Decision Theory

OVERVIEW

Chapter 1: Utility
Chapter 2: The Main Choice Paradoxes
Chapter 3: More Sophisticated Stuff
Chapter 1: UTILITY

1. Certainty: Utility
2. Uncertainty: Expected Utility
3. Risk Aversion
4. Mean-Variance(-Skewness) Preferences
5. Loss Aversion

1. Certainty

- Consider observed(!) choice between (bundles of) goods a, b, ...
- Symbols: $a \geq b$ if a is weakly preferred over b;
  $a \sim b$ if indifferent
- Three “axioms of rational choice”
1. Certainty: Axioms of Choice

A. Preference relation is complete: either \( a \geq b \) or \( a \leq b \) or both (i.e., \( a \sim b \)) for all \( a, b \)

B. Transitivity: \( a \geq b \) and \( b \geq c \) implies \( a \geq c \)

C. Continuity: close-by bundles (similar quantities) satisfy same preference relation

1. Certainty: Transitivity

- Example from Schultz’ lab: three gambles; same expected reward, reward variance; 
  \( a=\) positive skew, \( b=\) zero skew, \( c=\) negative skew; \( a \leq b, b \leq c \) yet \( a \geq c \)
1. Certainty: Main Theorem

- If choice satisfies three axioms, then choice can be represented as maximization of some function \( u(\cdot) \): \( a \geq b \) iff \( u(a) \geq u(b) \)
- Utility function is \textit{not unique}: any monotone transformation represents same choices
- As such, preferences are \textit{ordinal}

2. Uncertainty: Expected Utility

- Consider \textit{observed}(!) choice between gambles ("lotteries") \( a, b, \ldots \)
- Gamble consists of \((x, y; \pi)\): \( \£ x \) with probability \( \pi \), \( \£ y \) with probability \( 1-\pi \)
- Compound gambles: \( a=(x, y; \pi); b=(a, z, \pi') \) [same as \((x,y,z; \pi \pi', \pi(1-\pi)) \) - effectively independent]
- Symbols: \( a \geq b \) if \( a \) is weakly preferred over \( b \); \( a \sim b \) if indifferent
- "Axioms of rational choice"
2. Uncertainty: Axioms

A. Preferences are complete, transitive, and continuous
B. Satisfy independence axiom: \( a \geq b \) then for compound gambles: \((a,z,\pi) \geq (b,z,\pi)\)
C. Technical: There exists a best and worst gamble

2. Uncertainty: Von Neumann-Morgenstern Theorem

- Under these axioms, there exists a function \( u(\bullet) \) such that choices can be represented by maximization of \( U((x,y;\pi)) = \pi u(x) + (1-\pi) u(y) \)
  
  \[ = E u \]
2. Uncertainty: Uniqueness

- Utility \( u(\bullet) \) is unique up to a monotone linear transformation
- So, utility is cardinal
- So, can arbitrarily fix, say: \( u(x) = 0 \) and \( u(y) = 1 \) (but not \( u(z) \))

2. Uncertainty: Extensions

- Anscombe-Aumann: take a probability-theory approach (underlying states with probabilities \( \pi \) and acts that lead to consequences \( x, y, z \) in those states)
- Savage: allows for case with probabilities \( \pi \) are not known (ambiguity)
- Additional axiom: Sure-Thing Principle (gamble \( a \) pays strictly more than gamble \( b \) in a range of states, and the same in the remaining states, then \( a > b \) [strict])
- Leads to concept of Probabilistic Sophistication: rational decision maker assigns probabilities to states independently of payoffs to be received in states
3. Risk Aversion

- Back to situation where probabilities are known
- Two related problems:
  1. How to measure risk?
  2. What is risk aversion?

Intuitively, risk aversion means that you forego fair gambles

A fair gamble promises you £δ with π=1/2, otherwise -£δ

Expected utility (with base wealth \(w_o\)):
\[
U((w_o+\delta,w_o-\delta;1/2))=\frac{1}{2}u(w_o+\delta)+\frac{1}{2}u(w_o-\delta)
\]<\[
u(w_o) = U(w_o)
\]

✓ This means that \(u(\bullet)\) must be strictly concave
Notice that risk aversion is a property of $u(\bullet)$ and not of beliefs!

Concavity of the utility function is measured by how fast the first-order derivative of $u$ goes down with wealth:

- $u'$ measures marginal utility; $u' > 0$ (greed)
- $u''$ measures degree to which marginal utility goes down; $u'' < 0$ (decreasing marginal utility)
- $u''$ measures change in utility from incremental wealth and risk aversion!
3. Risk Aversion: Definitions

- **Certainty Equivalent**: Amount of wealth CE agent is willing to receive to forego gamble
  \[ U(CE) = U((w_1, w_2; \pi)) \]
- **Risk Premium** (RP):
  \[ RP = E[w] - CE = [\pi w_1 + (1-\pi) w_2] - CE \]

3. Risk Aversion: Scales

- **Arrow-Pratt Measure** of risk aversion: the risk premium (RP) required to accept a gamble
- **For small gambles**, this means that risk aversion can be measured entirely in terms of \( u'' \)
- Utility function is not unique, so adjust for \( u' \)
- **Absolute Risk Aversion Measure**:
  \[ ARA = -\frac{u''(w_o)}{u'(w_o)} \]
- **Relative Risk Aversion Measure**:
  \[ RRA = -\frac{u''(w_o)}{[w_o u'(w_o)]} \]
3. Risk Aversion: Standard Utility Functions

- **Constant Relative Risk Aversion**: power utility
  \[ u(w) = w^{1-g}/(1-g) \quad (g > 0) \]
  \[ u(w) = \ln(w) \]

- **Constant Absolute Risk Aversion**: exponential utility
  \[ u(w) = -b e^{-bw} \quad (b > 0) \]

3. Risk Aversion: Experiments


<table>
<thead>
<tr>
<th>Option A</th>
<th>Option B</th>
<th>Expected payoff difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/10 of $2.00, 9/10 of $1.60</td>
<td>1/10 of $3.85, 9/10 of $0.10</td>
<td>$1.17</td>
</tr>
<tr>
<td>2/10 of $2.00, 8/10 of $1.60</td>
<td>2/10 of $3.85, 8/10 of $0.10</td>
<td>$0.83</td>
</tr>
<tr>
<td>3/10 of $2.00, 7/10 of $1.60</td>
<td>3/10 of $3.85, 7/10 of $0.10</td>
<td>$0.50</td>
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<tr>
<td>4/10 of $2.00, 6/10 of $1.60</td>
<td>4/10 of $3.85, 6/10 of $0.10</td>
<td>$0.16</td>
</tr>
<tr>
<td>5/10 of $2.00, 5/10 of $1.60</td>
<td>5/10 of $3.85, 5/10 of $0.10</td>
<td>−$0.18</td>
</tr>
<tr>
<td>6/10 of $2.00, 4/10 of $1.60</td>
<td>6/10 of $3.85, 4/10 of $0.10</td>
<td>−$0.51</td>
</tr>
<tr>
<td>7/10 of $2.00, 3/10 of $1.60</td>
<td>7/10 of $3.85, 3/10 of $0.10</td>
<td>−$0.85</td>
</tr>
<tr>
<td>8/10 of $2.00, 2/10 of $1.60</td>
<td>8/10 of $3.85, 2/10 of $0.10</td>
<td>−$1.18</td>
</tr>
<tr>
<td>9/10 of $2.00, 1/10 of $1.60</td>
<td>9/10 of $3.85, 1/10 of $0.10</td>
<td>−$1.52</td>
</tr>
<tr>
<td>10/10 of $2.00, 0/10 of $1.60</td>
<td>10/10 of $3.85, 0/10 of $0.10</td>
<td>−$1.85</td>
</tr>
</tbody>
</table>
Portfolio theory (part of finance) does not work with expected utility theory because it is very hard to figure out optimal combination (portfolio) of gambles (securities) with dependent payoffs.

Instead, portfolio theory works with trade-offs between statistical moments:

- Expected payoff (return; reward)
- Payoff variance
- Skewness
- Kurtosis

“Markowitz Theory”

**Figure 3. Proportion of Safe Choices in Each Decision: Data Averages and Predictions**

*Note: Data averages for low real payoffs [solid line with dots] and 20x real payoffs [squares], with corresponding predictions for constant absolute risk aversion with $\alpha = 0.2$ [thick dashed lines] and risk neutrality [thin dashed line].*
4. MV Preferences

These preferences are also very useful for updating: learning can be based on simple adaptive expectations (Rescorla-Wagner) rule rather than Bayesian updating.

These preferences could be “justified” from expected utility theory using a Taylor series approximation:

\[ U(w) = u(E[w]) + \frac{1}{2} u''(E[w]) \text{var}(w) + \frac{1}{6} u'''(E[w]) \text{skew}(w) + \frac{1}{24} u''''(E[w]) \text{kurtosis}(w) \]

(Power/exponential utility: variance averse; negative skewness averse; kurtosis averse)

There does NOT exist an expected utility profile that matches the preferences exactly.
5. Loss Aversion

- Expected utility theory works with utility over final (total) wealth
- (Holt-Laury analysis ignored wealth outside lab!)
- Prospect theory: utility is evaluated for losses and gains relative to some reference point
- Since losses and gains are defined, one can talk about loss aversion

5. Loss Aversion

- Prospect theory is based on actual or hypothetical choice, not on axioms
- Yet it attempts to write choices as maximization of some expected-utility-like function
- One version has been axiomatized... (i.e., “rationalized”)
5. Loss Aversion: Prospect Theory

Loss aversion: utility function is “steeper” for losses than for gains:

- Strictly concave utility has the same feature (relative to any reference), so it is really the combination with other aspects of preferences that make prospect theory “special”
  - Risk aversion for gains (concavity)
  - Risk loving attitude towards losses (convexity)
- “Kink” (non-differentiability) is not crucial to generate loss aversion
5. Loss Aversion: Reference Point

- Where is the reference point (needed to define “losses“)?
  - Depends on “framing”
  - More formal approaches:
    - Internal habit: your own past wealth/consumption
    - External habit: other people’s wealth/consumption

5. Loss Aversion: Habit

- Problem with internal habit (or any other reference point that is under one’s control): optimal decisions are very difficult
- ... because agent wants to control reference point
5. Loss Aversion: Mathematics

Formula for prospect-theoretic preferences:
\[ u(w) = \frac{(w-w^*)^{(1-G)}}{1-G} \quad \text{for} \quad w \geq w^* \]
\[ u(w) = \frac{(w-w^*)^{(1-L)}}{1-L} \quad \text{for} \quad w < w^* \]
\[ U(w) = E[u(w)] \]
Where \( E[\bullet] \) is computed based on (biased) beliefs
And: \( L < 0 < G \)

Chapter 2: THE MAIN CHOICE PARADOXES

1. The Allais Paradox
2. The Ellsberg Paradox
1. The Allais Paradox

- Consider gamble A: £30 for sure, B=(45, 0; .8)
- Many prefer A
- Now “compound” gambles: (•, 0; .25), where •=A,B
- So: A’=(30, 0; .25) and B’=(45, 0; .2)
- People often switch: B’>A’

- This is a violation of the independence axiom
- Related to the certainty effect in psychology
1. The Allais Paradox: Disappointment Aversion

- Disappointment Aversion (DA) preferences overweigh outcomes below a threshold, namely, (fraction of) the certainty equivalent (CE) (Gul, *Econometrica* 1991)

Mathematics:

\[ U(CE) = E[u(x)] - \theta E_{x<\delta CE}[u(\delta CE) - u(x)] \]

- Notice: effective overweighing of extreme (unlikely?) and bad events
- Reference point is *endogenous*
- Explains why risk premia (⇒average returns on risky securities) in downturns are much higher than in upturns
1. The Allais Paradox: Related Approaches

- Probability weighting in prospect theory
- Rank-dependent utility
- Value-at-Risk preferences

1. The Allais Paradox: Probability Weighting
1. The Allais Paradox: NO Arbitrage Opportunity

- While the Allais paradox does not constitute a “bad” irrationality (since it does not imply a violation of the sure thing principle),
- ... some proposed “solutions“ imply one!
  - E.g., probability weighting functions
  - ... illustrating how delicate mathematical representation of preferences is

2. Ambiguity (The Ellsberg Paradox)

- = Probabilities are unknown
- Sense of ambiguity can also be induced socially:
  - Financial markets (my lab)
2. The Ellsberg Paradox

- Three states: r, g, b. Probability of r known; only joint probability of g, b known
- Typical preference ordering for lotteries that pay £1 if indicated event occurs:
  \[
  r > g \\
  \{r \text{ or } b\} < \{g \text{ or } b\}
  \]

Let \( u(0)=0, u(1)=1 \). Then choices reflect:

\[
U(r) > U(g) \Rightarrow \pi(r) > \pi(g) \\
U(\{r \text{ or } b\}) < U(\{g \text{ or } b\}) \Rightarrow \\
\pi(\{r \text{ or } b\}) = \pi(r) + \pi(b) < \pi(\{g \text{ or } b\}) = \pi(g) + \pi(b)
\]

- But the latter implies:
  \[
  \pi(r) < \pi(g)
  \]

- So, decision maker changes beliefs depending on bets!
2. Ambiguity: Savage

- Ambiguity sensitive decision makers assign different probabilities to states for each lottery
- In Savage’s theory, this is not allowed
- Savage does allow you to assign any probabilities

2. Ambiguity: Maxmin

- Maxmin ambiguity averse agents think Nature is malevolent
- Mathematics (setting $u(0)=0$, $u(1)=1$):
  \[ U(\{r \text{ or } b\}) = \pi + \min_{\rho} \{\rho \cdot 1 + (1-\rho) \cdot 0\} \]
  \[ (= \bar{\pi}) \]
- (Gilboa-Schmeidler, *J Math Econ* 1989)
2. Ambiguity: $\alpha$-Maxmin

- $\alpha$-Maxmin agents think Nature is malevolent with probability $\alpha$ and benevolent with probability $1-\alpha$
- Mathematics (setting $u(0)=0$, $u(1)=1$):
  $$U(\{r \text{ or } b\}) = \pi_1 + \alpha \min_{\rho \in [0,1]} \{\rho \text{ or } (1-\pi) 0\}$$
  $$+ (1-\alpha) \max_{\rho \in [0,1]} \{\rho \text{ or } (1-\pi) 0\}$$
  $$= \pi + \alpha (1-\alpha)(1-\pi)$$
- $\alpha$ measures ambiguity aversion ($\alpha=1$: maxmin; $\alpha=0.50$: ambiguity-neutral)
- Most subjects: $\alpha=0.75$ (study in my lab)
- (Ghirardato, ea, *JEc Theory* 2004)

2. Ambiguity: Partial Ambiguity

- Probabilities are not completely unknown
- Easily accommodated: constraint on choice of beliefs
- E.g. (again setting $u(0)=0$, $u(1)=1$):
  $$U(\{r \text{ or } b\}) = \pi_1 + \alpha \min_{\rho \in [0,0.1,1-\pi-0.05]} \{\rho \text{ or } (1-\rho) 0\}$$
  $$+ (1-\alpha) \max_{\rho \in [0,0.1,1-\pi-0.05]} \{\rho \text{ or } (1-\rho) 0\}$$
  $$= \pi + \alpha 0.1 + (1-\alpha)(1-\pi-0.05)$$
2. Ambiguity: Partial Ambiguity

- Example: paid $1 if red or green; $0 if blue; area of circle corresponds to probabilities
- Gray cover introduces ambiguity

Perceived probability of winning:
- $\alpha=1$: $\frac{1}{3}+\frac{1}{12}$ (Left); $\frac{1}{3}+\frac{1}{4}$ (Right)
- $\alpha=0.75$: $\frac{1}{3}+0.75\frac{1}{12}+0.25\frac{7}{12}$ (Left); $\frac{1}{3}+0.75\frac{3}{12}+0.25\frac{5}{12}$ (Right)
- $\alpha=0.5$: $\frac{1}{3}+0.5\frac{1}{12}+0.5\frac{7}{12}$ (Left); $\frac{1}{3}+0.5\frac{3}{12}+0.5\frac{5}{12}$ (Right)

$\alpha=1$ strongly prefers Right; $\alpha=0.75$ somewhat prefers Right; $\alpha=0.5$ is indifferent

2. Ambiguity: Probabilistic Sophistication

- Because beliefs of ambiguity averse agents change with lotteries, they violate probabilistic sophistication
- Probabilistic sophistication=utility based on two separate components:
  - probabilities assigned to states;
  - utility of wealth in state
- So, beliefs not to be affected by, a.o., risk aversion in state
Chapter 3: MORE SOPHISTICATED STUFF

1. Intertemporal Substitution
2. Discounting
3. Probabilistic Sophistication
4. Learning

1. Intertemporal Substitution

- Standard intertemporal preferences: time additive
- \( U(x_1, x_2, ...) = E[\sum \delta^t u(x_t)] \)
- Ignore discounting (\( \delta = 1 \))
1. Intertemporal Substitution: Smoothing

- Marginal rate of substitution of payoff (consumption) at $t$ and $t+1$ equals $u'(x_{t+1})/u'(x_t)$.
- Intertemporal substitution is controlled by $u'(\cdot)$.
- ... and hence, by decreasing marginal utility.
- So, the same parameter that controls decreasing marginal utility also determines demand for smoothing of payoffs (and temporal risk aversion, for that matter)!

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1. Intertemporal Substitution: Resolution of Uncertainty

- In addition, decision makers with time-additive preferences are indifferent to timing of resolution of uncertainty; e.g., they are indifferent to the following:

<table>
<thead>
<tr>
<th>t=1</th>
<th>t=2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Early:</td>
<td>£100</td>
</tr>
<tr>
<td></td>
<td>£100</td>
</tr>
<tr>
<td>Late:</td>
<td>£100</td>
</tr>
<tr>
<td></td>
<td>£25</td>
</tr>
</tbody>
</table>
1. Intertemporal Substitution: Recursive Utility

- To disentangle (temporal) risk aversion and intertemporal smoothing
- Uses a nonlinear function $W$
  \[ U_0(x_1,x_2) = W(x_1,E[U_1(x_1,x_2)|I_1]) \]
- Kreps-Porteus, *Econometrica* 1978
- Will at the same time generate demand for early/late resolution of uncertainty

*Preferring early/late resolution of uncertainty is NOT irrational!!*

2. Discounting

- Standard model: time-separable and exponential discounting:
  \[ U(x_1,x_2,...) = E[\Sigma \delta^t u(x_t)] \]
- Exponential discounting *necessary* (here) for time-consistent choices:
  - I choose $1.1$ in one year plus one month over $1$ in one year
  - When a year lapses, I still prefer to wait a month (I stick to my original plans)
    (Recursive, like in recursive utility)
2. Discounting: Time inconsistency

- Classical example:
  - I choose $1.1 in one year plus one month over $1 in one year
  - When a year lapses, I prefer not to wait a month – I switch
- Can be modeled using hyperbolic or quasi-hyperbolic ($\beta - \delta$) discounting [Phelps-Pollak R Ec Stud 1968; Laibson, QJE 1997]:
  \[ U(x_1, x_2, \ldots) = E[\sum_t (1+kt)^{-\delta}u(x_t)] \]
  \[ U(x_1, x_2, \ldots) = u(x_0) + \beta E[\sum_{t>0} \delta^t u(x_t)] \]

2. Discounting Without Uncertainty

- If there is no uncertainty, you cannot distinguish between form of discounting and curvature of utility function
- I prefer $1 now over $1.1 in one month because I have
  - A high discount rate and linear utility
  - A low discount rate but decreasing marginal utility
- Remember:
  - Utility is only ordinal under certainty, and hence marginal utility is not fixed
  - Under uncertainty, however, utility is cardinal; marginal utility is set by risk aversion!
3. Probabilistic Sophistication

- Separation from beliefs about states and utilities of payoffs in states (and hence, risk attitudes etc.)
- Subsumes that underlying (random) payoffs, there are STATES
- Random variables are but functions of those states (they're not inherently random)
- Like Kolmogorov Probability Theory (Machina-Schmeidler, Econometrica 1992)

Consider this game:
3. Probabilistic Sophistication

- Agents knows that colors determine outcomes
- Colors are like “states;” labels are like “random variables”
- Once the (underlying) colors are learned, no need to re-learn when only labels change
- Very often, states (colors) are hidden; then need to infer states = Bayesian learning

3. No Probabilistic Sophistication

- Ignore underlying colors
- Just follow the outcomes (payoffs) – reinforcement learning
- Like classical statistics: random variables are characterized by their moments (moment generating function) and distributions
- Need to re-learn every time labels change – because the distribution changes
4. Learning

- **Bayesian learning**: one perceives a “hidden state” or “hidden model” behind (random) outcomes
  - A Bayesian person believes that outcomes are generated by some hidden model
  - As you get to know this hidden model better, your learning rate changes
- **(Model-free) reinforcement learning**: One merely learns associations between observables

### 4. Learning

**In The Monty Hall Task**

[Diagram of the Monty Hall Task]

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Bayesian learning: one perceives a “hidden state” or “hidden model” behind (random) outcomes

A Bayesian person believes that outcomes are generated by some hidden model

As you get to know this hidden model better, your learning rate changes

(Model-free) reinforcement learning: One merely learns associations between observables
4. Learning
In The Monty Hall Task

- **Bayesian**: Infer from a model of actions of quiz master where item could be; as a result: always switch (see next slide)
- **Reinforcement learning**: repeatedly try an action and see which one ultimately works best (namely, switching)
- Humans are notoriously bad in the Monty Hall task (not Bayesian) (Yet they’re good at other, far more demanding tasks!)

4. (Bayesian) Learning
In The Monty Hall Task

- **Probability(Item behind Last Door | Quiz master opened first door and I chose second door)**
  
  $$\text{Probability(Item behind Last Door | Quiz master opened first door and I chose second door)} = \frac{\text{Probability(Item behind Last Door and Quiz master Opens first door | I chose second door)}}{\text{Probability(Quiz master opens first door | I chose second door)}} = \frac{1/3}{1/2} = \frac{2}{3}.$$  

- In 2/3 of cases, Quiz master has no choice and has to open a specific door to avoid opening the (other) door where the item is!