1 Theory (primer)

An auction is a game of incomplete information. Assume that there are \( N \) players, or bidders, indexed by \( i = 1, \ldots, N \). There are two fundamental random elements in any auction model.

- Bidders' private signals \( X_1, \ldots, X_N \).
- Bidders' utilities: \( u_i(X_i, X_{-i}) \), where \( X_{-i} \equiv \{X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_N\} \), the vector of signals excluding bidder \( i \)'s signal. Since signals are private, \( V_i \equiv u_i(X_i, X_{-i}) \) is a random variable from all bidders' point of view. In what follows, we will also refer to bidder \( i \)'s (random) utility as her \emph{valuation}.

Differing assumptions on the form of bidders’ utility function lead to an important distinction:

- **Private value** model: \( V_i = X_i \), \( \forall i \). Each bidder knows his own valuation, but not that of his rivals.\(^1\)

- **(Pure) common value** model: \( V_i = V, \forall i \), where \( V \) is in turn a random variable from all bidders’ point of view, and bidders’ signals are to be interpreted as their noisy estimates of the true but known common value \( V \). Therefore, signals will generally not be independent when common values are involved.

- More generally a common value model arises when \( u_i(X_i, X_{-i}) \) is functionally dependent on \( X_{-i} \). In CV models, each bidder does not know his valuation with certainty, but only has a noisy signal of it.

Examples:

- Symmetric independent private values (IPV) model: \( X_i \sim F, \text{ i.i.d. across all bidders } i, \) and \( V_i = X_i \). Therefore, \( F(X_1, \ldots, X_N) = F(X_1) \ast F(X_2) \cdots \ast F(X_N) \), and \( F(V_1, X_1, \ldots, V_N, X_N) = \prod_i [F(X_i)]^2 \).

- Conditional independent model: signals are independent, conditional on a common component \( V \). \( V_i = V, \forall i \), but \( F(V, X_1, \ldots, X_N) = F(V) \prod_i F(X_i | V) \).

\(^1\)More generally, in a private value model, \( u_i(X_i, X_{-i}) \) is restricted to be a function \emph{only} of \( X_i \).
Models also differ depending on the auction rules:

- **First-price** auction: the object is awarded to the highest bidder, at her bid.
- **Second-price** auction: awards the object to the highest bidder, but she pays a price equal to the bid of the second-highest bidder. (Sometimes second-price auctions are also called “Vickrey” auctions, after the late Nobel laureate William Vickrey.)
- In an **English or ascending** auction, the price is raised continuously by the auctioneer, and the winner is the last bidder to remain, and he pays an amount equal to the price at which all of his rivals have dropped out of the auction.
- In a **Dutch** auction, the price is *lowered* continuously by the auctioneer, and the winner is the first bidder to agree to pay any price.

### 1.1 Equilibrium bidding

In discussing equilibrium bidding in the different auction models, we will focus on the general symmetric affiliated model, used in the seminal paper of Milgrom and Weber (1982). The assumptions made in this model are:

- \( V_i = u_i(X_i, X_{-i}) \)
- Symmetry: the joint distribution function \( F(V_1, X_1, \ldots, V_N, X_N) \) is symmetric (i.e., exchangeable) in the indices \( i \) so that, for example, \( F(V_N, X_N, \ldots, V_1, X_1) = F(V_1, X_1, \ldots, V_N, X_N) \).
- The random variables \( V_1, \ldots, V_N, X_1, \ldots, X_N \) are affiliated. Consider a joint distribution function \( F(Z_1, \ldots, Z_M) \), and let \( \bar{Z} \equiv (Z_1, \ldots, Z_M) \) and \( \bar{Z}^* \equiv (Z_1^*, \ldots, Z_M^*) \) denote two independent draws from this distribution. Let \( \bar{Z} \) and \( \bar{Z}^* \) denote, respectively, the component-wise maximum and minimum. Then we say that \( Z_1, \ldots, Z_M \) are affiliated if \( F(\bar{Z})F(\bar{Z}) \geq F(Z_1, \ldots, Z_M)F(Z_1^*, \ldots, Z_M^*) \). In other words, large values for some of the variables make large values for the other variables most likely.

Some useful implications of affiliation:

- \( E[Z_1|Z_2] \geq 0 \).
- Let \( Y_i \equiv \max_{j \neq i} X_j \), the highest of the signals observed by bidder \( i \)'s rivals. Given affiliation, the conditional expectation \( E[V_i|X_i, Y_i] \) is increasing in both \( X_i \) and \( Y_i \).
- Rules out negative correlation between bidders’ valuations.

**Winner’s curse**  Another consequence of affiliation is the *winner’s curse*, which is just the fact that

\[ E[V_i|X_i] \geq E[V_i|X_i > Y_i] \]

where the conditioning event in the second expectation \((X_i > Y_i)\) is the event of winning the auction.

To see this, note that\(^2\)

\[
E[V_i|X_i] = E_{X_{–i}|X_i} E[V_i|X_i; X_{–i}] = \int \cdots \int E[V_i|X_i; X_{–i}] F(dX_1, \ldots, dX_{i–1}, dX_{i+1}, \ldots, dX_N|X_i) \\
\geq \int \cdots \int E[V_i|X_i; X_{–i}] F(dX_1 dX_{i–1}, dX_{i+1}, \ldots, dX_N|X_i) \\
= E[V_i|X_i > X_j, j \neq i] = E[V_i|X_i > Y_i].
\]

In other words, if bidder \(i\) “naively” bids \(E[V_i|X_i]\), her expected payoff from a first-price auction is negative for every \(X_i\).

\(^2\) Law of iterated expectation:

\[
Ex = \int xf(x)dx \\
= \int x \left[ \int f(x, y)dy \right] dx \\
= \int x \left[ \int f(x|y)f(y)dy \right] dx \\
= \int \left[ \int xf(x|y)dx \right] f(y)dy \\
= \int E[x|y]f(y)dy = E_y E[x|y].
\]
In equilibrium, therefore, rational bidders should “shade down” their bids by a factor to account for the winner’s curse. This winner’s curse intuition arises in many non-auction settings also. For example, in two-sided markets where traders have private signals about unknown fundamental value of the asset, the ability to consummate a trade is “good news” for sellers, but “bad news” for buyers, implying that, without ex-ante gains from trade, traders may not be able to settle on a market-clearing price. The result is the famous “lemons” result by Akerlof (1970), as well as a version of the “no-trade” theorem in Milgrom and Stokey (1982). Glosten and Milgrom (1985) apply the same intuition to explain bid-ask spreads in financial markets.

Next, we cover the first- and second-price auctions in some detail.
1.2 Second-price auctions

Let \( b^*(x) \) denote the equilibrium bidding strategy (which maps each bidder’s private information to his bid). Assume it is monotonic. Next we derive the functional form of this equilibrium strategy.

Given monotonicity, the price that bidder \( i \) will pay (if he wins) is \( b^*(Y_i) \): the bid submitted by his closest rivals. He only wins when his bid \( b < b^*(Y_i) \). Therefore, his expected profit from participating in the auction with a bid \( b \) and a signal \( X_i = x \) is:

\[
E_{Y_i} [(V_i - b^*(Y_i)) 1 (b^*(Y_i) < b) | X_i = x]
\]

\[= E_{Y_i} [(V_i - b^*(Y_i)) 1 (Y_i < X_i) | X_i = x]
\]

\[= E_{Y_i | X_i} E [(V_i - b^*(Y_i)) 1 (Y_i < X_i)]
\]

\[= E_{Y_i | X_i} [(E_{X_i | Y_i} (V_i | X_i, Y_i) - b^*(Y_i)) 1 (Y_i < X_i)]
\]

\[= \int_{-\infty}^{X_i} (v(x, Y_i) - b^*(Y_i)) f(Y_i | X_i = x).
\]

In equilibrium, bidder \( i \) also follows the equilibrium bidding strategy, so that \( b_i = b^*(X_i) \). Hence, the upper bound of integration above is \( X_i = (b^*)^{-1}(b) \), leading to

\[
\int_{-\infty}^{(b^*)^{-1}(b)} (v(x, Y_i) - b^*(Y_i)) f(Y_i | X_i = x).
\]

Bidder \( i \) chooses his bid \( b \) to maximize his profits. The first-order conditions are (using Leibniz’ rule):

\[0 = b^*-1(b) \frac{d}{db} \left[ v(x, b^*-1(b)) - b^*(b^*-1(b)) \right] f(b^*-1(b) | X_i) \Leftrightarrow \]

\[0 = \frac{1}{b^*(b)} [v(x, x) - b^*(x)] f(b^*-1(b) | X_i) \Leftrightarrow \]

\[b^*(x) = v(x, x) = E [V_i | X_i = x, Y_i = x].\]

In the PV case, the equilibrium bidding strategy simplifies to

\[b^*(x) = v(x, x) = x.\]

- “Truth-telling” is equilibrium
- “No-regret” feature
- Relation between English and second-price auctions
1.3 First-price auctions

We derive the symmetric monotonic equilibrium bidding strategy $b^*(\cdot)$ for first-price auctions. If bidder $i$ wins the auction, he pays his bid $b^*(X_i)$. His expected profit is

$$E[(V_i - b)1(b^*(Y_i) < b) | X_i = x]$$

$$=E_{Y_i|X_i}[(V_i - b)1(Y_i < b^{*-1}(b)) | X_i = x, Y_i]$$

$$=E_{Y_i|X_i}[(v(x, Y_i) - b)1(Y_i < b^{*-1}(b)) | X_i = x]$$

$$=\int_{-\infty}^{b^{*-1}(b)}(v(x, Y_i) - b)f(Y_i|x)dY_i.$$

The first-order conditions are

$$0 = -\int_{-\infty}^{b^{*-1}(b)} f(Y_i|x)dY_i + 1\frac{1}{b^{*'}(x)}[(v(x, x) - b) * f_{Y_i|X_i}(x|x)] \iff$$

$$0 = -F_{Y_i|x}(x|x) + 1\frac{1}{b^{*'}(x)}[(v(x, x) - b) * f_{Y_i|X_i}(x|x)] \iff$$

$$b^{*'}(x) = (v(x, x) - b^{*}(x))\left[\frac{f(x|x)}{F(x|x)}\right] \Rightarrow$$

$$b^{*}(x) = \exp\left(-\int_{\underline{x}}^{x} F(s|x)ds\right)b(\underline{x}) + \int_{\underline{x}}^{x} v(\alpha, \alpha)dL(\alpha|x).$$

where

$$L(\alpha|x) = \exp\left(-\int_{\underline{x}}^{x} F(s|x)ds\right).$$

Initial condition: $b(\underline{x}) = v(\underline{x}, \underline{x}).$

Alternatively, for a more intuitive expression, integrate the above by parts to obtain:

$$b^{*}(x) = v(x, x) - \int_{\underline{x}}^{x} L(\alpha|x)\frac{d}{d\alpha}v(\alpha, \alpha)d\alpha.$$

Roughly, the equilibrium bid is the winner’s curse-adjusted valuation, $v(x, x)$, minus a “markdown term” $\int_{\underline{x}}^{x} L(\alpha|x)\frac{d}{d\alpha}v(\alpha, \alpha)d\alpha \geq 0$.

For the IPV case:

$$V(\alpha, \alpha) = \alpha$$

$$F(s|s) = F(s)^{N-1}$$

$$f(s|s) = (n - 1)F(s)^{N-2}f(s)$$
Then from above:

\[
b^*(x) = \int_{x}^{\infty} \alpha \exp \left( - \left( n - 1 \right) \int_{\alpha}^{x} f(s) \frac{ds}{F(s)} \right) \left( n - 1 \right) \frac{f(\alpha)}{F(\alpha)} \, d\alpha
\]

\[
= \int_{x}^{\infty} \alpha \left( \frac{F(\alpha)}{F(x)} \right)^{n-1} \left( n - 1 \right) \frac{f(\alpha)}{F(\alpha)} \, d\alpha
\]

\[
= \int_{x}^{\infty} \alpha \cdot \frac{(n - 1)F(\alpha)^{n-2}f(\alpha)}{F(x)^{n-1}} \, d\alpha
\]

\[
= E[Y | Y < x]
\]

where \( Y \) denotes the maximum among bidder \( i \)'s rivals valuations. While this "looks" like the winner's curse, note that this is the case of independent valuations, so the winner's curse is absent.

**An example** \( X_i \sim U[0, 1], \ i.i.d. \) across bidders \( i \). Then \( F(s) = s, \ f(s) = 1 \). Then

\[
b^*(x) = 0 + \int_{0}^{x} \alpha \exp \left( - \left( n - 1 \right) \int_{\alpha}^{x} \frac{(n - 1)f(s)ds}{sF(s)} \right) \frac{(n - 1)f(\alpha)}{\alpha F(\alpha)} \, d\alpha
\]

\[
= \int_{0}^{x} \exp \left( -(n - 1)(\log \frac{x}{\alpha}) \right) (n - 1) \, d\alpha
\]

\[
= \int_{0}^{x} \left( \frac{\alpha}{x} \right)^{N-1} (N - 1) \, d\alpha
\]

\[
= \alpha \left( \frac{N - 1}{N} \right) \left( \frac{\alpha}{x} \right)^{N} \int_{0}^{x}
\]

\[
= \left( \frac{N - 1}{N} \right) x.
\]

Contrast this with the result in second-price auction.

### 1.3.1 Reserve prices

A reserve price just changes the initial condition of the equilibrium bid function. With reserve price \( r \), initial condition is now \( b(x^*(r)) = r \). Here \( x^*(r) \) denotes the screening value, defined as

\[
x^*(r) \equiv \inf \{ x : \ E[V_i | X_i = x, Y_i < x] \geq r \}.
\]

Conditional expectation in brackets is value of winning to bidder \( i \), who has signal \( x \). Screening value is lowest signal such that bidder \( i \) is willing to pay at least the reserve price \( r \).
(Note: in PV case, $x^*(r) = r$. In CV case, with affiliation, generally $x^*(r) > r$, due to winners curse.)

Equilibrium bidding strategy is now:

$$b^*(x) = \begin{cases} 
\exp \left( - \int_{x^*(r)}^{x} \frac{f(s|x)}{F(s|x)} ds \right) r + \int_{x^*(r)}^{x} V(\alpha, \alpha) dL(\alpha|x) & \text{for } x \geq x^*(r) \\
< r & \text{for } x < x^*(r) 
\end{cases}$$

For IPV, uniform example above:

$$b^*(x) = \left( \frac{N-1}{N} \right) x + \frac{1}{N} \left( \frac{r}{x} \right)^{N-1} r.$$ 

Role of reserve prices: limit entry by low-valuation bidders, to boost bids by high-valuation bidders. (Graph)

What about secret reserve prices??
2 Auction design: some well-known results

How to design an auction to raise revenue for the seller?

2.1 Revenue Equivalence

A surprising result is that: when bidders have independent valuations, the revenue from a first-price and second-price auction is equal on average. This is the “revenue-equivalence principle”.

**Sidetrack: mechanism design** Consider the auction as an example of a *mechanism*, which is a general scheme whereby agents with private types $v$ give a report $\tilde{v}$ of their type (where possibly $\tilde{v} \neq v$). The auction mechanism consists of two functions:

- $T(\tilde{v})$: payment agent must make if his report is $\tilde{v}$
- $P(\tilde{v})$: probability of winning the object if his report is $\tilde{v}$.

The *surplus* of an agent with type $v$, who reports that his type is $\tilde{v}$, is “quasilinear”:

$$ S(\tilde{v}; v) \equiv vP(\tilde{v}) - T(\tilde{v}). $$

The seller’s mechanism design problem is to choose the functions $P(\cdot)$ and $T(\cdot)$ to maximize his expected profits. By the “revelation principle” (for which the 2007 Nobel Laureate Roger Myerson is most well known for), we can restrict attention to mechanisms which satisfy an *incentive compatibility condition*, which requires that agents find it in their best interest to report the truth.

The revenue equivalence result falls out as an implication of quasilinearity and incentive compatibility. Assuming differentiability, incentive compatibility implies the following first-order condition:

$$ \frac{\partial S(\tilde{v}; v)}{\partial \tilde{v}} \bigg|_{\tilde{v}=v} = vP'(v) - T'(v) = 0. \quad (5) $$

Hence, under an IC mechanism, the surplus of an agent with type $v$ is just equal to

$$ \tilde{S}(v) \equiv S(v; v) = vP(v) - T(v). $$
Given quasilinearity, instead of viewing the seller as setting \( P \) and \( T \) functions, can view him as setting \( P \) and \( \bar{S} \) functions, and \( T(v) = vP'(v) - \bar{S}(v) \).

By Eq. (5), we see that IC places restrictions on the \( \bar{S} \) function that seller can choose. Namely, the derivative of the \( \bar{S} \) function must obey:

\[
\bar{S}'(v) = P(v) + vP'(v) - T'(v)
= P(v) + 0
\]  

(6)

where the second line follows from Eq. (5). This operation is the “envelope condition”. Hence, integrating up, the \( \bar{S} \) function obeys

\[
\bar{S}(v) = \bar{S}(\underline{v}) + \int_{\underline{v}}^{v} P(u)du
\]

(7)

where \( \underline{v} \) denotes the lowest type. From the above, we see that any IC-auction mechanism with quasilinear preferences is completely characterized by (i) the quantity \( \bar{S}(\underline{v}) \), the surplus gained by the lowest type; and (ii) \( P(\cdot) \), the probability of winning.

**Application to auctions**  Now note that, under independence of types, the monotonic equilibrium in the standard auction forms (including first-price and second-price auctions) have the same \( P(v) \) function. This is because the object is always awarded to the bidder with the highest type, so that for any \( v \), \( P(v) \) is just the probability that a bidder with type \( v \) has the highest type; that is, with \( N \) bidders, and each type \( v \) distributed i.i.d. from distribution \( F(v) \), we have \( P(v) = [F(v)]^{N-1} \). Furthermore, the standard auction forms also have the condition that \( b(\underline{v}) = \underline{v} \), so that \( \bar{S}(\underline{v}) = 0 \).

Hence, by quasilinearity, the standard auction forms also have the same \( T \) function, given by \( T(v) = S(v) - vP(v) \). Since the \( T \) function is the revenue function for the seller, for each type, we conclude that the standard auction forms give the seller the same expected revenue.

### 2.2 Optimal auctions

Now consider the seller’s optimal choice of mechanism. Let \( \vec{v} \equiv (v_1, \ldots, v_N) \) denote the vector of all the bidders’ valuations. As before, assume a “direct revelation mechanism”. The seller wants to choose the functions \( Q_1(\vec{v}), \ldots, Q_N(\vec{v}) \) and \( M_1(\vec{v}), \ldots, M_N(\vec{v}) \) to maximize his revenues, where \( Q_i(\vec{v}) \) denotes bidder \( i \)’s expected probability of winning for bidder \( i \),
as a function of all the reported valuations \( \vec{v} \) (so that \( \sum_i Q_i(\vec{v}) = 1 \)), and \( M_i(\vec{v}) \) likewise denotes the expected payment, as a function of all the valuations. Seller wants to maximize his expected revenue across all the bidders, which is:

\[
\sum_i \int \ldots \int M_i(\vec{v}) f(v_1) \cdots f(v_N) dv_1 \cdots dv_N.
\]

As shorthand, let \( \vec{v}_-i \equiv (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots v_N) \) and \( f(\vec{v}_-i) = \prod_{j \neq i} f(v_j) \). The relationship between the \( Q_i \) and \( M_i \) functions and the \( P \) and \( T \) functions from the previous section are:

\[
P(v_i) = \int Q_i(\vec{v}) f(v_{-i}) dv_i
\]

\[
T(v_i) = \int M_i(\vec{v}) f(v_{-i}) dv_i.
\]

Given these equivalences, seller’s expected revenue can be expressed as:

\[
\sum_i \int \ldots \int M_i(\vec{v}) f(v_1) \cdots f(v_N) dv_1 \cdots dv_N
\]

\[
= \sum_i \int \int T(v_i) f(v_i) dv_i
\]

\[
= \int \int [v_i P(v_i) - \bar{S}(v_i)] f(v_i) dv_i
\]

\[
= \int \int v_i P(v_i) f(v_i) dv_i - \int \int f(v_i) \int P(x) dx dv_i - \int \int \bar{S}(v_i) f(v_i) dv_i
\]

\[
= \int \int P(v_i) f(v_i) \left[ v_i - \frac{1 - F(v_i)}{f(v_i)} \right] dv_i - \bar{S}(v_i)
\]

\[
= \int \cdots \int Q_i(\vec{v}) f(\vec{v}) v_i^* dv_i
\]

The third equality follows by substituting in the IC constraint \((7)\). The fourth equality uses integration by parts to simplify the second term, with the substitution \( u = \int v P(x) dx \) and \( dv = f(v_i) dv_i \). In the above, \( v_i^* \) is called the “virtual type” corresponding to a type \( v_i \). The interpretation is that the revenue that the seller makes from a bidder with type \( v_i \) is \( v_i^* \leq v_i \), so that the deviation \( \frac{1 - F(v_i)}{f(v_i)} \) is the bidder’s “information rent” which arises because only the bidder knows his type.

Under the additional assumption:

\[
(\ast) \quad \frac{1 - F(v)}{f(v)} \text{ is decreasing in } v
\]
then \( v^* \) is increasing in \( v \). In this case, it is clear from inspecting the above that, for all sets of valuations \((v_1, v_2, \ldots, v_N)\), because \( v^* \) is increasing in \( v \), the seller maximizes the above by awarding the item to the bidder with the highest valuation, i.e. setting

\[
Q_i(\vec{v}) = \begin{cases} 
1 & v_i > v_j, j \neq i \\
0 & \text{otherwise}
\end{cases}
\]

Correspondingly, he sets \( P(v_i) = Pr(v_i \text{ is highest out of } N \text{ signals}) = F(v_i)^{N-1} \). This is just like in the standard auction forms.

However, because of the information rent, he will not be willing to sell to all bidders, but only those with non-negative \( v^* \). Thus he will set a reserve price, such that all bidders with \( v^* \leq 0 \) are excluded. Let \( r^* \) denote this optimal reserve price. Recall that, with a reserve price \( r \) and IPV, the lower bound on bids is given by the condition \( b(r) = r \), that is, the reserve price \( r \) screens out bidders with valuations \( v \leq r \). Hence, the optimal reserve price \( r^* \) satisfies

\[
r^* - \frac{1 - F(r^*)}{f(r^*)} = 0.
\]

\( r^* \) depends on the distribution of types \( f(v) \). For example, if \( v \sim U[0,1] \), then

\[
0 = r^* - \frac{1 - F(r^*)}{f(r^*)} = r^* - \frac{1 - r^*}{1} = 2r^* - 1 \Rightarrow r^* = \frac{1}{2}.
\]

The crucial assumption underlying the revenue equivalence property is independence of the bidders’ valuations. In the affiliated case, the expected revenue in the second-price auction is usually larger than the revenue in the first-price auction: this is called the linkage principle.

### 2.3 Complete surplus extraction

When bidders have independent valuations, the revenue equivalence property implies that a bidder with valuation \( x \), only needs to pay \( x^* = x - \frac{1 - F(x)}{f(x)} \), implying that he gets an “information rent” equal to the inverse hazard rate, \( \frac{1 - F(x)}{f(x)} \).

When types are no longer independent (as in the general affiliated case of Milgrom and Weber (1982)), revenue equivalence of the standard auction forms no longer holds. However,

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3In statistics, the quantity \( f(v)/[1 - F(v)] \) is called the hazard function, so that assumption (*) is equivalently that the hazard function of \( v \) is increasing.

4The highest type \( \bar{x} \) gets zero information rent; this is the familiar “no distortion at the top” result from optimal mechanism design.
a remarkable series of papers (Cremer and McLean (1985), Cremer and McLean (1988), McAfee and Reny (1992)) argued that the seller can potentially exploit dependence among bidder types to extract all the surplus in the auction.

We consider this insight for a two-bidder correlated private values auction, as considered in McAfee and Reny (1992) (pp. 397–398). Bidders 1 and 2 have values $v^1$ and $v^2$, which are both discrete-valued from the finite set $\{v_1, \ldots, v_n\}$. Let $P$ denote the $n \times n$ matrix of conditional probabilities: the $(i, j)$-th element of $P$ is the probability that (from bidder 1’s point of view) bidder 2 has value $v_j$, given that bidder 1 has value $v_i$. Clearly, the rows of $P$ should sum to one.

Cremer and McLean focus on simple two-part mechanisms. First, bidders find out their values, and then choose from among a menu of participation fee schedules offered by the seller. These schedules specify the fixed fee that each bidder must pay, as a function of both bidders’ reports. Second, the two bidders participate in a Vickrey (second-price) auction. Truth-telling is assured in the Vickrey auction, so we focus on the optimal design of the participation fee schedules assuming that both bidders report the truth, i.e., report $v^1$ and $v^2$.

In what follows, we focus on bidder 1 (argument for bidder 2 is symmetric). Intuitively, the seller wants to design the fee schedules so that bidder 1 pays an amount arbitrarily close to his expected profit $\pi_1$ from participating which, given his value $v^1$, is $v^1 - E[v^2|v^1]$ (due to second-stage Vickrey auction). Given the discreteness in values, without loss of generality we can consider fee schedules of the form where, for the $m$-th schedule, bidder 1 must pay an amount $z_{mij}$ if he announces $v^1 = v_i$, but bidder 2 announces $v^2 = v_j$. Therefore bidder 1’s expected profit from schedule $m$, given his type $v^1 = v_i$, is $v_i - E[v^2|v^1 = v_i] - \sum_{j=1}^{n} p_{ij} z_{mij}$.

Assume that the seller offers $n$ different schedules, indexed $m = 1, \ldots, n$. Schedule $m$ is targeted to be chosen by a bidder who has valuation $v_1 = v_m$. Also assume that $z_{mij} = v_m - E[v^2|v^1 = v_m] + \alpha_m * x_{mij}$. Hence, the net expected payoff to bidder 1, who has valuation $v_i$, chooses schedule $m$, and then the bidders announce $(v_i, v_j)$, are:

$$v_i - E[v^2|v^1 = v_i] - (v_m - E[v^2|v^1 = v_m]) - \alpha_m \sum_{j=1}^{n} p_{ij} x_{mij}.$$ 

It turns out that the parameter $\alpha$ will be the same across all schedules, and so we eliminate the $m$ subscript from these quantities.

In order for the seller to design schedules which extract all of bidder 1’s surplus, we require
that:

1. For each \( i = 1, \ldots, n \), there exists a vector \( \vec{x}_i \equiv (x_{i1}, \ldots, x_{in})' \) such that
   \[ \sum_{j=1}^{n} p_{ij} x_{ij} = 0, \quad \sum_{j=1}^{n} p_{ij'} x_{ij} \leq 0, \quad \forall i' \neq i : \]
   that is, the vector \( \vec{x}_i \) “separates” the \( i \)-th row of \( P \) from the other rows, at 0.

2. \( \alpha \) must be set sufficiently large so that, for all \( i = 1, \ldots, n \), if bidder 1 has value \( v^1 = v_i \) he will choose schedule \( i \). This implies that his expected profit from choosing schedule \( i \) exceeds that from choosing any schedule \( i' \neq i \). These incentive compatibility conditions pin down \( \alpha \):
   \[ -\alpha \sum_{j=1}^{n} p_{ij} x_{ij} > v_i - E[v^2|v^1 = v_i] - (v_{i'} - E[v^2|v^1 = v_{i'}]) - \alpha \sum_{j=1}^{n} p_{ij} x_{ij'}, \quad \forall i' \neq i \]
   \[ \iff 0 > v_i - E[v^2|v^1 = v_i] - (v_{i'} - E[v^2|v^1 = v_{i'}]) - \alpha \sum_{j=1}^{n} p_{ij} x_{ij'}, \quad \forall i' \neq i \]
   \[ \iff \alpha > \max_{i' \neq i} \left( \frac{v_i - E[v^2|v^1 = v_i]}{\sum_{j=1}^{n} p_{ij} x_{ij'}} - (v_{i'} - E[v^2|v^1 = v_{i'}]) \right). \]

A sufficient condition for the first point above is that, for all \( i = 1, \ldots, n \), the \( i \)-th row of \( P \) is not in the \((n\text{-dimensional})\) convex set spanned by all the other rows of \( P \).\(^5\) Then we can apply the separating hyperplane theorem to say that there always exists a vector \((x_{i1}, \ldots, x_{in})'\) which separates the \( i \)-th row of \( P \) from the convex set composed of convex combinations of the other \( n - 1 \) rows of \( P \).

In order for the \( i \)-th row of \( P \) is not in the \((n\text{-dimensional})\) convex set spanned by all the other rows of \( P \), we require that the \( i \)-th row of \( P \) cannot be written as a linear combination of the other \( n - 1 \) rows, i.e., that all the rows of \( P \) are linear independent. As Cremer and McLean (1988) note, this is a full rank condition on \( P \).

As an example, consider the case where the \( P \) matrix is diagonal (that is, \( p_{ii} = 1 \) for all \( i = 1, \ldots, n \)). This is the case of “Stalin’s food taster”. In this case, the vector \( \vec{x}_i \) with element \( x_{ii} = 0 \) and \( x_{ij} = -1 \) for \( j \neq i \) satisfies point 1. The numerator of the expression for \( \alpha \) is equal to 0. Thus any arbitrarily small participation fee will lead to full surplus extraction!

\(^5\)Or, using terminology in McAfee and Reny (1992), the \( i \)-th row of \( P \) is not in the convex hull of the other \( n - 1 \) rows of \( P \).
References


