Problem Set #4: Miscellaneous, and Large Sample Theory

1. (Probability limits) Consider a sequence of random variables \( \{X_i\} \), \( i = 1, 2, \ldots \) which are i.i.d. (i.e., independently and identically distributed) with \( EX_i = \mu \) and \( \text{Var}X_i = \sigma^2 \), for all \( i \).

Let \( \bar{X}_n \equiv \frac{1}{n} \sum_i X_i \). For each of the following random sequences, derive the probability limit (if it exists). Be rigorous and explicit about the theorems used in each step of your argument.

(a) \( \bar{X}^2_n \)
(b) \( \frac{1}{\bar{X}^2_n} \)
(c) \( \exp(\bar{X}_n) \)
(d) \( \frac{1}{\bar{X}_n - \mu} \)

2. (Limit distribution) Consider a sequence of random variables \( \{X_i\} \), \( i = 1, 2, \ldots \) which are i.i.d. (i.e., independently and identically distributed) with \( EX_i = \mu \) and \( \text{Var}X_i = \sigma^2 \), for all \( i \).

Let \( \bar{X}_n \equiv \frac{1}{n} \sum_i X_i \). For each of the following random sequences, derive the limit distribution (if it exists). Be rigorous and explicit about the theorems used in each step of your argument.

(a) \( \bar{X}^2_n \)
(b) \( \frac{1}{\bar{X}_n} \)
(c) \( \exp(\bar{X}_n) \)
(d) \( \frac{1}{\bar{X}_n - \mu} \)

3. (Asymptotic behavior of binomial experiments) Consider two sequences of independent and identical Bernoulli experiments. In the first sequence of experiments, the outcomes are

\[
Y_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p \end{cases}, \quad i = 1, 2, 3, \ldots
\]

In the second sequence, the outcomes are

\[
Z_i = \begin{cases} 1 & \text{with probability } q \\ 0 & \text{with probability } 1 - q \end{cases}, \quad i = 1, 2, 3, \ldots
\]

For each of the following, derive the probability limit and limit distribution (if they exist). Be rigorous and explicit about the theorems used in each step of your argument.

(a) \( \frac{1}{n} \sum_{i=1}^{n} Y_i \)
(b) \( \frac{1}{n} \sum_{i=1}^{n} Y_i + Z_i \)
(c) \( \frac{1}{n} \sum_{i=1}^{n} (1 - Y_i) \)
(d) \( \frac{1}{n} \sum_{i=1}^{n} Y_i + (1 - Z_i) \)
(e) \( \frac{1}{n} \sum_{i=1}^{n} Y_i + X_i \), where \( X_i \equiv 1 - Y_i \).
4. (Almost sure convergence) Assume that $X \sim U[0, 1]$. Define the random sequences

$$S_i = \begin{cases} 
X & \text{if } X \in \{1, 1/2, 1/4, 1/8, 1/16 \ldots \} \\
\frac{X}{i} & \text{if } X \notin \{1, 1/2, 1/4, 1/8, 1/16 \ldots \}
\end{cases}, \ i = 1, 2, 3, \ldots$$

Does the random sequence $S_i$ converge almost surely? If so, to what?