\end{aligned}$$

So, the centralizer of $(xg)^2$ includes $g^{-1}S_{x,y}$, which has a positive density. So, the centralizer of $(xg)^2$ has finite index. This implies that $(xg)^2 = e$, since G is ICC, and so only the identity can have a finite index centralizer. Hence $xg \in A$ for all $g \in S_{x,y}$. So $xS_{x,y} \subseteq A$. Hence $S_{x,y}$ also has zero density, which is a contradiction. $\square$

2.3. **A Heavy-Tailed Probability Distribution on $\mathbb{N}$.** Here we state and prove a lemma about the existence of a probability distribution on $\mathbb{N}$ such that infinite i.i.d. samples from this measure have certain properties. We will use this distribution in the proof of Proposition 2.2.

Lemma 2.5. *There exists a probability distribution p on $\mathbb{N}$ with finite Shannon entropy and the following property: for any $\varepsilon > 0$ there exist constants $K_\varepsilon, N_\varepsilon \in \mathbb{N}$ such that for any natural number $m \geq K_\varepsilon$ there exists an $E_\varepsilon^m \subseteq \mathbb{N}^m$ such that:*

(1) $p^{\times m}(E_\varepsilon^m) \geq 1 - \varepsilon$, where $p^{\times m}$ is the m-fold product measure $p \times \cdots \times p$.
(2) For any $s = (s_1, \ldots, s_m) \in E_\varepsilon^m$, the maximum of $\{s_1, \ldots, s_{K_\varepsilon}\}$ is at most N_ε.

(3) *For any* $s = (s_1, \dots, s_m) \in E_\varepsilon^m$ *and for any* $K_\varepsilon \le k \le m$, *the maximum of* $\{s_1, \dots, s_k\}$ *is at least* k^2.
(4) *For any* $s = (s_1, \dots, s_m) \in E_\varepsilon^m$ *and for any* $K_\varepsilon \le k \le m$, *the maximum of* $\{s_1, \dots, s_k\}$ *appears in* $(s_1, \dots, s_k)$ *only once.*

Proof. Let p be the following probability measure on $\mathbb{N}$: $p(n) = cn^{-5/4}$, where $1/c = \sum_{n=1}^{\infty} n^{-5/4}$. It is straightforward to see that p has finite entropy.

Let $s = (s_1, s_2, \dots) \in \mathbb{N}^\infty$ have distribution $p^{\times\infty}$; i.e., s is chosen i.i.d. p. Since each s_i has distribution p, for each $n \in \mathbb{N}$ we have:

(2.2)
$$\mathbb{P}\left[s_i \ge n\right] = \sum_{m=n}^{\infty} p(m) = c \sum_{m=n}^{\infty} m^{-5/4} \ge c \int_n^\infty x^{-5/4} \mathrm{d}x = 4cn^{-1/4}.$$

For $k \ge 1$, let

$$\max\nolimits_s^k \coloneqq \max\{s_1, \dots, s_k\}.$$

Let A_k be the event that $\max_s^k < k^2$. We have:

$$\begin{aligned}\mathbb{P}\left[A_k\right] &= \mathbb{P}\left[s_i < k^2 \ \forall i \in \{1, \dots, k\}\right] \\ &= \left(1 - \mathbb{P}\left[s_1 < k^2\right]\right)^k \\ &\le (1 - 4c(k^2)^{-1/4})^k \\ &\le e^{-4ck^{1/2}}.\end{aligned}$$

Since the sum of these probabilities is finite, by Borel-Cantelli we get that

$$\mathbb{P}\left[A_k \text{ infinitely often}\right] = 0.$$

This means that we have a random index ind'_s which is almost surely finite, and for $A = \{s \mid \max_s^k \ge k^2 \quad \forall k \ge \mathrm{ind}'_s\}$ we have $\mathbb{P}\left[A\right] = 1$.

For $k \ge 1$, let

$$\mathrm{next}_s^k \coloneqq \min\{i > k \mid s_i \ge \max\nolimits_s^k\}.$$

In other words, next_s^k is the first index $i > k$ for which s_i is at least as large as $\max_s^k$. Let B_k be the event that

$$s_{\mathrm{next}_s^{\mathrm{ind}'_s + k}} = \max\nolimits_s^{\mathrm{ind}'_s + k}.$$

This is the event that the first entry after $i = \mathrm{ind}'_s + k$ that is at least as large as $\max_s^i$, is in fact equal to $\max_s^i$. We would like to show that

B_k only happens for finitely many k. To this end, let $C_k^{i,j}$ be the event that $\mathrm{next}_s^{\mathrm{ind}'_s+k} = i$ and $\max_s^{\mathrm{ind}'_s+k} = j$, and note that for any $i, j \in \mathbb{N}$

$$\mathbb{P}\left[B_k \middle| C_k^{i,j}\right] = \mathbb{P}\left[s_i = j | s_i \geq j\right] \leq \frac{cj^{-5/4}}{4cj^{-1/4}} = \frac{1}{4j}.$$

The equality follows from the independence of s_i from $s_1, s_2, \ldots, s_{i-1}$, and the inequality is a consequence of (2.2). By the definition of ind'_s, we have that $C_k^{i,j}$ can only have positive probability if $j \geq k^2$. Therefore, and since the union $\cup_{i,j} C_k^{i,j}$ has measure one, we have that $\mathbb{P}\left[B_k\right] \leq k^{-2}/4$. So, again by Borel-Cantelli, we get

$$\mathbb{P}\left[B_k \text{ infinitely often}\right] = 0.$$

This means that we have a random index ind''_s which is almost surely finite, and for $B = \{s \mid s_{\mathrm{next}_s^k} > \max_s^k \quad \forall k \geq \mathrm{ind}'_s + \mathrm{ind}''_s\}$ we have $\mathbb{P}\left[B\right] = 1$.

Let $\mathrm{ind}_s = \max_s^{\mathrm{ind}'_s+\mathrm{ind}''_s} + 1$. Note that ind_s is almost surely finite and for $k \geq \mathrm{ind}_s$ we can prove the following properties for s.

(1) Since $k > \max_s^{\mathrm{ind}'_s+\mathrm{ind}''_s} \geq (\mathrm{ind}'_s+\mathrm{ind}''_s)^2 \geq \mathrm{ind}'_s$, by the definition of ind'_s, we have that $\max_s^k \geq k^2$.

(2) We know that $\max_s^k \geq k^2 > \max_s^{\mathrm{ind}'_s+\mathrm{ind}''_s}$. So we have indices

$$\mathrm{next}_s^{\mathrm{ind}'_s+\mathrm{ind}''_s} = i_1 < i_2 < \cdots < i_l < i_{l+1} = \mathrm{next}_s^k$$

such that $i_j \in \{1, \ldots, k\}$ for $j = 1, 2, \ldots, l$, $i_{l+1} > k$, and $i_j = \mathrm{next}_s^{i_j-1}$ for $j = 2, 3, \ldots, l+1$.
Since each $i_j > \mathrm{ind}'_s + \mathrm{ind}''_s$, by the definition of ind''_s, for $j = 1, \ldots, l$ we get that $s_{i_j} = \max_s^{i_j} = \max_s^{i_{j+1}-1}$ appears in $(s_1, \ldots, s_{i_{j+1}-1})$ only once. In particular $\max_s^k = s_{i_l}$ appears in $(s_1, \ldots, s_k)$ only once.

Fix $\varepsilon > 0$. Since ind_s is almost surely finite, there are constants $K_\varepsilon \in \mathbb{N}$ and $N_\varepsilon \in \mathbb{N}$ such that the event $E_\varepsilon = \{\mathrm{ind}_s \leq K_\varepsilon \text{ and } \max_s^{K_\varepsilon} \leq N_\varepsilon\}$ has probability at least $1 - \varepsilon$.

Let $m \geq K_\varepsilon$ be a natural number and E_ε^m be the projection of E_ε to the first m coordinates. We have:

(1) $p^{\times m}(E_\varepsilon^m) \geq \mathbb{P}\left[E_\varepsilon\right] \geq 1 - \varepsilon$.

(2) Any $(s_1, \ldots, s_m) \in E_\varepsilon^m$ has an extension $s = (s_1, s_2, \ldots) \in E_\varepsilon \subseteq \mathbb{N}^\infty$. Since $s \in E_\varepsilon$, we have $\max_s^{K_\varepsilon} \leq N_\varepsilon$, which means that the maximum of $\{s_1, \ldots, s_{K_\varepsilon}\}$ is at most N_ε.

(3) Any $(s_1, \ldots, s_m) \in E_\varepsilon^m$ has an extension $s = (s_1, s_2, \ldots) \in E_\varepsilon \subseteq \mathbb{N}^\infty$. Since $s \in E_\varepsilon$, we have $K_\varepsilon \geq \mathrm{ind}_s$. So for $K_\varepsilon \leq k \leq m$

we have $k \geq \mathrm{ind}_s$. The first property we proved for s gives us $\max_s^k \geq k^2$, i.e., the maximum of $\{s_1, \ldots, s_k\}$ is at least k^2.

(4) Any $(s_1, \ldots, s_m) \in E_\varepsilon^m$ has an extension $s = (s_1, s_2, \ldots) \in E_\varepsilon \subseteq \mathbb{N}^\infty$. Since $s \in E_\varepsilon$, we have $K_\varepsilon \geq \mathrm{ind}_s$. So for $K_\varepsilon \leq k \leq m$ we have $k \geq \mathrm{ind}_s$. The second property we proved for s gives us that the maximum of $\{s_1, \ldots, s_k\}$ appears in $(s_1, \ldots, s_k)$ only once.

This completes the proof. □

2.4. **Proof of Proposition 2.2.** Let $G = \{a_1, a_2, \ldots\}$; $\varepsilon > 0$; $p \in P(\mathbb{N})$, $K_\varepsilon, N_\varepsilon \in \mathbb{N}$, and $E_\varepsilon^m \subseteq \mathbb{N}^m$ be the probability measure, the constants, and the events from Lemma 2.5. To simplify notation let $N = N_\varepsilon$ and $K = K_\varepsilon$. Choose $g_1, g_2, \ldots, g_N$ arbitrarily in G. For any $n \leq N$ let $A_n = \{g_n, g_n^{-1}, a_n, a_n^{-1}\}$ and $B_n = \cup_{i \leq n} A_i \cup \{e\}$. Choose $h \in G$ such that $h \notin (B_N)^{2N}$, which is a finite set.

Now, for $n = N+1, N+2, \ldots$ we define g_n inductively and set $A_n = \{g_n, g_n^{-1}, a_n, a_n^{-1}\}$, $B_n = \cup_{i \leq n} A_i \cup \{e\}$, and note that A_n and B_n are symmetric. Let $n+1 > N$, and assume that $g_1, \ldots, g_n \in G$ are defined. Let $g_{n+1} \in G$ be a super-switching element for $\big((B_n)^n\{h, h^{-1}\}(B_n)^n,\ (B_n)^{2n}\big)$ which is not in $(B_n)^{4n}\{h, h^{-1}\}(B_n)^{4n}$. The existence of such super-switching element is guaranteed by Proposition 2.4 and the fact that $(B_n)^n\{h, h^{-1}\}(B_n)^n$ and $(B_n)^{2n}$ are finite symmetric subsets of G and $(B_n)^{4n}\{h, h^{-1}\}(B_n)^{4n}$ is finite.

For $n \in \mathbb{N}$, define a symmetric probability measure μ_n on A_n with

$$\mu_n = \varepsilon 2^{-n}\left(\frac{1}{2}\delta_{a_n} + \frac{1}{2}\delta_{a_n^{-1}}\right) + (1 - \varepsilon 2^{-n})\left(\frac{1}{2}\delta_{g_n} + \frac{1}{2}\delta_{g_n^{-1}}\right).$$

Here δ_g is the point mass on $g \in G$. Finally, let

$$\mu = \sum_{n=1}^{\infty} p(n)\mu_n.$$

Obviously μ is symmetric and has full support, and therefore is generating. Since p has finite entropy and each μ_n has support of size at most 4, it follows easily that μ has finite entropy.

We want to show that for μ and h we have

$$\liminf_{M \to \infty} \left\| \frac{1}{M} \sum_{m=1}^{M} h\mu^{*m} - \frac{1}{M} \sum_{m=1}^{M} \mu^{*m} \right\| > 0.$$

Fix $M \in \mathbb{N}$ larger than K and N. For each $n \in \mathbb{N}$ define $f_n : \{1, 2, 3, 4\} \to A_n$ by

$$f_n(1) = a_n,\ f_n(2) = a_n^{-1},\ f_n(3) = g_n,\ f_n(4) = g_n^{-1}.$$

Let

$$\Omega = \{(m, s, w) \mid m \in \{1, \dots, M\},\ s \in \mathbb{N}^m,\ w \in \{1, 2, 3, 4\}^m\}.$$

Let $\nu : \Omega \to [0, 1]$ be defined by

$$\nu(m, s, w) = \frac{1}{M}\ p^{\times m}(s)\ \mu_{s_1}(f_{s_1}(w_1))\ \mu_{s_2}(f_{s_2}(w_2))\ \dots\ \mu_{s_m}(f_{s_m}(w_m)).$$

Obviously ν is a probability measure on Ω.

Define $r : \Omega \to G$ by

$$r(m, s, w) = f_{s_1}(w_1) f_{s_2}(w_2) \dots f_{s_m}(w_m).$$

It is not difficult to see that $r_*\nu = \frac{1}{M}\sum_{m=1}^{M} \mu^{*m}$, and so we need to show that $\|hr_*\nu - r_*\nu\|$ is bounded away from zero for large M.

For $m = 1, 2, \dots, M$, recall that $E^m_\varepsilon \subseteq \mathbb{N}^m$ is the event given by Lemma 2.5. Fix $\max\{K, N\} \le m \le M$ and $s \in E^m_\varepsilon$. Define

$$\begin{aligned}
i_{s,1} =\ & \min\{j \in \{1 \dots, m\} \mid s_j > N\},\\
i_{s,2} =\ & \min\{j > i_{s,1} \mid s_j \ge s_{i_{s,1}}\},\\
& \vdots\\
i_{s,l(s)} =\ & \min\{j > i_{s,l(s)-1} \mid s_j \ge s_{i_{s,l(s)-1}}\} = \max\{s_1, \dots, s_m\}.
\end{aligned}$$

So, $i_{s,1}$ is the index of the first entry of $(s_1, \dots, s_m)$ which is larger than N, $i_{s,2}$ is the index of the next entry that is at least as large, etc. Note that by the second property of E^m_ε in Lemma 2.5, we know that

$$K < i_{s,1} < i_{s,2} < \dots < i_{s,l(s)},$$

and by the fourth property,

$$N < s_{i_{s,1}} < s_{i_{s,2}} < \dots < s_{i_{s,l(s)}}.$$

Let $W^{m,s}_\varepsilon = \{w \in \{1, 2, 3, 4\}^m \mid \forall k \le l(s)\ w_{i_{s,k}} = 3, 4\}$. Note that for $\Omega_{m,s} = \{m\} \times \{s\} \times \{1, 2, 3, 4\}^m \subseteq \Omega$ we have

$$\begin{aligned}
\mathbb{P}\left[W^{m,s}_\varepsilon | \Omega_{m,s}\right] &= 1 - \mathbb{P}\left[\neg W^{m,s}_\varepsilon | \Omega_{m,s}\right]\\
&= 1 - \mathbb{P}\left[w_{i_{s,1}} = 1, 2;\ \text{or } w_{i_{s,2}} = 1, 2;\ \dots;\ \text{or } w_{i_{s,l(s)}} = 1, 2\ \middle|\Omega_{m,s}\right]\\
&\ge 1 - \sum_{k=1}^{l(s)} \mathbb{P}\left[w_{i_{s,k}} = 1, 2\middle|\Omega_{m,s}\right]\\
&= 1 - \sum_{k=1}^{l(s)} \varepsilon 2^{-s_{i_{s,k}}}\\
&\ge 1 - \sum_{j=1}^{\infty} \varepsilon 2^{-j} = 1 - \varepsilon.
\end{aligned}$$

The last inequality is because $s_{i_{s,1}} < s_{i_{s,2}} < \cdots < s_{i_{s,l(s)}}$.

Finally, let

$$E_\varepsilon = \{(m, s, w) \in \Omega \mid \max\{K, N\} < m \le M,\ s \in E_\varepsilon^m,\ w \in W_\varepsilon^{m,s}\}.$$

Note that

$$\begin{aligned} \nu(E_\varepsilon) &\ge (1 - \max\{K, N\}/M)(1 - \varepsilon)(1 - \varepsilon) \\ &> 1 - 2\varepsilon - \max\{K, N\}/M. \end{aligned}$$

Claim 2.6. *For any $\alpha, \beta \in E_\varepsilon$, we have $hr(\alpha) \neq r(\beta)$.*

We prove this claim after we finish the proof of the Proposition.

Let ν_1 be equal to ν conditioned on E_ε, and ν_2 be equal to ν conditioned on the complement of E_ε. We have $\nu = \nu(E_\varepsilon)\nu_1 + (1 - \nu(E_\varepsilon))\nu_2$, and by the above claim we know $\|hr_*\nu_1 - r_*\nu_1\| = 2$. So

$$\begin{aligned} \left\| \frac{1}{M} \sum_{m=1}^{M} h\mu^{*m} - \frac{1}{M} \sum_{m=1}^{M} \mu^{*m} \right\| &= \|hr_*\nu - r_*\nu\| \\ &= \|\nu(E_\varepsilon)(hr_*\nu_1 - r_*\nu_1) + (1 - \nu(E_\varepsilon))(hr_*\nu_2 - r_*\nu_2)\| \\ &\ge \nu(E_\varepsilon)\, \|hr_*\nu_1 - r_*\nu_1\| - 2(1 - \nu(E_\varepsilon)) \\ &\ge 2(1 - 2\varepsilon - \max\{K, N\}/M) \\ &\quad - 2(2\varepsilon + \max\{K, N\}/M) \\ &= 2 - 8\varepsilon - 4\max\{K, N\}/M, \end{aligned}$$

which is bounded away from zero for ε small enough and all M large enough. This completes the proof of Proposition 2.2.

Proof of Claim 2.6. Let $\alpha = (m, s, w)$, $\beta = (n, t, v) \in E_\varepsilon$. Hence $\max\{K, N\} < m, n \le M$, $s \in E_\varepsilon^m$, $t \in E_\varepsilon^n$, $w \in W_\varepsilon^{m,s}$, and $v \in W_\varepsilon^{n,t}$. Assume that $hr(\alpha) = r(\beta)$. So, we have

$$hf_{s_1}(w_1) \cdots f_{s_m}(w_m) = f_{t_1}(v_1) \cdots f_{t_n}(v_n).$$

Let $K < i_1 < i_2 < \cdots < i_{l(s)}$ and $K < j_1 < j_2 < \cdots < j_{l(t)}$ be the indices we defined for s and t in the proof of Proposition 2.2. We remind the reader that the unique maximum of $(s_1, \ldots, s_m)$ is attained at $i_{l(s)}$, with a corresponding statement for $(t_1, \ldots, t_n)$ and $j_{l(t)}$. So we have

$$\begin{aligned} &h\, \overbrace{f_{s_1}(w_1) \cdots f_{s_{i_{l(s)}-1}}(w_{i_{l(s)}-1})}^{b_1}\, f_{s_{i_{l(s)}}}(w_{i_{l(s)}})\, \overbrace{f_{s_{i_{l(s)}+1}}(w_{i_{l(s)}+1}) \cdots f_{s_m}(w_m)}^{b_2} \\ &= \underbrace{f_{t_1}(v_1) \cdots f_{t_{j_{l(t)}-1}}(v_{j_{l(t)}-1})}_{c_1}\, f_{t_{j_{l(t)}}}(v_{j_{l(t)}})\, \underbrace{f_{t_{j_{l(t)}+1}}(v_{j_{l(t)}+1}) \cdots f_{t_n}(v_n)}_{c_2}. \end{aligned}$$

Let $p = s_{i_{l(s)}} = \max\{s_1, \ldots, s_m\}$ and $q = t_{j_{l(t)}} = \max\{t_1, \ldots, t_n\}$. Since $w \in W_\varepsilon^{m,s}$ and $v \in W_\varepsilon^{n,t}$, we know $f_{s_{i_{l(s)}}}(w_{i_{l(s)}}) = g_p^{\pm 1}$ and $f_{t_{j_{l(t)}}}(v_{j_{l(t)}}) = g_q^{\pm 1}$, so

$$hb_1 g_p^{\pm 1} b_2 = c_1 g_q^{\pm 1} c_2. \tag{2.3}$$

Since $p = \max\{s_1, \ldots, s_m\}$, and since $m \geq K$, we know that $m \leq m^2 \leq p$. So $b_1, b_2 \in (B_{p-1})^{p-1}$. Similarly $c_1, c_2 \in (B_{q-1})^{q-1}$.

Consider the case that $p > q$. Then $c_1, c_2, g_q^{\pm 1} \in (B_q)^q \subseteq (B_{p-1})^{p-1}$. Hence $g_p^{\pm 1} = [b_1^{-1}]h^{-1}[c_1 g_q^{\pm 1} c_2 b_2^{-1}]$ by (2.3), and so

$$g_p \in (B_{p-1})^{4(p-1)}\{h, h^{-1}\}(B_{p-1})^{4(p-1)},$$

which is a contradiction with our choice of g_p. Similarly, if $p < q$, we get a contradiction. So we can assume that $p = q$.

If $p = q$, then by (2.3) we have

$$hb_1 g_p^{\pm 1} b_2 = c_1 g_p^{\pm 1} c_2,$$

and $c_1, c_2, b_1, b_2 \in (B_{p-1})^{p-1}$. So, for $x = c_1^{-1} h b_1 \in (B_{p-1})^{p-1}\{h, h^{-1}\}(B_{p-1})^{p-1}$ we have $g_p^{\pm 1} x g_p^{\pm 1} = c_2 b_2^{-1} \in (B_{p-1})^{2(p-1)}$. By the fact that g_p is a super-switching element for $\big((B_{p-1})^{p-1}\{h, h^{-1}\}(B_{p-1})^{p-1}, \, (B_{p-1})^{2(p-1)}\big)$, we get that x is the identity.

So $hb_1 = c_1$, i.e.

$$hf_{s_1}(w_1) \cdots f_{s_{i_{l(s)}-1}}(w_{i_{l(s)}-1}) = f_{t_1}(v_1) \cdots f_{t_{j_{l(t)}-1}}(v_{j_{l(t)}-1}).$$

By the exact same argument, we can see this leads to a contradiction unless

$$hf_{s_1}(w_1) \cdots f_{s_{i_{l(s)-1}-1}}(w_{i_{l(s)-1}-1}) = f_{t_1}(v_1) \cdots f_{t_{j_{l(t)-1}-1}}(v_{j_{l(t)-1}-1}).$$

And again, this leads to a contradiction unless

$$hf_{s_1}(w_1) \cdots f_{s_{i_{l(s)-2}-1}}(w_{i_{l(s)-2}-1}) = f_{t_1}(v_1) \cdots f_{t_{j_{l(t)-2}-1}}(v_{j_{l(t)-2}-1}).$$

By continuing this, we get a contradiction unless

$$h\overbrace{f_{s_1}(w_1) \cdots f_{s_{i_1-1}}(w_{i_1-1})}^{b'} = \overbrace{f_{t_1}(v_1) \cdots f_{t_{j_1-1}}(v_{j_1-1})}^{c'}.$$

(1) Recall that $s_1, \ldots, s_{i_1-1} \leq N$, which means $\max\{s_1, \ldots, s_{i_1-1}\} \leq N$.
(2) On the other hand $s_{i_1} > N$ and we know that $s_1, \ldots, s_K \leq N$. So, $i_1 > K \implies i_1 - 1 \geq K$. So we get that $\max\{s_1, \ldots, s_{i_1-1}\} \geq (i_1 - 1)^2 \geq (i_1 - 1)$.
(3) Combining the last two results we get $N \geq i_1 - 1$.

(4) Again, since $s_1, \dots, s_{i_1-1} \le N$, we get that
$$f_{s_1}(w_1), \dots, f_{s_{i_1-1}}(w_{i_1-1}) \in B_N.$$
So,
$$b' = f_{s_1}(w_1) \cdots f_{s_{i_1-1}}(w_{i_1-1}) \in (B_N)^{i_1-1} \subseteq (B_N)^N.$$

A similar argument shows that $c' = f_{t_1}(v_1) \cdots f_{t_{j_1-1}}(v_{j_1-1}) \in (B_N)^N$. But we know that $hb' = c' \implies h = c'b'^{-1} \in (B_N)^{2N}$, which is in contradiction with our choice of h.

$\square$

3. Groups with no ICC quotients

In this section we prove the following.

Proposition 3.1. *Let G be a countable discrete group with no ICC quotients. Then $\Pi(G, \mu)$ is trivial for any generating measure μ on G.*

This claim has appeared previously in the literature [19], and we provide a proof for completeness. The following theorem is in particular well known.

Theorem 3.2 (Choquet-Deny Theorem)**.** *Let G be a group and let μ be any generating probability measure on G. Then the center of G acts trivially on $\Pi(G, \mu)$.*

Proof. Let Z denote the center of G. We show that the restriction to Z of any bounded μ-harmonic on G is constant.

Endow $[0,1]^Z$ with the product topology, and let $F \subset [0,1]^Z$ denote the set of all the restriction to Z of μ-harmonic functions which are bounded by 0 and 1. Then F is a compact convex space. The *Krein-Milman Theorem* states that F is the closure of the convex hull of its extremal points. Hence it is enough to show that the extremal points in F are constant functions, and hence the bounded μ-harmonic functions are Z-invariant.

Consider the G-action on μ-harmonic functions given by $f^g(x) = f(gx)$, and define the Z-action on F similarly. Let f be an extremal function in F. So $f = \overline{f}|_Z$ is the restriction to Z of some μ-harmonic function $\overline{f}$. Then for any $z \in Z$ we get
$$f(z) = \overline{f}(z) = \sum_g \mu(g)\overline{f}(zg) = \sum_g \mu(g)\overline{f}(gz) = \sum_g \mu(g)\overline{f}^g(z).$$

Since the equation above holds for any $z \in Z$, we have shown that $f = \sum_g \mu(g)\overline{f}^g|_Z$. Since f is extremal, and since $\overline{f}^g|_Z \in F$, $f = \overline{f}^g|_Z$ for all g in the support of μ. By applying the same argument to convolution powers of μ, and since μ is generating, we have that $f = \overline{f}^g|_Z$ for all $g \in G$, and in particular $f = \overline{f}^z|_Z = f^z$ for all $z \in Z$. This means that f is constant.

$\square$

Let $H \leq G$ be a subgroup of finite index, and fix some generating μ on G. It is well known that the subgroup H is *μ-recurrent*, namely, that the μ-random walk will hit H with probability 1. We denote the distribution of the first hit by θ. So θ is a generating measure on H which is called *the μ-hitting measure*. The stopping time theorem implies that the restriction map gives a canonical H-equivariant isomorphism between bounded μ-harmonic functions on G and bounded θ-harmonic functions on H. It follows that $\Pi(G, \mu)$ and $\Pi(H, \theta)$ are isomorphic as H-spaces.

Lemma 3.3. *Let G be a countable discrete group, μ be a generating probability measure on G, and let $g \in G$ have finite conjugacy class. Then g acts trivially on $\Pi(G, \mu)$.*

Proof. Let $g \in G$ be an element with a finite conjugacy class. Then its centralizer $H = C_G(g)$ is of finite index in G. Denote by θ the hitting measure on H. So $\Pi(G, \mu)$ and $\Pi(H, \theta)$ are H-isomorphic. Since g in in the center of H, by the Choquet-Deny Theorem, it acts trivially on $\Pi(H, \theta)$, and hence it acts trivially on $\Pi(G, \mu)$. $\square$

In particular, if there exists some generating measure μ such that the G-action on $\Pi(G, \mu)$ is faithful then G is an ICC group.

Proof of Proposition 3.1. Assume that there exists some μ on G for which $\Pi(G, \mu)$ is non-trivial. Let N denote the kernel of the action. So G/N acts faithfully on $\Pi(G/N, \bar{\mu})$ where $\bar{\mu}$ is the μ-projected measure on G/N via $G \to G/N$. By the Lemma 3.3, G/N is an ICC group. $\square$

Remark 3.4. *More generally, by applying an induction argument and Lemma 3.3, one can show that the* hyper-FC-center [5, 22] *acts trivially on $\Pi(G, \mu)$ for every generating μ on G.*

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