Lecture Note on Random Choice for Summer School at the University of Tokyo 2017

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1 Introduction

Random Choice is

- related with many fields (Pyschology, Emprical IO, Math).
- an active topic in the decision theory literature.

Why choices can be random ?

- Consider a population of agents. From the view point of an outside observer, there is unobservable heterogeneity across agents. Because of the heterogeneity, we observe a distribution of choices in the population.
- Each agent is making a random choice because of random preferences or utilities. For example, intertemporal choice of the agent can be affected by many subjective elements such as the expectation of inflation, temptation, future consumption plan etc. Those subjective elements are unobservable so the choice of the agent looks random.

2 Model

Let X be a finite set. X is the set of outcomes. Let $\mathcal{D} \equiv 2^X \setminus \{\emptyset\}$.

Definition 1. *A function $\rho : \mathcal{D}\times X \to [0,1]$ is called random choice function if $\sum_{x\in D}\rho(D,x) = 1$. The set of random choice functions is denoted by $\mathcal{P}$.*

$\rho(D,x)$ is the probability that an element x is chosen from a choice set D. Let $T \equiv |\mathcal{D}| \times |X|$. Note that $|\mathcal{D}| = 2^{|X|} - 1$. A random choice function ρ is an element of $\mathbf{R}^T$.

2.1 Random Utility Model

Let V be the set of all functions $v : X \to \mathbf{R}$. Notice that v is $|X|$ dimensional real vector. Let $\mathcal{V}$ be the Borel algebra of V. Denote the finitely additive probability measure by $\Delta(V)$.

Definition 2. *A random choice function ρ is called random utility function if there exists a probability measure $\mu \in \Delta(\mathcal{V})$ such that for all $(D,x) \in \mathcal{D} \times X$,*

$$\rho(D,x) = \mu(v \in \mathcal{V} | v(x) \geq v(D)).$$

Remark 1. *Suppose that a random choice function ρ is random utility function with μ. Then for any $x, y \in X$ such that $\mu(v \in \mathcal{V}|v(x) = v(y)) = 0$.*

Remark 2. *By the finite additivity of μ and the finiteness of X, the above remark implies that the measure of utilities which allows tie is zero.*

2.2 Random Ranking Model

Let U be the set of bijection between $X \to \{1, \ldots, |X|\}$. If $u(x) = k$, I interpret that x is $|X| + 1 - k$ th best element of X with respect to u. So if $u(x) > u(y)$, then x is better than y with respect to u. For all $(D,x) \in \mathcal{D} \times X$ if $u(x) > u(y)$ for all $y \in D \setminus \{x\}$, I write $u(x) \geq u(D)$. There are $|X|!$ elements in U.

I denote the set of probability measures over U by $\Delta(U)$. Since U is finite, $\Delta(U) = \left\{(\nu_1, \ldots, \nu_{|U|}) \in \mathbf{R}_+^{|U|} \middle| \sum_{i=1}^{|U|} \nu_i = 1\right\}$.

Definition 3. *A random choice function ρ is called random ranking function if there exists a probability measure $\nu \in \Delta(U)$ such that for all $(D,x) \in \mathcal{D} \times X$*

$$\rho(D,x) = \nu(u \in U | u(x) \geq u(D)).$$

The set of random utility functions is denoted by $\mathcal{P}_r$.

Remark 3. *For any random choice function ρ, ρ is a random ranking function if and only if ρ is a random utility function*

2.3 Logit Model

Definition 4. *A random choice function ρ is called Luce function if there exists a function $w : X \to \mathbf{R}_+$ such that for all $(D, x) \in \mathcal{D} \times X$*

$$\rho(D, x) = \frac{w(x)}{\sum_{w \in D} w(y)}.$$

Definition 5. *A random choice function ρ is called logit function if there exists a function $v : X \to \mathbf{R}$ such that for all $(D, x) \in \mathcal{D} \times X$*

$$\rho(D, x) = \frac{e^{v(x)}}{\sum_{y \in D} e^{v(y)}}.$$

The set of logit functions is denoted by $\mathcal{P}_l$.

Remark 4. *ρ is a Luce function if and only if ρ is a logit function.*

Axiom 1. *ρ satisfies Independence from Irrelevant Alternative (IIA) if for any $D \in \mathcal{D}$ and $x, y \in D$ and $z \in X$,*

$$\frac{\rho(D, x)}{\rho(D, y)} = \frac{\rho(D \cup z, x)}{\rho(D \cup z, y)}.$$

Axiom 2. *ρ satisfies Positivity $\rho(D, x) > 0$ if for any $D \in \mathcal{D}$ and $x \in D$.*

Theorem 1. *ρ satisfies IIA and Positivity if and only if ρ is a logit function.*

Lemma 1. *If ρ is a logit function, then ρ is a random utility function. Moreover, ρ is a random ranking function with $\nu \in \Delta(U)$ such that $\nu(u) > 0$ and ν rationalizes ρ.*

Lemma 2. *If ρ is a random utility function with a random utility vector $v(x) + \varepsilon_x$, where ε_x is Type I extreme value distribution, for each $x \in X$, then ρ is a logit function with w.*

In empirical literature, an element of X is considered as a real vector, which is an explanatory variable for the element. For example in Berry et al. (1995), X consists of cars available in the market. Then for each car, $x \in X$ describes the car's price, weight, size, fuel efficiency, and etc.

Definition 6. *Let $X \subset \mathbf{R}^k$. A random choice function ρ is called linear logit function if there exists $\beta \in \mathbf{R}^k$ such that for all $(D, x) \in \mathcal{D} \times X$*

$$\rho(D, x) = \frac{e^{\beta \cdot x}}{\sum_{y \in D} e^{\beta \cdot y}}.$$

We denote ρ by ρ_l^{β}. The set of linear logit functions is denoted by $\mathcal{P}_{ll}$.

3 Axiomatization of Random Utility Model

There are two axomatizations. One is by McFadden and Richter (1990). The other is by Falmagne (1978).

3.1 Axiom of Revealed Stochastic Preference

McFadden and Richter (1990) provide the following axiomatization.

Definition 7. *For any sequence $(D_i, x_i)_{i=1}^n$*

$$B((D_i, x_i)_{i=1}^n, \rho) = \max_{u \in U} \sum_{i=1}^{n} 1(u(x_i) \geq u(D_i)) - \sum_{i=1}^{n} \rho(D_i, x_i).$$

Theorem 2. *For any $\rho \in \mathcal{P}$, ρ is a random utility function if and only if $B((D_i, x_i)_{i=1}^n, \rho) \geq 0$ for any sequence $(D_i, x_i)_{i=1}^n$.*

3.2 Axiomatization by Block-Marschak Polynomial

Axiom 3. *(Regularity) For any $D, E \in \mathcal{D}$ such that $x \in D \subset E$, $\rho(D, x) \geq \rho(E, x)$.*

Remark 5. *(i) A random utility function ρ satisfies Regularity. (ii) For any $D \in \mathcal{D}$ such that $x \in D$ and any $z, z' \in X \setminus D$,*

$$\rho(D, x) - \left[\rho(D \cup z, x) + \rho(D \cup z', x)\right] + \rho(D \cup \{z, z'\}, x) \geq 0$$

The following axiom generalize this idea by considering $\{z_i\} = X \setminus D$.

Definition 8. *For any $\rho \in \mathcal{P}$ and $(D, x) \in \mathcal{D} \times X$ such that $x \in D$,*

$$K(\rho, D, x) = \sum_{E : D \subset E} (-1)^{|E \setminus D|} \rho(E, x).$$

Falmagne (1978) shows that the following axiom characterizes random utility model.

Axiom 4. *For any* $(D, x) \in \mathcal{D} \times X$ *such that* $x \in D$, $K(\rho, D, x) \geq 0$.

Theorem 3. *(Falmagne (1978)) For any* $\rho \in \mathcal{P}$, ρ *is a random utility function if and only if* ρ *satisfies Axiom 4.*

4 Random Expected Utility Model by Gul and Pesendorfer (2006)

Let $N = \{1, 2, \ldots, n+1\}$ for $n \geq 1$. N is the set of outcomes. Let $P = \{x \in \mathbf{R}_+^{n+1} | \sum_{i=1}^{n+1} x^i = 1\}$. P is the set of lotteries. Let $\mathcal{B}$ be the Borel algebra of P.

Let $\mathcal{D}$ be the set of all finites subsets of P. $\mathcal{D}$ is the set of choice sets. Let Π be the set of all probability measures on the measurable space $(P, \mathcal{B})$.

For any $D \in \mathcal{D}$, $\rho(D)$ is a probability measure in the measurable space $(P, \mathcal{B})$ which gives nonnegative probability only on D.

For any $x \in P$ and $u \in \mathbf{R}^{n+1}$, I have $(u^1 - u^{n+1}, \ldots, u^n - u^{n+1}) \cdot x = \sum_{i=1}^{n+1} u^i x_i - u_{n+1}$.

Remark 6. $u \cdot x \geq u \cdot y$ *if and only if* $(u^1 - u^{n+1}, \ldots, u^n - u^{n+1}) \cdot x \geq (u^1 - u^{n+1}, \ldots, u^n - u^{n+1}) \cdot y$.

Hence, we can normalize the set of utilities as $U = \{u \in \mathbf{R}^{n+1} | u^{n+1} = 0\}$.

For any $D \in \mathcal{D}$ and $x \in D$, define

$$N(D, x) = \{u \in U | u \cdot x \geq u \cdot y \text{ for all } y \in D\},$$
$$N^+(D, x) = \{u \in U | u \cdot x > u \cdot y \text{ for all } y \in D \setminus x\}.$$

$N(D, x)$ is the set of utilities such that x is an optimal choice from D is x. $N^+(D, x)$ is the set of utilities such that x is the unique optimal choice from D.

Let $\mathcal{F}$ be the smallest algebra that contains $N(D, x)$ for all (D, x).

Definition 9. *A random utility function is a function* $\mu : \mathcal{F} \to [0, 1]$ *such that* $\mu(U) = 1$ *and*

$$\mu(F \cup F') = \mu(F) + \mu(F')$$

if $F \cap F' = \emptyset$ *and* $F, F' \in \mathcal{F}$. *The RUF is countably additive if* $\sum_{i=1}^{\infty} \mu(F_i) = \mu(\bigcup_{i=1}^{\infty} F_i)$ *whenever* F_i *is a countable collection of pairwise disjoint sets in* $\mathcal{F}$ *such that* $\bigcup_{i=1}^{\infty} F_i$.

Definition 10. *A random utility function μ is regular if in every decision problem with probability 1, the realized utility function has a unique maximizer. Formally, μ is regular if $\mu\Big(\bigcup_{x\in D} N^+(D,x)\Big) = 1$ for all $D \in \mathcal{D}$.*

In other words, a regular random utility function give zero probability to the utilities which allow ties (i.e., $u \in U$ such that $u \cdot x = u \cdot y$ for some $x, y \in X$).

Definition 11. *A random choice function ρ is called regular random expected utility function if there exists a regular random utility function μ such that $\rho^D(x) = \mu(N(D,x))$ for all $D \in \mathcal{D}$ and $x \in P$. We say ρ is countably additive if μ is countably additive.*

Theorem 4. *For a random choice function ρ, if there exist two regular random utility functions μ and μ' such that $\mu(N(D,x)) = \rho^D(x) = \mu'(N(D,x))$ for all $D \in \mathcal{D}$ and $x \in P$, then $\mu = \mu'$.*

For any $D, D' \in \mathcal{D}$, Hausdorff Distance between D and D' is

$$d_h(D,D') = \max\Big\{\max_{x\in D}\min_{x'\in D'} \|x - x'\|, \max_{x'\in D'}\min_{x\in D} \|x - x'\|\Big\}.$$

For all $D, D' \subset \mathcal{D}$ and $\lambda \in [0,1]$,

$$\lambda D + (1-\lambda)D' = \{\lambda x + (1-\lambda)y | x \in D, y \in D'\} \in \mathcal{D},$$

where the mixture is the probability mixture.

Definition 12. *The RCR ρ is*

- *mixture continuous if $\rho^{\alpha D+(1-\alpha)D'}$ is continuous in α for all $D, D' \in \mathcal{D}$.*
- *continuous if $\rho : \mathcal{D} \to \Pi$ is continuous.*
- *monotone if $x \in D \subset D'$, then $\rho^{D'}(x) \leq \rho^D(x)$.*
- *linear if*

$$\rho^{\lambda D+(1-\lambda)\{y\}}(\lambda x + (1-\lambda)y) = \rho^D(x).$$

- *extreme if $\rho^D(extD) = 1$.*

Theorem 5. *The RCR ρ is mixture continuous, monotone, linear, and extreme if and only if it is a regular random expected utility function.*

Theorem 6. *The RCR ρ is continuous, monotone, linear, and extreme if and only if it is a countably additive regular random expected utility function.*

Lemma 3. *If ρ is monotone, linear, and extreme then $x \in D$, $x' \in D'$, and*

$$N(D,x) = N(D',x') \implies \rho^D(x) = \rho^{D'}(x').$$

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