Invariant random subgroups of semidirect products

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Abstract

We study invariant random subgroups (IRSs) of semidirect products \( G = A \rtimes \Gamma \). In particular, we characterize all IRSs of parabolic subgroups of \( \text{SL}_d(\mathbb{R}) \), and show that all ergodic IRSs of \( \mathbb{R}^d \rtimes \text{SL}_d(\mathbb{R}) \) are either of the form \( \mathbb{R}^d \rtimes K \) for some IRS of \( \text{SL}_d(\mathbb{R}) \), or are induced from IRSs of \( \Lambda \rtimes \text{SL}(\Lambda) \), where \( \Lambda < \mathbb{R}^d \) is a lattice.

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1 Introduction

Let \( G \) be a locally compact, second countable group and let \( \text{Sub}_G \) be the space of closed subgroups of \( G \), considered with the Chabauty topology [9].

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Definition 1. An invariant random subgroup (IRS) of $G$ is a random element of $\text{Sub}_G$ whose law is a conjugation invariant Borel probability measure.

The term IRS was introduced by Abért–Glasner–Virág in [2], but the mathematical object has been studied earlier by Vershik [19]. Examples of IRSs include normal subgroups, as well as random conjugates $g\Gamma g^{-1}$ of a lattice $\Gamma < G$, where the conjugate is chosen by selecting $\Gamma g$ randomly against the given finite measure on $\Gamma \setminus G$. More generally, any IRS of a lattice $\Lambda < G$ induces an IRS of $G$: if $\mu_\Gamma$ is the law of the original IRS and $\eta$ is a $G$-invariant probability measure on $\Gamma \setminus G$, the new law $\mu_G$ is given by the integral

$$\mu_G = \int_{\Gamma \subset \Gamma \setminus G} g_* \mu_\Gamma \, d\eta,$$

where $\mu_\Gamma$ is regarded as a measure on $\text{Sub}_\Gamma \subset \text{Sub}_G$, and $g$ acts on $\text{Sub}_G$ by conjugation. Informally, we conjugate the IRS of $\Gamma$ by an '$\eta$-random' element of $G$. Since $\text{Sub}_G$ is compact [4, Lemma E.1.1], the space of (conjugation invariant) Borel probability measures on $\text{Sub}_G$ is weak* compact, by Riesz's representation theorem and Alaoglu's theorem. Hence, IRSs compactify the set of lattices in $G$. There is a growing literature on IRSs (see, e.g., [3, 5, 6, 8, 17]) and their applications, see especially [1, 7, 12, 18].

Our goal in this note is to develop an understanding of IRSs of semidirect products $G = A \rtimes \Gamma$. There are few general constructions of such IRSs: there is the trivial IRS $\{e\}$, and IRSs of the form $A \rtimes K$, where $K$ is an IRS of $\Gamma$. When the kernel $\Gamma_{\text{triv}}$ of the action $\Gamma \rtimes A$ is nontrivial, one can also construct IRSs of the form $H \rtimes K$, where $H$ is an IRS of $A$ and $K$ is an IRS of $\Gamma$ that lies in $\Gamma_{\text{triv}}$, but additional examples are hard to find.

The kernel of our work are Theorems 2.5 and 2.6, in which we study ‘transverse’ IRSs of $G = A \rtimes \Gamma$ when $A$ is torsion-free abelian or simply connected nilpotent. Here, an IRS $H < G$ is transverse if $H \cap A = \{0\}$. This theorem has two parts: when $A$ is torsion-free abelian, we prove that that the projection of $H$ to $\Gamma$ acts trivially on $A$ almost surely, and if $A$ is a simply connected nilpotent Lie group, we show that an (often large) subgroup of $\Gamma$ acts precompactly on the Zariski closure of the set of all first coordinates of elements $(v, M) \in H$, as $H$ ranges through the support of the IRS.

As applications of Theorems 2.5 and 2.6, we study IRSs of two familiar semidirect products: the special affine groups $\mathbb{R}^d \rtimes \text{SL}_d(\mathbb{R})$ and the parabolic subgroups of $\text{SL}_d(\mathbb{R})$.

1.1 IRSs of special affine groups

We are particularly interested in IRSs of $\mathbb{R}^d \rtimes \text{SL}_d(\mathbb{R})$. In addition to the examples $\{e\}$ and $\mathbb{R}^d \rtimes K$ mentioned above, one can construct an IRS from a lattice $\Lambda \subset \mathbb{R}^d$. Namely, the subgroup $\text{SL}(\Lambda) < \text{SL}_d(\mathbb{R})$ stabilizing $\Lambda$ is also a lattice, see [15], so the semidirect product $\Lambda \rtimes \text{SL}(\Lambda)$ is a lattice in $\mathbb{R}^d \rtimes \text{SL}_d(\mathbb{R})$, and hence a random conjugate of it is an IRS.

Theorem 1.1. Let $H$ be a non-trivial ergodic IRS of $\mathbb{R}^d \rtimes \text{SL}_d(\mathbb{R})$. Then either

1. $H = \mathbb{R}^d \rtimes K$ for some IRS $K < \text{SL}_d(\mathbb{R})$, or
2. $H$ is induced from an IRS of $\Lambda \rtimes \text{SL}(\Lambda)$, for some lattice $\Lambda < \mathbb{R}^d$.

Here, an IRS is **ergodic** if its law is an ergodic measure for the conjugation action of $G$ on $\text{Sub}_G$. By Choquet’s theorem [14], every IRS can be written as an integral of ergodic IRSs. Note that by transitivity of the action of $\text{SL}_d(\mathbb{R})$ on the space of lattices of a fixed covolume, we can actually choose $\Lambda$ in 2. to be a scalar multiple of $\mathbb{Z}^d$.

As a corollary, any normal subgroup of $\mathbb{R}^d \rtimes \text{SL}_d(\mathbb{R})$ is of the form $\mathbb{R}^d \rtimes K$ where $K$ is a normal subgroup of $\text{SL}_d(\mathbb{R})$. (Here, $K = \{e\}$, $\text{SL}_d(\mathbb{R})$ or $\{\pm I\}$, where the last option is only available when $d$ is even.) Similarly, it follows that every lattice of $\mathbb{R}^d \rtimes \text{SL}_d(\mathbb{R})$ is a finite index subgroup of some $\Lambda \rtimes \text{SL}(\Lambda)$. We expect that these results are not entirely surprising, although we note that Theorem 4.8 of [10] is that $\mathbb{R}^d \rtimes \text{SL}_d(\mathbb{R})$ has no uniform lattices, which follows trivially from this classification.

Stuck–Zimmer [16] show that for $d > 2$, every ergodic IRS of $\text{SL}_d(\mathbb{R})$ is either a lattice or a normal subgroup. This result, together with Theorem 1.1, implies that for $d > 2$ every ergodic IRS of $\mathbb{R}^d \rtimes \text{SL}_d(\mathbb{R})$ is likewise either a lattice or a normal subgroup.

In light of Theorem 1.1, to understand IRSs in special affine groups it suffices to study those of $G = \mathbb{Z}^d \rtimes \text{SL}_d(\mathbb{Z})$. There are the usual examples $\{e\}$ and $\mathbb{Z}^d \rtimes K$, where $K$ is an IRS of $\text{SL}_d(\mathbb{Z})$, but in general, some subtle finite group theory appears. For instance, let

$$\pi_n : G \rightarrow (\mathbb{Z}/n\mathbb{Z})^d \rtimes \text{SL}_d(\mathbb{Z}/n\mathbb{Z})$$

be the reduction map and setting $d = 2$, consider the subgroup

$$H = \left\{ \left( (t, 0), (\frac{1}{11})^t \right) \bigg| t \in \mathbb{Z}/n\mathbb{Z} \right\} < (\mathbb{Z}/n\mathbb{Z})^d \rtimes \text{SL}_d(\mathbb{Z}/n\mathbb{Z}).$$

The preimage $\pi_n^{-1}(H)$ is a finite index subgroup of $G$, and therefore can be considered as an IRS, but it does not have the form $\Lambda \rtimes K$ for any $\Lambda < \mathbb{Z}^d, K < \text{SL}_d(\mathbb{Z})$. However, we will show that all IRSs of $G$ are semidirect products up to some ‘finite index noise’. Namely, let

$$G_n = \text{Ker} \pi_n = n\mathbb{Z}^d \rtimes \Gamma(n),$$

where $\Gamma(n)$ is the kernel of the reduction map $\text{SL}_d(\mathbb{Z}) \rightarrow \text{SL}_d(\mathbb{Z}/n\mathbb{Z})$. We prove:

**Theorem 1.2.** Let $H$ be a non-trivial ergodic IRS of $\mathbb{Z}^d \rtimes \text{SL}_d(\mathbb{Z})$. Then there is some $n \in \mathbb{N}$ such that $H_n = H \cap G_n$ is of the form $n\mathbb{Z}^d \rtimes K$, where $K$ is an IRS of $\text{SL}_d(\mathbb{Z})$.

### 1.2 IRSs of parabolic subgroups of $\text{SL}_d(\mathbb{R})$

Suppose that $W = \mathbb{R}^d$ is a finite dimensional real vector space, written as a direct sum

$$W = S_1 \oplus \cdots \oplus S_n$$

of subspaces, and that $\mathcal{F}$ is the associated flag

$$0 = W_0 < W_1 < \cdots < W_n = W, \quad W_k = \oplus_{i=1}^k S_i.$$
Let $P < SL(W)$ be the corresponding parabolic subgroup, i.e. the stabilizer of the flag $\mathcal{F}$, and let $V < P$ be the associated unipotent subgroup, consisting of all $A \in P$ that act trivially on each of the factors $W_i/W_{i-1}$. We then have

$$P = V \rtimes R, \quad R = \left\{ (A_1, \ldots, A_n) \in \prod_{i=1}^n GL(S_i) \mid \prod_i \det A_i = 1 \right\}.$$  

Elements of $P$ can be considered as upper triangular $n \times n$-matrices, where the $ij^{th}$ entry is an element of $\mathcal{L}(S_i, S_j)$, the vector space of linear maps $S_i \rightarrow S_j$. Elements of $R$ are diagonal matrices, and elements of $V$ are upper unitriangular.

Take a subset $E \subset \{1, \ldots, n\}^2$ consisting of pairs $(i, j)$ with $i < j$ and such that if $(i, j) \in E$, then $(i', j), (i, j') \in E$ for $i' < i$ and $j' > j$. So, imagining elements of $E$ as corresponding to matrix entries, we are considering subsets of entries above the diagonal, that are closed under 'going up' and 'going to the right'. Let $V_E < P$ be the normal subgroup consisting of all matrices that are equal to the identity matrix except at entries corresponding to elements of $E$, and let $K_E < R$ be the kernel of the $R$-action (by conjugation) on $V/V_E$.

**Theorem 1.3** (IRSs of parabolic subgroups). The ergodic IRSs of $P$ are exactly the random subgroups of the form $V_E \rtimes K$, where $K$ is an ergodic IRS of $K_E$.

The subgroups $V_E$ above are exactly the normal subgroups of $P$ that lie in $V$. So, a special case of the theorem is that an ergodic IRS of $P$ that is contained in $V$ is a normal subgroup of $P$. In fact, when proving Theorem 1.3, one first proves this special case, and then applies it to $H \cap V$ when $H$ is a general ergodic IRS of $P$. Once one knows $H \cap V = V_E$, the statement of Theorem 1.3 is not a surprise, since the only obvious way to construct an IRS $H$ with $H \cap V = V_E$ is to take a semidirect product with an IRS of $K_E$.

The group $K_E$ can be described explicitly via matrices. Let $\mathcal{J}$ be the set of all $i \in \{1, \ldots, n\}$ such that if $i < n$, then $(i, i+1) \in E$, and if $i > 1$, then $(i-1, i) \in E$. Then $(A_1, \ldots, A_n)$ acts trivially on $V/V_E$ exactly when for each maximal interval $\{i, \ldots, j\} \subset \{1, \ldots, n\} \setminus \mathcal{J}$, there is some $\lambda \in \mathbb{R} \setminus \{0\}$ such that $A_i = \cdots = A_j = \lambda I$. In a picture, if $E$ consists of the starred entries below, then $(A_1, \ldots, A_n) \in K_E$ can be any diagonal matrix with the diagonal entries below, subject to the additional condition $\prod_i \det A_i = 1$.

\[
\begin{pmatrix}
\lambda I & 0 & * & * & * & * & * \\
0 & \lambda I & * & * & * & * & * \\
0 & 0 & A_3 & * & * & * & * \\
0 & 0 & 0 & \mu I & 0 & 0 & * \\
0 & 0 & 0 & 0 & \mu I & 0 & * \\
0 & 0 & 0 & 0 & 0 & \mu I & * \\
0 & 0 & 0 & 0 & 0 & 0 & \mu I \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_8
\end{pmatrix} \quad (1)
\]

This means that $K_E$ is isomorphic to the quotient by a determinant condition of a direct product of general linear groups, some which are copies of $GL(\mathbb{R}) \cong \mathbb{R} \setminus \{0\}$. Note that the
conjugation action of every element of \( R \) on \( \mathcal{K}_\xi \) is equal to a conjugation by an element of \( \mathcal{K}_\xi \), since \( R \) is generated by \( \mathcal{K}_\xi \) and its centralizer. So, every IRS of \( \mathcal{K}_\xi \) is an IRS of \( R \).

1.3 Plan of the paper

The paper is organized as follows. In §2, we establish some preliminary results: we introduce in §2.1 a useful co-cycle associated to an IRS in \( A \rtimes \Gamma \), prove two facts about finite measure preserving linear actions in §2.2, and prove the result about transverse IRSs in §2.3. Section 3 concerns IRSs of parabolic subgroups, and in §4 we prove Theorems 1.1 and 1.2.

2 IRSs in general semidirect products

In this section we study semidirect products \( G = A \rtimes \Gamma \), where \( \Gamma \) acts on \( A \) by automorphisms. As above, \( \text{pr} \) is the natural projection \( G \to \Gamma \).

2.1 The cocycle \( S_H \)

Let \( H \) be a subgroup of \( G \). For each \( M \in \text{pr}H \) let
\[
S_H(M) = \{ v \in A : (v, M) \in H \}.
\]
Then \( S_H(I) = H \cap A \) is a subgroup of \( A \) where \( I \in \Gamma \) denotes the identity element.

Let \( (v, M), (w, N) \in H \). Then \( (v, M)(w, N) = (v \cdot Mw, MN) \in H \). It follows that
\[
S_H(MN) = S_H(M) \cdot MS_H(N), \tag{2}
\]
where multiplication here denotes that of sets: \( B \cdot C = \{ b \cdot c : b \in B, c \in C \} \).

Claim 2.1. The image of \( S_H \) is in \( A/S_H(I) \), and \( S_H : \text{pr}H \to A/S_H(I) \) is a cocycle.

Proof. By (2), it suffices to show that the image of \( S_H \) is in \( A/S_H(I) \).

If we substitute \( M = I \) into (2) then we get
\[
S_H(N) = S_H(I) \cdot S_H(N),
\]
and so each \( S_H(N) \) must be a union of cosets of the group \( S_H(I) \). Substituting \( N = M^{-1} \) into (2) yields
\[
S_H(I) = S_H(M) \cdot MS_H(M^{-1}).
\]
Hence \( S_H(M) \) and \( MS_H(M^{-1}) \) are cosets of \( S_H(I) \) in \( A \) — i.e. elements of \( A/S_H(I) \) — with
\[
S_H(M)^{-1} = MS_H(M^{-1}). \tag{3}
\]

We end this section with a useful observation. As we will apply it only when \( A \) is abelian, we use additive notation here. Let \( (w, N) \) be an arbitrary element of \( G \), and let \( (v, M) \in H \). Then \( (v, M)^{(w, N)} = (N^{-1}v + N^{-1}(M - I)w, MN) \in H^{(w, N)} \). (Here, \( a^b = b^{-1}ab \).) Hence
\[
S_{H(w, N)}(MN) = N^{-1}S_H(M) + N^{-1}(M - I)w. \tag{3}
\]
2.2 Group actions preserving finite measures

Here are three brief lemmas we will need in the next section.

**Lemma 2.2.** Suppose a group $Z$ acts linearly on $\mathbb{R}^d$ preserving a finite measure $m$, and $V = \text{Span}(\text{supp } m)$. Then the image of the map $Z \to \text{GL}(V)$ is precompact.

The proof is similar to an argument of Furstenberg used in his proof of the Borel density theorem [11, Lemma 3].

*Proof.* Restricting, it suffices to prove the lemma when $\text{Span}(\text{supp } m) = \mathbb{R}^d$. Let $(z_n)$ be a sequence in $Z$. After passing to a subsequence, we can assume that there is some subspace $W \subset \mathbb{R}^d$ such that the maps $z_n|_W$ converge to some linear map $z : W \to \mathbb{R}^d$, while $z_n(x) \to \infty$ if $x \in \mathbb{R}^d \setminus W$. For instance, one can take $W$ to be any subspace that is maximal among those for which there exists a subsequence $(z_{n_k})$ with the property that $z_{n_k}(x)$ is bounded for all $x \in W$, and then pass to a subsequence of such a subsequence.

If in the above, we always have $W = \mathbb{R}^d$, we are done. So, assume $W \neq \mathbb{R}^d$. Pick a metric inducing the one-point compactification topology on $\mathbb{R}^d \cup \infty$ and let $D : \mathbb{R}^d \cup \infty \to \mathbb{R}$ be the distance to the closed set $z(W) \cup \infty$. By the dominated convergence theorem,

$$\int D(x) \, dm(x) = \int D(z_n(x)) \, dm(x) \to 0,$$

so $m$ is supported on $z(W)$. But as $W$ is a proper subspace, so is $z(W)$. This contradicts our assumption that $\text{Span}(\text{supp } m) = \mathbb{R}^d$. \(\blacksquare\)

**Lemma 2.3.** Suppose that $G$ is a locally compact second countable group, and the induced action of $Z \leq \text{Aut}(G)$ on the space $\text{Sub}_G$ preserves a finite measure $\mu$ that is supported on lattices. Then $Z$ preserves the Haar measure of $G$.

*Proof.* For some $n$, the set $S$ of lattices with covolume in $[\frac{1}{n}, n]$ has positive measure. If $Z$ does not preserve Haar measure $\nu$, there is some $A \in Z$ with $A_* \nu = c \nu$ with $c > n^2$. The sets $A^i S$, where $i \in \mathbb{Z}$, are then all disjoint and have the same positive measure. This is a contradiction. \(\blacksquare\)

**Lemma 2.4.** Suppose that $\mathbb{R}^d = \bigoplus_i \mathcal{L}_i$, a direct sum of subspaces, and that $\mu$ is a finite Borel measure on the Grassmannian of $k$-dimensional subspaces of $\mathbb{R}^d$. Suppose that for each $j$, there is a linear map $A_j : \mathbb{R}^d \to \mathbb{R}^d$ that acts as a scalar map $v \mapsto \lambda_i v$ on each subspace $\mathcal{L}_i$, satisfies $\lambda_j > \lambda_i$ for $i \neq j$, and induces a map on the Grassmannian that preserves $\mu$. Then $\mu$ is concentrated on subspaces that are direct sums of the $\mathcal{L}_i$.

*Proof.* The argument is similar to that of Lemma 2.2. Under iteration by $A_j$, every $k$-dimensional subspace $P$ of $\mathbb{R}^d$ converges to a subspace of the form $\mathcal{L}_j \oplus P'$, where $P' \subset \bigoplus_{i \neq j} \mathcal{L}_i$. Applying the dominated convergence theorem, it follows that $\mu$ is concentrated on such subspaces. This works for all $j$, so the lemma follows. \(\blacksquare\)
2.3 Transverse IRSs

Let $A$ and $\Gamma$ be locally compact, second countable topological groups, and suppose $\Gamma$ acts by continuous automorphisms on $A$. Let $\Gamma_{\text{triv}}$ be the kernel of the action, and let $G = A \rtimes \Gamma$ be the associated semidirect product.

We call a subgroup $H \leq G$ transverse if $H \cap A = \{0\}$. For example, in the direct product $A \times A$, the diagonal subgroup is transverse, as is the second factor.

**Theorem 2.5** (Structure of transverse IRSs in semidirect products, part 1). Suppose $G = \mathbb{R}^d \rtimes \Gamma$ and $H$ is a transverse IRS of $G = \mathbb{R}^d \rtimes \Gamma$. Then $\text{pr} \, H \leq \Gamma_{\text{triv}}$ almost surely.

**Remark 1.** Theorem 2.5 also applies when $G = S \rtimes \Gamma$ and $S$ is a closed subgroup of $\mathbb{R}^d$. Indeed, the $\Gamma$-action on such an $S$ extends to the span of $S$ to which Theorem 2.5 applies, and any transverse IRS of $G = S \rtimes \Gamma$ induces a transverse IRS of $G = \text{span}(S) \rtimes \Gamma$.

**Remark 2.** If the action $\Gamma \circ A$ is faithful (as it is, for example, in the case of the special affine groups), then Theorem 2.5 implies there are no nontrivial transverse IRSs of $G$. Also, note that the theorem fails when $A$ is not torsion-free abelian. For instance, if $A$ is finite then a random conjugate of $\Gamma$ is an IRS of $A \rtimes \Gamma$. And if $A$ is not abelian, the antidiagonal

$$\{(g, g^{-1}) \mid g \in A\} \subset A \rtimes A,$$

where $a \in A$ acts on $x \in A$ by $a(x) = a^{-1}xa$, is a normal subgroup of $A \rtimes A$ that does not project into $A_{\text{triv}} = Z(A)$. However, we expect that for general $A$, if $H$ is a transverse IRS of $A \rtimes \Gamma$, then the action of any element of $\text{pr} \, H$ on $A$ is well-approximated by inner automorphisms of $A$ in some sense.

**Proof of Theorem 2.5.** Let $H$ be a nontrivial transverse IRS of $G$. In order to get a contradiction, suppose that it is not the case that $\text{pr} \, H \leq \Gamma_{\text{triv}}$ almost surely. Then there is an open subset $U \subset \Gamma$ with compact closure such that $U \cap \Gamma_{\text{triv}} = \emptyset$, and $\text{pr} \, H \cap U \neq \emptyset$ with positive probability. In addition we choose $U$ small enough so that for some $w \in \mathbb{R}^d$, some $0 < b_1 < b_2 \in \mathbb{R}_+$ and some linear $L : \mathbb{R}^d \rightarrow \mathbb{R}$, we have that

$$b_1 \leq L((M - I)w) \leq b_2, \text{ for all } M \in U. \quad (4)$$

Choose a left Haar measure $\mu_H$ on $\text{pr} \, H$. By [5, Claim A.2], this can be done so that the $\mu_H$ vary continuously with $H \in \text{Sub}_G$, when regarded as measures on $\Gamma \geq \text{pr} \, H$.

Because $H$ is transverse, $S_H(M)$ is a single element of $\mathbb{R}^d$ for any $M \in \text{pr} \, H$. Selecting first a random $H \in \text{Sub}_G$ with $\text{pr} \, H \cap U \neq \emptyset$, and then a $\mu_H$-random $M \in \text{pr} \, H \cap U$, we can interpret the cocycle $S_H(M)$ as a $\mathbb{R}^d$-valued random variable. Here, note that $\mu_H(\text{pr} \, H \cap U)$ is always finite and nonzero, since $\text{pr} \, H \cap U$ is nonempty, pre-compact and open in $H$.

Taking $w \in \mathbb{R}^d$ as in the first paragraph of the proof, let $H^w = (w, I)^{-1}H(w, I)$. Since $\text{pr} \, H = \text{pr} \, H^w$, we get a map $(H, M) \mapsto (H^w, M)$ defined on the domain

$$\{(H, M) \mid H \in \text{Sub}_G, \text{pr} \, H \cap U \neq \emptyset, M \in \text{pr} \, H \cap U\} \quad (5)$$
of the random variable $S_H(M)$. As $H$ is an IRS, this map is measure preserving, so the distributions of $S_{Hw}(M)$ and $S_H(M)$ are equal, say to a probability measure $m_U$ on $\mathbb{R}^d$.

By (3), we have $S_{Hw}(M) = S_H(M) + (M - I)w$ for all $M \in \text{pr}H = \text{pr}H^w$. Iterating the conjugation by $w$ and using (4),

$$L(S_H(M)) + nb_1 \leq L(S_{Hw}(M)) \leq L(S_H(M)) + nb_2, \forall n \in \mathbb{N}. \quad (6)$$

This contradicts the fact that $m_U$ is a probability measure. For suppose $[a_1, a_2] \subset \mathbb{R}$ is an interval with $m_U(L^{-1}([a_1, a_2])) > 0$. For a sufficiently sparse sequence $n_k \in \mathbb{N}$, the intervals $[a_1 + n_k b_1, a_2 + n_k b_2] \subset \mathbb{R}$ are all disjoint. Hence,

$$1 \geq \sum_k m_U(L^{-1}[a_1 + n_k b_1, a_2 + n_k b_2]) \geq \sum_k m_U(L^{-1}[a_1, a_2]) = \infty.$$

This contradiction proves the theorem.

\section*{Theorem 2.6 (Structure of transverse IRSs in semidirect products, part 2)}

Suppose $G = A \rtimes \Gamma$, $A$ is a simply connected nilpotent Lie group, $H$ is a transverse IRS of $G = A \rtimes \Gamma$ and $\lambda$ is the law of $H$. Let

$$\mathcal{H} = \cup_{H \in \text{supp} \lambda} H.$$

If $\mathcal{V} \subset A$ is the Zariski closure of the set of first coordinates of all $(v, M) \in \mathcal{H}$, then $\mathcal{V}$ is $\Gamma$-invariant and the image of the map $Z(\text{pr} \mathcal{H}) \longrightarrow \text{Aut}(\mathcal{V})$ is precompact.

Here $Z(\text{pr} \mathcal{H})$ denotes the centralizer of $\text{pr} \mathcal{H}$ in $\Gamma$, and the Zariski closure of a subset of $A$ is the smallest connected Lie subgroup of $A$ containing that subset.

\begin{remark}
Theorem 2.6 also applies when $G = S \rtimes \Gamma$ and $S$ is a closed subgroup of some simply connected nilpotent Lie group $A$. Indeed, the $\Gamma$-action on such an $S$ extends to the Zariski closure $\overline{S}$ [15, Theorem 2.11], to which Theorem 2.6 applies, and any transverse IRS of $G = S \rtimes \Gamma$ induces a transverse IRS of $G = \overline{S} \rtimes \Gamma$. See [15, Chapter II] for more information about the ‘Zariski closure’ operation in simply connected nilpotent Lie groups, which behaves very similarly to ‘span’ in $\mathbb{R}^d$.
\end{remark}

\begin{remark}
To illustrate Theorem 2.6, suppose $A = \Gamma = \mathbb{R}^2$ and $(s, t) \in \Gamma$ acts by a rotation on $A$ with angle $s$. Then if

$$H_{\theta} = \left\{ ((t \cos \theta, t \sin \theta), (0, t)) \mid t \in \mathbb{R} \right\} \leq A \rtimes \Gamma,$$

we obtain a transverse IRS of $G = A \rtimes \Gamma$ by randomly picking $\theta \in [0, 2\pi]$ against Lebesgue measure. Here, the centralizer $Z(\text{pr} \mathcal{H})$ is all of $\Gamma$, which acts compactly on $A$.
\end{remark}

\begin{proof}[Proof of Theorem 2.6] The $\Gamma$-invariance of $\mathcal{V}$ is immediate. For if $N \in \Gamma$ and $(v, M) \in \mathcal{H}$,

$$(e, N)^{-1}(v, M)(e, N) = (N^{-1}v, N^{-1}MN). \quad (7)$$

\end{proof}
Here, we write $e$ for the identity element since $A$ is not necessarily abelian. As $\text{supp} \lambda$ is conjugation invariant, the set of all $v \in A$ such that $(v, M) \in \mathcal{H}$ for some $M$ is $\Gamma$-invariant. Hence, its Zariski closure $V$ is also $\Gamma$-invariant.

As in the proof of Theorem 2.5, choose $U \subset \Gamma$ with compact closure such that $\text{pr} H \cap U \neq \emptyset$ with positive probability. Let $N \in Z(\text{pr} \mathcal{H})$ and write $H^N = (e, N)^{-1} H(e, N)$. Substituting $N^{-1}MN = M$ in (7) we see that $\text{pr} H = \text{pr} H^N$, so as before the distribution of $S_{H^N}(M)$ is the same as $m_U$, the distribution of $S_H(M)$. Now, though, (7) implies that $S_{H^N}(M) = N^{-1}(S_H(M))$. So, the measure $m_U$ on $A$ is $Z(\text{pr} \mathcal{H})$-invariant.

Since $A$ is a simply connected nilpotent Lie group, there is a diffeomorphism $\text{log} : A \to \mathfrak{a}$ to the Lie algebra $\mathfrak{a}$ that is an inverse for the Lie group exponential map [13, 1.127]. Then $\text{log}^* m_U$ is a probability measure on $\mathfrak{a}$ that is invariant under the induced action of $Z(\text{pr} \mathcal{H})$ on $\mathfrak{a}$. By Lemma 2.2, $Z(\text{pr} \mathcal{H})$ acts precompactly on the span $V_U = \text{Span}(\text{supp} \text{log}^* m_U)$, and therefore it acts precompactly on the sum $V$ of all $V_U$, as $U$ ranges over all possible choices. But the Zariski closure $V = \exp(V)$, so then $Z(\text{pr} \mathcal{H})$ acts precompactly on $V$ as well. □

We present an easy corollary of Theorem 2.5:

**Corollary 2.7.** The only ergodic IRSs of the affine group $\mathbb{R} \rtimes \mathbb{R}^+$ are the point masses on its closed, normal subgroups: $\{e\}, \mathbb{R}, \mathbb{R} \rtimes \mathbb{R}^+$ and $\mathbb{R} \rtimes \{\alpha^n \mid n \in \mathbb{Z}\}$, where $\alpha > 0$.

Note that this stands in contrast to other metabelian groups (e.g., lamplighter groups) that have a rich set of invariant random subgroups [8].

**Proof of Corollary 2.7.** Let $H$ be a non-trivial ergodic IRS of $\mathbb{R} \rtimes \mathbb{R}^+$. If $H$ is transverse, then $\text{pr} H = \{1\} \in \mathbb{R}^+$, by Theorem 2.5. Hence $H = \{e\}$.

Otherwise, the random subgroup $H \cap \mathbb{R} \subset \mathbb{R}$ is nontrivial almost surely, and its law is invariant under the $\mathbb{R}^+$ action (i.e., multiplication by a scalar). So, $H \cap \mathbb{R} = \mathbb{R}$ almost surely, and $H = \mathbb{R} \rtimes \text{pr} H$. But $\text{pr} H$ is an ergodic IRS of $\mathbb{R}^+$, and thus must be a point mass on either $\{1\}, \mathbb{R}^+$ or $\mathbb{R} \rtimes \{\alpha^n \mid n \in \mathbb{Z}\}$, where $\alpha > 0$. We have thus proved the claim. □

### 3 IRSs of parabolic subgroups

To recap our notation: $W = S_1 \oplus \cdots \oplus S_n$ is a real vector space, $\mathcal{F}$ is the associated flag

$$0 = W_0 < W_1 < \cdots < W_n = W, \quad W_k = \oplus_{i=1}^k S_i,$$

$P < \text{SL}(W)$ is the parabolic subgroup stabilizing $\mathcal{F}$, $V < P$ is the unipotent subgroup of all $A \in P$ that act trivially on each of the factors $W_i/W_{i-1}$, and

$$P = V \rtimes R, \quad R = \left\{ (A_1, \ldots, A_n) \in \prod_{i=1}^n \text{GL}(S_i) \mid \prod_i \det A_i = 1 \right\}.$$
Also, $\mathcal{E} \subset \{1, \ldots, n\}^2$ will denote a subset of pairs $(i, j)$ with $i < j$ that is closed under ‘going up’ and ‘going to the right’, and we will let $V_\mathcal{E} < P$ be the normal subgroup consisting of all matrices that are equal to the identity matrix except at entries corresponding to elements of $\mathcal{E}$. Let $\mathcal{K}_\mathcal{E} < R$ be the kernel of the $R$-action (by conjugation) on $V/V_\mathcal{E}$.

The goal of this section is to prove Theorem 1.3, i.e. that the ergodic IRSs of $P$ are exactly the random subgroups of the form $V_\mathcal{E} \rtimes K$, where $K$ is an ergodic IRS of $\mathcal{K}_\mathcal{E}$.

We start with the following lemma.

**Lemma 3.1.** Suppose that $H$ is an invariant random subgroup of $P$ that lies in $V$. Then almost surely, $H = V_\mathcal{E}$ for some $\mathcal{E}$.

**Proof.** Regard $V$ as the space of upper unitriangular block matrices, where the $ij$th entries is in $\mathcal{L}(S_i, S_j)$. It suffices to show that almost surely, $H$ is a ‘matrix entry subgroup’, i.e. a subgroup determined by prescribing that some fixed subset of the matrix entries are all zero. As there are only finitely many such subgroups, it will follow that almost surely, $H$ is a matrix entry subgroup of $V$ that is a normal subgroup of $P$. A quick computation with elementary matrices shows that the only such subgroups are the $V_\mathcal{E}$ described above.

Let $H_0$ and $\overline{H}$ be the identity component and Zariski closure of $H$, respectively, recalling that the *Zariski closure* of a subgroup is the smallest connected Lie subgroup of $V$ containing it. (See [15, Chapter II].) Then $H_0$ and $\overline{H}$ are both $R$-invariant random subgroups of $V$.

Let $\mathfrak{h}_0$ and $\overline{\mathfrak{h}}$ be the associated Lie algebras, which are $R$-invariant random subspaces of the Lie algebra $\mathfrak{v}$ of $V$. One can identify $\mathfrak{v}$ with the set of all strictly upper triangular block matrices, where the $ij$th entry is an element of $\mathcal{L}(S_i, S_j)$. If we identify $\mathcal{L}(S_i, S_j)$ with the subspace of $\mathfrak{v}$ consisting of matrices that are nonzero at most in the $ij$th entry, then

$$
\mathfrak{v} = \oplus_{i < j} \mathcal{L}(S_i, S_j).
$$

The action $R \circ \mathfrak{v}$ leaves all the factors $\mathcal{L}(S_i, S_j)$ invariant. Moreover, if $k < l$ the matrix in $R$ that has a $2I$ in the $kk$th entry and a $\frac{1}{2}I$ in the $ll$th entry (and is otherwise equal to the identity matrix) acts as a scalar matrix $\lambda I$ on each $\mathcal{L}(S_i, S_j)$, where

$$
\lambda = \begin{cases} 
4 & (i, j) = (k, l) \\
2 & i = k, j \neq l \text{ or } j = l, i \neq k \\
\frac{1}{2} & i = l \text{ or } j = k, \text{ and } i \neq j \\
1 & \text{otherwise.}
\end{cases}
$$

(8)

So, by Lemma 2.4, almost surely both $\mathfrak{h}_0$ and $\overline{\mathfrak{h}}$ are direct sums of some of the factors $\mathcal{L}(S_i, S_j)$, i.e. the groups $H_0$ and $\overline{H}$ are matrix entry subgroups. If they are the same, we are done since then $H = H_0 = \overline{H}$ is a matrix entry subgroup.

Passing to a positive measure $R$-invariant subset, we may thus assume that almost surely $H_0$ and $\overline{H}$ are fixed matrix entry subgroups and that $H_0 \subsetneq H$. As $H$ is an IRS of $P$, $H_0$ is a normal subgroup of $P$. We can then project $H$ to a $P$-invariant random lattice of the quotient group $\overline{H}/H_0$. Lemma 2.3 implies that the $P$ action on $\overline{H}/H_0$ preserves Haar
measure. But if $\mathcal{D}$ is the set of matrix entries that are free to take on any value in $\overline{H}$ and prescribed to be zero in $H_0$, there is a diffeomorphism

$$\overline{H}/H_0 \longrightarrow \oplus_{(i,j) \in \mathcal{D}} \mathcal{L}(S_i, S_j)$$

that takes a matrix in $\overline{H}$ to the list of its $\mathcal{D}$-entries. If Lebesgue measures are chosen on the Euclidean spaces $\mathcal{L}(S_i, S_j)$, the resulting product measure pulls back to a Haar measure on $\overline{H}/H_0$. So, one can witness that the action $R \circ \overline{H}/H_0$ does not preserve Haar measure as follows. Let $i_{\min}$ be the minimum $i$ such that there is some $(i, j) \in \mathcal{D}$, and $i_{\max}$ be the maximum $i$ such that there is some $(j, i) \in \mathcal{D}$, and define $A \in R$ by letting

$$A_{ii} = \begin{cases} 2I & i = i_{\min} \\ \frac{1}{2}I & i = i_{\max} \\ I & \text{otherwise.} \end{cases}$$

Then there are no entries of $\mathcal{D}$ directly above the $i_{\min}$ diagonal entry, and no entries to the right of the $i_{\max}$ diagonal entry. Hence, all eigenvalues of the (linear) action of $A$ on $\oplus_{(i,j) \in \mathcal{D}} \mathcal{L}(S_i, S_j)$ are equal to either 1 or 2, so $A$ cannot preserve Lebesgue measure.

Now suppose that $H$ is an ergodic IRS of $P = V \rtimes R$. Lemma 3.1 implies that there is some $\mathcal{E}$ such that $H \cap V = V_{\mathcal{E}}$ almost surely. Applying Theorem 2.5 to the transverse IRS that is the projection of $H$ to $(V/V_{\mathcal{E}})^{ab} \rtimes R$, where $(\cdot)^{ab}$ is abelianization, we see that $\text{pr} H \subset R$ almost surely acts trivially on $(V/V_{\mathcal{E}})^{ab}$. But if $\mathcal{A}$ is the set of super diagonal entries in our block matrices that do not lie in $\mathcal{E}$, there is an isomorphism

$$(V/V_{\mathcal{E}})^{ab} \longrightarrow \oplus_{(i,j) \in \mathcal{A}} \mathcal{L}(S_i, S_j)$$

that comes from taking a matrix in $V$ to its list of $\mathcal{A}$-entries. It follows that a matrix in $R$ acts trivially on $(V/V_{\mathcal{E}})^{ab}$ if and only if it acts trivially on $V/V_{\mathcal{E}}$: triviality of the $(V/V_{\mathcal{E}})^{ab}$-action is enough to force the conditions on diagonal entries indicated in the matrix (1) from the introduction. Hence, $\text{pr} H$ almost surely lies in the kernel $\mathcal{K}_{\mathcal{E}}$ of the $V/V_{\mathcal{E}}$-action as desired.

We now know that $H \cap V = V_{\mathcal{E}}$ and $\text{pr} H \subset \mathcal{K}_{\mathcal{E}}$ almost surely. We would like to conclude that $H$ has the form $V_{\mathcal{E}} \rtimes K$ for some IRS $K < K_{\mathcal{E}}$. Note that this is not immediately obvious—the diagonal in $\mathbb{R}^2$ is a normal subgroup that intersects the first factor trivially, but does not split as a product of subgroups of the two factors. By Theorem 2.6, we know that the centralizer $Z(\text{pr} H) \subset R$ acts precompactly on $\mathcal{X} \subset V/V_{\mathcal{E}}$, where $\mathcal{X}$ is the Zariski closure in $V/V_{\mathcal{E}}$ of the projections of all first coordinates of elements $(v, M) \in H$. If $\mathcal{X} = \{V_{\mathcal{E}}\}$, we are done, since then the first coordinates of all $(v, M) \in H$ lie in $V_{\mathcal{E}} = H \cap V$ and $H$ must have the form $V_{\mathcal{E}} \rtimes K$ for some IRS $K < \mathcal{K}_{\mathcal{E}}$.

So, we may assume that $\mathcal{X} V_{\mathcal{E}} \supseteq V_{\mathcal{E}}$. Picking a matrix $B$ in the difference, there is some entry $(i, j) \notin \mathcal{E}$ in which $B$ is nonzero. The centralizer $Z(\text{pr} H)$ contains all elements of $R$ all of whose diagonal entries are scalars, so in particular it contains the matrix whose eigenvalues $\lambda$ are listed in (8) above. The action of this matrix on $B$ scales the $(i, j)$ entry by 4, so $Z(\text{pr} H)$ does not act pre-compactly on $\mathcal{X}$, and we have a contradiction.
4 IRSs of special affine groups

Using Theorems 2.5 and 2.6, it is now fairly easy to prove the results on IRSs of special affine groups stated in the introduction.

**Proof of Theorem 1.1.** Let $H$ be a nontrivial ergodic IRS of $\mathbb{R}^d \rtimes \text{SL}_d(\mathbb{R})$. Suppose that $H \cap \mathbb{R}^d = \{0\}$ almost surely. As the action $\text{SL}_d(\mathbb{R}) \circlearrowleft \mathbb{R}^d$ is faithful, Theorem 2.5 implies that $H$ is trivial. So, $H \cap \mathbb{R}^d$ is almost surely some nontrivial subgroup of $\mathbb{R}^d$.

In order to prove $H \cap \mathbb{R}^d$ is either a lattice or $\mathbb{R}^d$, it suffices to prove that the Zariski closure of $H \cap \mathbb{R}^d$ is almost surely $\mathbb{R}^d$. If not, we get for some $1 \leq k \leq d-1$, a $\text{SL}_d(\mathbb{R})$-invariant probability measure on the Grassmannian of $k$-dimensional subspaces of $\mathbb{R}^d$. In the terminology of Furstenberg [11], $\text{SL}_d(\mathbb{R})$ is a m.a.p. group, so this measure must be concentrated on $\text{SL}_d(\mathbb{R})$-invariant points. (Apply [11, Lemma 3] to the $k$-th exterior power of $\mathbb{R}^d$.) However, no nontrivial subspaces of $\mathbb{R}^d$ are $\text{SL}_d(\mathbb{R})$-invariant.

Now suppose $H \cap \mathbb{R}^d$ is a lattice (almost surely). Let $\mu$ denote the law of $H$. By decomposing $\mu$ over the map $H \mapsto H \cap \mathbb{R}^d$, we can write $\mu = \int \mu_A \, d\nu(\Lambda)$ where $\nu$ is the pushforward of $\mu$ under $H \mapsto H \cap \mathbb{R}^d$ and $\mu_A$ is concentrated on the set of subgroups $H$ such that $H \cap \mathbb{R}^d = \Lambda$. By ergodicity $\nu$ is supported on the set of lattices of some fixed covolume $c > 0$. Moreover $\nu$ is $\text{SL}_d(\mathbb{R})$ invariant since the map $H \mapsto H \cap \mathbb{R}^d$ is equivariant. Since $\text{SL}_d(\mathbb{R})$ acts transitively on this set of lattices, it follows that $\nu$ must be the Haar measure.

By equivariance, we must have $\mu_{g\Lambda} = g_* \mu_A$ for $g \in \text{SL}_d(\mathbb{R})$ and $\nu$-a.e. $\Lambda$. Because $\text{SL}_d(\mathbb{R})$ acts transitively on the set of lattices with fixed covolume, we can assume without loss of generality that $\mu_{g\Lambda} = g_* \mu_A$ holds for every $g \in \text{SL}_d(\mathbb{R})$ and lattice $\Lambda$.

We claim that $\mu_{\Lambda}$-a.e. $H$ is contained in $\Lambda \rtimes \text{SL}(\Lambda)$. First let $(v, M) \in H$. For any $w \in \Lambda$ we have that $(w, I) \in H$, and so

$$(v, M)(w, I)(v, M)^{-1} = (Mw, I) \in H \cap \mathbb{R}^d = \Lambda.$$  

Because $w \in \Lambda$ is arbitrary, $M \in \text{SL}(\Lambda)$. Next observe that the law of $H$ is invariant under conjugation by $\Lambda \rtimes \text{SL}(\Lambda)$. So if there exists $M \in \text{SL}(\Lambda)$ such that $\text{SL}_d(M) \neq \Lambda$ with positive probability then $M\text{SL}_d(M)^{-1} \cap \mathbb{R}^d \neq \Lambda$ with positive probability. This contradiction shows that $\text{SL}_d(M) = \Lambda$ almost surely which implies $H \leq \Lambda \rtimes \text{SL}(\Lambda)$. Thus $\mu_{\Lambda}$ is the law of an IRS of $\Lambda \rtimes \text{SL}(\Lambda)$. This IRS must be ergodic because $\mu$ is ergodic.

**Proof of Theorem 1.2.** Let $H$ be a non-trivial, ergodic IRS of $G = \mathbb{Z}^d \rtimes \text{SL}_d(\mathbb{Z})$. Then $H \cap \mathbb{Z}^d$ is a random subgroup of $\mathbb{Z}^d$ whose law is invariant to the $\text{SL}_d(\mathbb{Z})$ action. Note that since the action $\text{SL}_d(\mathbb{Z}) \circlearrowleft \mathbb{Z}^d$ is faithful, Theorem 2.5 implies that $H \cap \mathbb{Z}^d \neq \{0\}$. Since there are only countably many subgroups of $\mathbb{Z}^d$, the distribution of $H \cap \mathbb{Z}^d$ must be concentrated on a single, finite $\text{SL}_d(\mathbb{Z})$-orbit. So, $H \cap \mathbb{Z}^d$ is almost surely finite index in $\mathbb{Z}^d$.

Let $O = \{M(H \cap \mathbb{Z}^d) \setminus \mathbb{Z} \setminus \mathbb{Z} : M \in \text{SL}_d(\mathbb{Z})\}$ be the orbit of $H \cap \mathbb{Z}^d$ under the $\text{SL}_d(\mathbb{Z})$ action. Now, the intersection of the groups in this orbit is also finite index in $\mathbb{Z}^d$, and is furthermore $\text{SL}_d(\mathbb{Z})$-invariant, and so must equal $n\mathbb{Z}^d$ for some $n \in \mathbb{N}$.

Recall that $G_n = (n\mathbb{Z}^d) \times \Gamma(n)$, and let $H_n = H \cap G_n$, a finite index subgroup of $H$. Using the cocycle notation of §2.1, for any $M \in \text{pr} \ H_n$ it holds that $S_H(M) = S_{H}(I) := H \cap \mathbb{Z}^d$, since
otherwise $S_H(M)$ is a non-trivial coset of $S_H(I)$, and its intersection with $n\mathbb{Z}^d$, a subgroup of $S_H(I)$, is trivial, thus excluding $M$ from $\text{pr } H_n$. It follows that $H_n = (n\mathbb{Z}^d) \rtimes (\text{pr } H_n)$. This completes the proof of Theorem 1.2.

\section*{References}


