NORMAL AMENABLE SUBGROUPS OF THE AUTOMORPHISM GROUP OF THE FULL SHIFT

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Abstract. We show that every normal amenable subgroup of the automorphism group of the full shift is contained in its center. This follows from the analysis of this group's Furstenberg topological boundary, through the construction of a minimal and strongly proximal action.

We extend this result to higher dimensional full shifts. This also provides a new proof of Ryan's Theorem and of the fact that these groups contain free groups.

Contents

1. Introduction

For $n \geq 2$, let $A = \{0, 1, \ldots, n-1\}$ be a finite alphabet. Equip the countable product $A^{\mathbb{Z}}$ with the product topology. Let $\sigma \colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be the left shift, and let $\mathrm{aut}(A^{\mathbb{Z}})$ be the group of homeomorphisms of $A^{\mathbb{Z}}$ that commute with the shift σ. The space $A^{\mathbb{Z}}$ is called the *full shift*, and $\mathrm{aut}(A^{\mathbb{Z}})$ is called the *automorphism group of the full shift*. The elements of this group are known as (invertible) *cellular automata*.

Date: December 1, 2015.

This research was partially conducted at Microsoft Research, New England.

This group has been studied extensively, starting with Hedlund [10], who showed that it is countable, and that in many senses it very large; in particular, it contains free groups of every rank, and hence it is non-amenable[1] Ryan [13] showed that its center $Z(\mathrm{aut}(A^{\mathbb{Z}}))$ is equal to $\Sigma(A^{\mathbb{Z}})$, the group consisting of the powers of the shift σ. Boyle, Lind and Rudolph [1] made further progress, extend many results to automorphism groups of shifts of finite type, and note that $\mathrm{aut}(2^{\mathbb{Z}})$ and $\mathrm{aut}(4^{\mathbb{Z}})$ are not algebraically isomorphic[2].

Our main theorem is a strengthening of Ryan's:

Theorem 1.1. *Every normal amenable subgroup of* $\mathrm{aut}(A^{\mathbb{Z}})$ *is contained in* $\Sigma(A^{\mathbb{Z}})$.

For every group G there exists a maximal normal amenable subgroup called the *amenable radical* (see for example [12]); we denote it by $\sqrt{G}$. Thus this theorem in fact states that $\sqrt{\mathrm{aut}(A^{\mathbb{Z}})} = \Sigma(A^{\mathbb{Z}})$.

Furman [7] showed that the amenable radical is the kernel of a group's action on its *Furstenberg topological boundary.* A *topological boundary* of a group G is a compact G-space X such that the continuous G action on X is minimal and strongly proximal [8, 9]. The Furstenberg topological boundary $B(G)$ (or the *maximal boundary*) is the universal topological boundary, in the sense that it admits a G-equivariant map to any G-boundary. Since $Z(G) \subseteq \sqrt{G}$, and since the extension of a faithful action is faithful, it follows that

Theorem 1.2 (Furman [7])**.** *If a group G has a topological boundary X such that the action of $G/Z(G)$ is faithful, then $\sqrt{G} = Z(G)$.*

To prove our main result, Theorem 1.1, we construct a topological boundary of $\mathrm{aut}(A^{\mathbb{Z}})$ whose kernel is equal to $\Sigma(A^{\mathbb{Z}})$; our theorem then follows from Furman's. To show that our action is strongly proximal we use Glasner's notion of an *extremely proximal* action [9]. We define these terms precisely in the next section.

One can replace $\mathbb{Z}$ with $\mathbb{Z}^d$ in the discussion above. In this case the shift σ and the group it generates are replaced with the d shifts which generate $\Sigma(A^{\mathbb{Z}^d})$. The automorphism group $\mathrm{aut}(A^{\mathbb{Z}^d})$ is defined to be group of homeomorphisms of $A^{\mathbb{Z}^d}$ that commutes with $\Sigma(A^{\mathbb{Z}^d})$.

Hochman [11] proves in this setting an analogue of Ryan's Theorem, namely that $Z(\mathrm{aut}(A^{\mathbb{Z}^d})) = \Sigma(A^{\mathbb{Z}^d})$, and in fact shows that the same holds for the automorphism groups of a large class of shifts. We likewise strengthen his theorem, for the case of the full shift.

[1] Hedlund attributes most of these results to Curtis, Lyndon and Hedlund.

[2] It is not known if $\mathrm{aut}(2^{\mathbb{Z}})$ and $\mathrm{aut}(3^{\mathbb{Z}})$ are isomorphic.

Theorem 1.3. *Every normal amenable subgroup of* $\mathrm{aut}(A^{\mathbb{Z}^d})$ *is contained in* $\Sigma(A^{\mathbb{Z}^d})$.

Recently, Cyr and Kra [4] showed that some subshifts[3] of $A^{\mathbb{Z}}$ with sub-exponential growth have amenable automorphism groups. Their work follows a number of papers that show that the automorphism group of "small" shifts is indeed "small" [2, 3, 5, 6, 14, 15]. A natural question is the following: for which subshifts of $A^{\mathbb{Z}}$ does it still hold that the amenable radical of the automorphism group is equal to its center?

2. A boundary of $\mathrm{aut}(A^{\mathbb{Z}})$

In this section we denote $G = \mathrm{aut}(A^{\mathbb{Z}})$ and $\Sigma = \Sigma(A^{\mathbb{Z}})$.

Let $A^{\mathbb{Z}}_p \subset A^{\mathbb{Z}}$ be the set of configurations (as we shall refer to elements of $A^{\mathbb{Z}}$) that have a constant infinite prefix:

$$A^{\mathbb{Z}}_p = \{x \in A^{\mathbb{Z}} \,:\, \exists m \in \mathbb{Z}, a \in A \text{ s.t. } x_k = a \text{ for all } k \le m\}.$$

Note that this set is invariant to the G-action.

Let $A^{\mathbb{Z}}_* \subset A^{\mathbb{Z}}_p$ be the result of the exclusion from $A^{\mathbb{Z}}_p$ of the n constant configurations. This set is still G-invariant. Given $x \in A^{\mathbb{Z}}_*$, let $\ell(x)$ denote the last coordinate of the constant prefix:

$$\ell(x) = \min\{m \in Z \,:\, x_m \neq x_{m+1}\}.$$

Note that $\ell(\sigma x) = \ell(x) - 1$, in general $\ell(\sigma^k x) = \ell(x) - k$, and in particular $\ell(\sigma^{\ell(x)} x) = 0$.

Let Ω be given by

$$\Omega = \{x\colon \{0, 1, 2, \ldots\} \to A \,:\, x_0 \neq x_1\}.$$

This is the space of one-sided infinite configurations, in which the first symbol is different than the second. We equip it with the natural topology induced from the product topology.

To define a G action on Ω, let $\varphi\colon \Omega \to A^{\mathbb{Z}}_*$ assign to $x \in \Omega$ the two-sided configuration in which a one-sided, infinite constant x_0 prefix precedes $x_1 x_2 \ldots$. Formally:

$$[\varphi(x)]_m = \begin{cases} x_0 & m \le 0 \\ x_m & \text{otherwise} \end{cases}.$$

Note that the image of φ is all the configurations in $A^{\mathbb{Z}}_*$ for which $\ell(x) = 0$. Hence $\varphi^{-1}(\sigma^{\ell(x)} x)$ is well defined for every $x \in A^{\mathbb{Z}}_*$. Accordingly,

[3] A *shift* or *subshift* of $A^{\mathbb{Z}}$ is a closed, shift-invariant subset of $A^{\mathbb{Z}}$.

define $\psi\colon A_*^{\mathbb{Z}} \to \Omega$ by $\psi(x) = \varphi^{-1}(\sigma^{\ell(x)}x)$; note that $\psi \circ \varphi$ is the identity. Now, given $g \in G$ and $x \in \Omega$ let

$$gf = \psi\left(g\varphi(x)\right).$$

It is straightforward to verify that this is indeed a G-action on Ω. Note also that this action factors through G/Σ, since $\sigma f = \sigma$ for all $x \in \Omega$. Additionally, it is easy to see that the G/Σ-action is faithful; this is due to the fact that $\psi^{-1}(\Omega)$ is dense in $A^{\mathbb{Z}}$.

For $a \in A$, let $\Omega^a = \{x \in \Omega : x_0 = a\}$. Let $G^a \subset G$ be the finite index subgroup that fixes Ω^a, and let $G^* = \cap_a G^a$.

2.1. **Extreme proximality.** An action $H \curvearrowright X$ of a discrete group on a compact metric space is said to be *extremely proximal* if, for any closed $Y \subsetneq X$, there exists a sequence $\{h_k\} \subset H$ such that $\lim_k h_k X$ is a singleton, where the limit is taken in the Hausdorff topology [9].

An action $H \curvearrowright X$ of a discrete group on a compact Hausdorff space is said to be *strongly proximal* if, for any Borel probability measure μ on X, there exists a sequence $\{h_k\} \subset H$ such that $\lim_k h_k\mu$ is a point mass, where the limit is taken in the weak* topology [8, 9]. We prove the following theorem in Section 5.

Theorem 2.1. *The action $G^* \curvearrowright \Omega^a$ is extremely proximal for all $a \in A$.*

We can now conclude that G^* is not amenable, and in fact includes a free group with two generators. This follows from the following theorem.

Theorem 2.2 (Glasner [9])**.** *If a group has a non-trivial minimal extremely proximal action then it contains a free subgroup on two generators.*

The fact that G^* is contains a free subgroup on two generators was already shown in [10], and thus this provides a new proof of that fact.

2.2. **Induction to a strongly proximal, minimal action.** Recall that Ω^a is the subset of all $x \in \Omega$ such that $x_0 = a$. Let $\bar{\Omega}$ be the collection of subsets of Ω which intersect each Ω^a in exactly one element. Hence every element of $\bar{\Omega}$ is a set of size $n = |A|$, and can be written as

$$\{x^0, x^1, \ldots, x^{n-1}\}$$

with $x_0^a = a$ for all $a \in A$. The topology on $\bar{\Omega}$ is inherited from Ω in the obvious way.

There is a natural G action on $\bar{\Omega}$, derived from the action on Ω, and hence on the subsets of Ω.

Since the elements of G^* preserve x_0 - that is, $[gx]_0 = x_0$ for all $x \in \Omega$ and $g \in G^*$ - the action of G^* on $\bar{\Omega}$ is isomorphic to the diagonal action $G^* \curvearrowright \Omega^0 \times \Omega^1 \times \cdots \times \Omega^{n-1}$. Since each action $G^* \curvearrowright \Omega^a$ is extremely proximal (Theorem 2.1), the product action $G^* \curvearrowright \Omega^0 \times \Omega^1 \times \cdots \times \Omega^{n-1}$ is strongly proximal. This follows from the facts that (i) extremely proximal actions are strongly proximal and (ii) that a product of strongly proximal actions is likewise strongly proximal [9]. Hence we have shown the following theorem.

Theorem 2.3. *The G^* action on $\bar{\Omega}$ is strongly proximal.*

We next show that this action is also minimal.

Theorem 2.4. *The G^* actions on both $\bar{\Omega}$ and Ω^0 are minimal.*

We prove this theorem in Section 5.

3. The full shift over $\mathbb{Z}$

Given the construction of the previous section, the proof of our main theorem is immediate.

Proof of Theorem 1.1. Note that $\bar{\Omega}$ is a topological boundary of $\mathrm{aut}(A^{\mathbb{Z}})$, since it is strongly proximal (Theorem 2.3) and minimal (Theorem 2.4). Since this action is a faithful action of $\mathrm{aut}(A^{\mathbb{Z}})/\Sigma(A^{\mathbb{Z}})$, the claim follows by Theorem 1.2. □

4. The full shift over $\mathbb{Z}^d$

In this section we extend our result to show that the amenable radical of $\mathrm{aut}(A^{\mathbb{Z}^d})$ is the group of shifts. We do this by essentially reducing the higher dimensional case to the one dimensional case.

Fix a dimension d. For $k \in \mathbb{N}$, let M_k be the basis for $\mathbb{Z}^d$ given by the rows of the following matrix:

$$\begin{pmatrix} 1 & k & 0 & \cdots & 0 & 0 \\ 0 & 1 & k & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & k \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Let $\mathbf{v}$ be the unit vector of the d^{th} coordinate (which is also the last vector in M_k), and let $U_k \subset \mathbb{Z}^d$ the span of the first $d-1$ vectors in M_k. Then every element of $\mathbb{Z}^d$ can be uniquely written as $\mathbf{u} + \ell \cdot \mathbf{v}$ where $\mathbf{u} \in U_k$ and $\ell \in \mathbb{Z}$.

The important property of U_k is the following.

Claim 4.1. *Every nonzero element* $\mathbf{u} \in U_k$ *has norm greater than* k.

Proof. Represent $\mathbf{u}$ as an integral linear combination of the first $d-1$ vectors in M_k, and note that for $\mathbf{u}$ to be nonzero there must be a largest index i such that the coefficient given by the i^{th} basis vector is nonzero. This implies that the $i+1^{\text{st}}$ coordinate is a nonzero multiple of k, which implies that the norm of $\mathbf{u}$ is at least k. □

Let $A^{\mathbb{Z}^d}_{U_k}$ be the subset of $A^{\mathbb{Z}^d}$ which is periodic mod U_k; that is, under the natural shift action of $\mathbb{Z}^d$ on $A^{\mathbb{Z}^d}$, $A^{\mathbb{Z}^d}_{U_k}$ is the set of U_k-invariant elements of $A^{\mathbb{Z}^d}$. Note that $A^{\mathbb{Z}^d}_{U_k}$ is a closed subset of $A^{\mathbb{Z}^d}$. We endow it with the induced topology. After proving a simple claim, we will proceed to show how $A^{\mathbb{Z}^d}_{U_k}$ can be identified with $A^{\mathbb{Z}}$.

The next claim follows directly from the definition of $A^{\mathbb{Z}^d}_{U_k}$ and Claim 4.1.

Claim 4.2. *The projection of* $A^{\mathbb{Z}^d}_{U_k}$ *to* B_k*, the ball of radius* k *in* $\mathbb{Z}^d$*, is equal to* A^{B_k}.

That is, any $x \in A^{B_k}$ can be completed to an element of $A^{\mathbb{Z}^d}_{U_k}$.

Using the obvious group isomorphism between $\mathbb{Z}$ and $\mathbb{Z}\cdot\mathbf{v} \subset \mathbb{Z}^d$ (recall that $\mathbf{v}$ is the last vector in M_k), we obtain the homeomorphism $\pi\colon A^{\mathbb{Z}^d}_{U_k} \to A^{\mathbb{Z}}$ given by

$$[\pi(x)]_n = x_{n\cdot\mathbf{v}}.$$

Note that this is indeed a bijection since $\mathbb{Z}\cdot\mathbf{v}$ is a set of representatives $\mathbb{Z}^d/U_k$, the cosets of U_k in $\mathbb{Z}^d$. It is straightforward to check that π is also continuous.

We thus define a group homomorphism $\phi_k\colon \operatorname{aut}(\mathbb{Z}^d) \to \operatorname{aut}(A^{\mathbb{Z}})$ by setting $\phi_k(g) = \pi\circ g\circ \pi^{-1}$. Note that $\pi^{-1}\circ\sigma\circ\pi$, the conjugation of the shift on $A^{\mathbb{Z}}$ by π, is a shift on $A^{\mathbb{Z}^d}_{U_k}$. It follows that $\phi_k(g)$ commutes with the shift σ, and hence the image of ϕ_k is indeed in $\operatorname{aut}(\mathbb{Z})$.

Let $L_r \subset \operatorname{aut}(\mathbb{Z}^d)$ be the set of cellular automata with memory less than r. That is, $g \in L_r$ if $[g(x)]_0$ is determined by the restriction of x to some ball of radius less than r around 0.

Claim 4.3. *If* $g \in L_k$ *then* g *is the unique element in* $\phi_k^{-1}(g) \cap L_k$.

Proof. Every $g \in L_k$ is uniquely determined, among elements of L_k, by its action on $A^{\mathbb{Z}^d}_{U_k}$; this follows from Claim 4.2. Since the kernel of ϕ_k is the kernel of the action $\operatorname{aut}(\mathbb{Z}^d) \curvearrowright A^{\mathbb{Z}^d}_{U_k}$, it follows that if g has memory less than k then g is the unique element in L_k that is mapped to $\phi(g)$. □

Note that there may, however, be other elements of $\mathrm{aut}(\mathbb{Z}^d)$, which will not be in L_k, whose action on $A_{U_k}^{\mathbb{Z}^d}$ is the same as that of g.

We will exploit these homomorphisms ϕ_k for varying k in order to prove our theorem. We first note the following easy lemma.

Lemma 4.4. *Let $\phi : H \to K$ be a group homomorphism, and let $\sqrt{H}$ denote the amenable radical of H. Then $\sqrt{H} \subseteq \phi^{-1}(\sqrt{K})$.*

Proof. This follows immediately from the fact that both amenable groups and normal subgroups are preserved under quotients. □

We are now in a position to prove the main theorem of this section.

Proof. Choose $g \in \sqrt{\mathrm{aut}(\mathbb{Z}^d)}$. Then g has memory less than k for some large enough k, i.e., $g \in L_k$. From 4.4 it follows that $\phi_k(g) \in \sqrt{\mathrm{aut}(\mathbb{Z})}$, and so $\phi_k(g)$ is a shift σ^m, for some $m \in \mathbb{Z}$, by Theorem 1.1.

We now claim that since $\phi_k(g)$ is a shift and since $g \in L_k$, then g is a shift. Showing this will conclude the proof that the amenable radical of $\mathrm{aut}(\mathbb{Z}^d)$ is equal to the shifts. To see this, note that since $\phi_k(g) = \sigma^m$ then for every $x \in A^{\mathbb{Z}}$, $[\phi_k(g)x]_0 = x_m$. Hence, by the definition of ϕ_k, for every $y \in A_{U_k}^{\mathbb{Z}^d}$ it holds that $[gy]_{\mathbf{0}} = y_{m\cdot\mathbf{v}}$. By the definition of $A_{U_k}^{\mathbb{Z}^d}$, $y_{m\cdot\mathbf{v}} = y_{m\cdot\mathbf{v}+\mathbf{u}}$ for every $\mathbf{u} \in U_k$. Since $g \in L_k$, it follows that the norm of $m \cdot \mathbf{v} + \mathbf{u}$ is at most k for some $\mathbf{u}$. Therefore the shift by $m \cdot \mathbf{v} + \mathbf{u}$ is also in L_k. But by Claim 4.3 the unique element that is both in L_k and $\phi^{-1}(g)$ is g, and hence g is the shift by $m \cdot \mathbf{v} + \mathbf{u}$.

□

5. Construction of cellular automata

5.1. **Defining cellular automata.** To define invertible cellular automata on $A^{\mathbb{Z}}$ we will use the following general scheme[4]. First, we fix a *start marker* S and an *end marker* E, where $S \in A^k$ and $E \in A^{k'}$ for some $k, k' \in \mathbb{N}$. We choose an $n \in \mathbb{N}$ and call some subset $\mathcal{D} \subseteq A^n$ the set of possible *data*. Finally, let π be a bijection $\pi\colon \mathcal{D} \to \mathcal{D}$. In our constructions, π will always be an involution. We then define a cellular automaton $g\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ by the mapping $SDE \to S\pi(D)E$, where the *data* D is an element of $\mathcal{D}$. That is, let $x \in A^{\mathbb{Z}}$, and denote $x_{m,m'} = x_m x_{m+1} \dots x_{m'-1}$. Then if $x_{m,m+k+n+k'} = SDE$ for some $D \in \mathcal{D}$ then $[g(x)]_{m,m+k+n+k'} = S\pi(D)E$, and everywhere else g is the identity. We will use the following notation to define particular automata. For example, if $S = 000$, $E = 111$, $\mathcal{D} = \{2332, 3223\}$ and

[4] This scheme is a generalization of the one used in [1, Section 2].

$\pi(2332) = (3223)$ then we will define g by the diagram, in which the data appear in boldface.

$$\begin{array}{c} 000\mathbf{2332}111 \\ \big\updownarrow g \\ 000\mathbf{3223}111 \end{array}$$

For such an automaton to be well defined, it suffices to show that no two data matches overlap; that is, if $x_{m,m+k+n+k'} = SDE$ and $x_{m',m'+k+n+k'} = SD'E$ for some $m \neq m'$ and $D, D' \in \mathcal{D}$, then the data match intervals $[m+k, m+k+n)$ and $[m'+k, m'+k+n)$ do not intersect. To show that such an automaton is invertible, if suffices to furthermore show that no data match overlaps a marker match. That is, the data match interval $[m+k, m+k+n)$ does not intersect either of the marker match intervals $[m', m'+k)$ and $[m'+k+n, m'+k+n+k')$. We refer to these conditions below as the *overlap* conditions.

We will need to slightly generalize this construction to cellular automata where there is a collection of start markers $(S_1, \ldots, S_\ell)$, corresponding end markers $(E_1, \ldots, E_\ell)$ corresponding data sets $(\mathcal{D}_1, \ldots, \mathcal{D}_\ell)$ and corresponding bijections $(\pi_1, \ldots, \pi_\ell)$. As before, if $x_{m,m+k+n+k'} = S_i D E_i$ for some $D \in \mathcal{D}_i$ then $[g(x)]_{m,m+k+n+k'} = S_i \pi_i(D) E_i$. To ensure well-defindedness and invertibility, similar overlap conditions need to apply. That is, that no data match overlaps another data or marker match, whether of the same index i or not. To specify such automata we will use similar diagrams, for example

$$\begin{array}{ccc} 000\mathbf{2332}111 & \quad & 1000\mathbf{43}00 \\ \big\updownarrow g & & \big\updownarrow g \\ 000\mathbf{3223}111 & & 1000\mathbf{34}00 \end{array}$$

Note that in this example the markers do overlap, but the data cannot.

5.2. **Proof of Theorem 2.4.**

Proof of Theorem 2.4. We show that the G^* action on $\bar{\Omega}$ is minimal. The proof that the action on Ω^0 is minimal follows by the same argument.

To this end, we choose arbitrary $\bar{\omega}, \bar{\eta} \in \bar{\Omega}$ and show that there exists a sequence $g_k \in G^*$ such that $\lim_k g_k \bar{\omega} = \bar{\eta}$.

For each $a \in A$, define $x^a, y^a \in \Omega^a$ by $\bar{\omega} \cap \Omega^a = \{x^a\}$ and $\bar{\eta} \cap \Omega^a = \{y^a\}$. Denote $x^a_{[k]} = x^a_1 \ldots x^a_k \in A^k$ and likewise $y^a_{[k]} = y^a_1 \ldots y^a_k \in A^k$.

Define the transformation g_k as follows.

$$\begin{array}{ll} a^{2^k}\mathbf{a^k}ax^a_{[k]} & \text{for each } a \text{ s.t. } x^a_{[k]} \neq y^a_{[k]} \\ \quad\Big\updownarrow{\scriptstyle g_k} & \\ a^{2^k}\mathbf{y^a_{[k]}}ax^a_{[k]} & \end{array}$$

Note that the choice of 2^k in the start marker is arbitrary; we could have chosen any function that is sufficiently larger than k.

Note also that x^a_1 and y^a_1 are both not equal to a. Using this, and that fact that the end marker starts with a, it is straightforward (if tedious) to verify that the overlap conditions of Section 5.1 are satisfied.

Finally, it is likewise easy to see that $\lim_k g_k x^a = y^a$ in particular, $[g_k(x^a)]_{[k]} = y_{[k]}$. Hence $\lim_k g_k\bar{\omega} = \bar{\eta}$.

$\square$

5.3. **Proof of Theorem 2.1.** Let $o \in \Omega^0$ be given by $o_0 = 0$ and $o_m = 1$ for all $m > 0$. Given $f \in \Omega^0$, let $r(f)$ measure the length of the initial sequence of ones in f:

$$r(f) = \min\{m \geq 0 \,:\, f_{m+1} \neq 1\}.$$

Note that $r(f)$ is well defined for $f \in \Omega^0$ except o; we define $r(o) = \infty$. For $m \in \mathbb{N}$ define

$$C_m = r^{-1}(m) = \{f \in \Omega^0 \,:\, r(f) = m\}.$$

Note that $\cup_{m=1}^{\infty} C_m = \Omega^0 \setminus \{o\}$, that each C_m is closed, and that $\lim_m C_m = \{o\}$ for all m.

We now define a sequence $\{g_k\}_{k>0} \subset G^*$ as follows:

$$\begin{array}{ll} 0^{2^k}\mathbf{0^k}1^y0 & \text{for each } 0 < y < k \\ \quad\Big\updownarrow{\scriptstyle g_k} & \\ 0^{2^k}\mathbf{1^k}1^y0 & \end{array}$$

If $|A| \geq 3$, then in addition we let a be any symbol in A that does not equal 0 or 1, and add to g_k the following transformations:

$$\begin{array}{ll} 0^{2^k}\mathbf{0^k}1^ya & \text{for each } 0 \leq y < k \text{ and } a \in A \setminus \{0,1\} \\ \quad\Big\updownarrow{\scriptstyle g_k} & \\ 0^{2^k}\mathbf{1^k}1^ya & \end{array}$$

For example, two transformations performed by g_3 are

$$
\begin{array}{ccc}
00000000\mathbf{000}110 & \qquad & 00000000\mathbf{000}2 \\
\updownarrow{\scriptstyle g_3} & & \updownarrow{\scriptstyle g_3} \\
00000000\mathbf{111}110 & & 00000000\mathbf{111}2
\end{array}
$$

Using the fact that y is strictly less than k, it is straightforward to check the overlap conditions of Section 5.1, and hence each g_k is a well defined involution.

Now, note that if $r(f) < k$ then $r(g_k f) = r(f) + k$. Hence, for all $k > m$ we have that $g_k C_m \subseteq C_{m+k}$. Hence

Claim 5.1. $\lim_k g_k C_m = \lim_k C_{m+k} = \{o\}$.

Proof of Theorem 2.1. Without loss of generality, assume $a = 0$. Let $C \subsetneq \Omega^0$ be closed. Then there exists a $g_0 \in G^*$ such that $g_0 C$ does not include o, by the minimality of the G^* action on Ω^0 (Theorem 2.4). Since $g_0 C$ is closed, it is disjoint from some neighborhood of o, and so it is contained in the finite union $\cup_{i=1}^m C_i$, for some m large enough. Let $\bar{g}_k = g_k g_0$, where g_k is as defined above. Then

$$\lim_k \bar{g}_k C = \lim_k g_k g_0 C \subseteq \lim_k g_k \cup_{i=1}^m C_i = \cup_{i=1}^m \lim_k g_k C_i = \{o\},$$

where the last equality follows from Claim 5.1. But the first limit cannot be an empty set, and so

$$\lim_k \bar{g}_k C = \{o\}.$$

□

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