Exercise 8.6

(i) The monopolist’s profit function is

\[ P_i = q^A(1 - q^A - c) + q^B(1 - q^B - (c - \lambda q^A)) = q^A(d - q^A) + q^B(d + \lambda q^A - q^B). \]

The FOCs for maximization are

\[ d - 2q^A + \lambda q^B = 0, \quad d + \lambda q^A - 2q^B = 0, \]

which when solved yield

\[ q^A = q^B = \frac{d}{2 - \lambda}. \]

(ii) **Approach 1:** We use the concept of a rational expectations equilibrium. Firm 2 anticipates the quantity produced by firm 1 in period A, even though he cannot observe it. We then enforce the fact that in equilibrium this expectation must be correct.

Call firm 2’s expectation of firm 1’s production \( \bar{q}_1^A \). Then in period B he will produce

\[ q^B_2 = \frac{d - \lambda \bar{q}_1^A}{3}, \]

(essentially anticipating the Cournot competition and resulting Nash equilibrium assuming his guess is correct). So firm 1’s period-B profit function is

\[ \Pi^B_1 = q^B_1(d - q^B_2 - q^B_2 + \lambda q^A_1), \]

where \( q^B_2 \) is as given above. The FOC for maximization is

\[ d + \lambda q^A_1 - q^B_2 - 2q^B_1 = 0, \]

with solution (after plugging in our expression for \( q^B_2 \))

\[ q^B_1 = \frac{d + \lambda(\bar{q}_1^A + 3q^A_1)/2}{3}. \]

Corresponding period-B profits are

\[ \Pi^B_1 = \frac{(d + \lambda(\bar{q}_1^A + 3q^A_1)/2)^2}{9}. \]

This is firm 1’s reduced-form profit in period B. Note that it depends both on firm 1’s production choice in period A, as well as firm 2’s expectation of this production. The significant feature of this setup is that firm 1 can change only his own production, not firm 2’s expectation. His overall profit function is then

\[ \Pi_1 = q^A_1(d - q^A_1) + \frac{(d + \lambda(\bar{q}_1^A + 3q^A_1)/2)^2}{9}, \]

with FOC

\[ d - 2q^A_1 + \frac{\lambda}{3}(d + \lambda(\bar{q}_1^A + 3q^A_1)/2) = 0. \]
Now, in equilibrium the expectation is correct: \( \bar{q}_A^1 = q_A^1 \). Then we find that

\[
q_A^1 = \frac{\lambda + 3}{2(3 - \lambda^2)} d.
\]

**Approach 2:** We treat the setup as a standard one-shot game where firm 1 must choose \( q_A^1 \) and \( q_B^1 \), while firm 2 simultaneously chooses \( q_B^2 \). Firm 1’s profit function is

\[
\Pi_1 = q_A^1(d - q_A^1) + q_B^1(d + \lambda q_A^1 - q_B^1 - q_B^2).
\]

Taking the FOCS (taking firm 2’s production as given) and solving the resulting system yield reaction curves

\[
R_A^1(q_B^2) = \frac{d(2 + \lambda) - \lambda q_B^2}{4 - \lambda^2}, \quad R_B^1(q_B^2) = \frac{d(2 + \lambda) - 2q_B^2}{4 - \lambda^2}.
\]

Meanwhile, firm 2’s profit function is

\[
\Pi_2 = q_B^2(d - q_B^2 - q_B^1).
\]

Taking the FOC (with firm 1’s production a given) yields reaction curve

\[
R_B^2(q_1^B) = \frac{d - q_B^1}{2}.
\]

Now, in equilibrium each firm correctly anticipates the other’s choice and best-responds. Solving the resulting system of equations yields

\[
q_A^1 = \frac{\lambda + 3}{2(3 - \lambda^2)} d, \quad q_B^1 = \frac{\lambda + 1}{3 - \lambda^2} d, \quad q_B^2 = \frac{2 - \lambda - \lambda^2}{2(3 - \lambda^2)} d.
\]

**(iii)** This is a more familiar setup - now firm 1 can anticipate not only the direct effect of a period-A investment on period-B marginal cost, but also the strategic effect on firm 2’s production. Firm 1’s reduced-form profit function is

\[
\Pi_1 = q_A^1(d - q_A^1) + \frac{(d + 2\lambda q_A^1)^2}{9},
\]

with FOC

\[
d - 2q_A^1 + \frac{4\lambda}{9}(d + 2\lambda q_A^1) = 0
\]

and resulting production

\[
q_A^1 = \frac{4\lambda + 9}{2(9 - 4\lambda^2)} d.
\]

To compare this quantity with the previous result, note that

\[
\frac{d}{da} \left( \frac{\lambda + a}{a - \lambda^2} \right) = -\frac{\lambda^2 + \lambda}{(a - \lambda^2)^2} < 0.
\]

Therefore

\[
\frac{\lambda + 4/9}{9/4 - \lambda^2} > \frac{\lambda + 3}{3 - \lambda^2} \Rightarrow q_{obs}^A > q_{unobs}^A.
\]
So firm 1 produces more when this quantity can be observed by firm 2. This result shouldn’t be surprising - if firm 1 acts “tough” by investing a lot toward lowering his marginal cost in period B, then firm 2 will produce less and firm 1 will enjoy softer competition. But firm 1 enjoys this strategic effect only if his investment is observable; otherwise he must take firm 2’s expectation as a given and can’t boost period-A production to soften firm 2 in period B.

(iv) Now there exists the possibility that firm 2 can be driven out of the market entirely. If firm 1 boosts its period-A production enough, he can force firm 2’s period-B profits below the fixed cost of entry. Then firm 2 will decline to compete at all and firm 1 will enjoy a period-B monopoly. Whether it is profitable for firm 1 to deter rather than accommodate depends on the size of firm 2’s fixed cost; if the fixed cost is large enough, we would expect deterrence to be profitable, and we would see $q_1^4$ larger than in part (iii) in order to deter firm 2’s entry.
2 Exercise 8.9

Let \((p_1, p_2)\) be the equilibrium prices in the absence of a price floor. Now suppose that the price floor forces firm 1 to raise his price to \(p_1 + \varepsilon\), where \(\varepsilon\) is relatively small. (Because products are differentiated, the profit function varies smoothly with a small price increase, and does not abruptly drop to 0 as would happen in pure Bertrand competition.) The net effect on firm 1’s profit, taking into account the strategic reaction by firm 2, is

\[
\Delta \Pi_1 = \Pi_1(p_1 + \varepsilon, p_2(p_1 + \varepsilon)) - \Pi_1(p_1, p_2) \simeq \varepsilon \frac{d\Pi_1}{dp_1}(p_1, p_2) = \varepsilon \left( \frac{\partial \Pi_1}{\partial p_1}(p_1, p_2) + \frac{\partial \Pi_1}{\partial p_2}(p_1, p_2) \frac{dp_2}{dp_1} \right). 
\]

Now, given that \((p_1, p_2)\) are equilibrium prices, we must have

\[
\frac{\partial \Pi_1}{\partial p_1}(p_1, p_2) = 0
\]

(this is essentially the envelope theorem). So we find that

\[
\Delta \Pi_1 \simeq \varepsilon \frac{\partial \Pi_1}{\partial p_1}(p_1, p_2) \frac{dp_2}{dp_1}. 
\]

Given that products are substitutes we have

\[
\frac{\partial \Pi_1}{\partial p_2} > 0, \quad \frac{dp_2}{dp_1} > 0. 
\]

(The direct effect of a change in a competitor’s price is to increase your demand and therefore your profit. In response, you would increase your own price to take advantage of the higher demand.) Therefore \(\Delta \Pi_1 > 0\), and so for small enough \(\varepsilon\) (so that a first-order Taylor expansion is valid), firm 1’s profits increase when a price floor is enforced.

The effect is purely a strategic one - the price floor acts as a commitment by firm 1 not to lower his price. As a result, firm 2 raises his price, the prevailing prices are closer to the monopoly price than in Bertrand equilibrium, and profits increase. Note that at these prices firm 1 is not best-responding - he would like to lower his prices a bit to capture more of the market. But this is myopic, for in response firm 2 would lower his prices, and the resulting equilibrium would be less profitable than if firm 1 hadn’t best-responded.

This result is a repetition of the common theme that naive best-responding does not always produce optimal outcomes. If a firm could commit to not best-respond, he could actually raise his equilibrium profits in some circumstances. A government-imposed price floor is one such form of commitment.
3 Exercise 7.1

(i) We first solve for the indifference point, where a consumer is indifferent between purchasing from either firm. At this point the net cost to the consumer must be the same for purchasing from either firm:

\[ p_1 + t(x - a)^2 = p_2 + t(x - (1 - b))^2. \]

Some algebra yields

\[ x = a + \frac{1 - a - b}{2} + \frac{p_2 - p_1}{t(1 - a - b)}. \]

All consumers to the left of this point purchase from firm 1, and all to the right purchase from firm 2. (We assume \( \bar{s} \) is high enough that all consumers purchase, and that the price differential is small enough that \( 0 < x < 1 \).) So we find that

\[ D_1(p_1, p_2) = x = a + \frac{1 - a - b}{2} + \frac{p_2 - p_1}{2t(1 - a - b)}. \]

and

\[ D_2(p_1, p_2) = 1 - x = b + \frac{1 - a - b}{2} + \frac{p_1 - p_2}{2t(1 - a - b)}. \]

(ii) Firm 1’s profit function is

\[ \Pi_1 = D_1(p_1, p_2)(p_1 - c) = \left( a + \frac{1 - a - b}{2} + \frac{p_2 - p_1}{2t(1 - a - b)} \right)(p_1 - c). \]

Taking \( p_2 \) as given and maximizing wrt \( p_1 \) yields the reaction curve

\[ R_1(p_2) = \frac{p_2 + c}{2} + \frac{t}{2}(1 + a - b)(1 - a - b). \]

A similar computation for firm 2 produces

\[ R_2(p_1) = \frac{p_1 + c}{2} + \frac{t}{2}(1 - a + b)(1 - a - b). \]

Solving for the Nash equilibrium produces (after some algebra)

\[ p_1 = c + t(1 - a - b) \left( 1 + \frac{a - b}{3} \right), \quad p_2 = c + t(1 - a - b) \left( 1 + \frac{b - a}{3} \right). \]
Exercise 7.2

(i) We first find the demand curves faced by each firm. We find the indifference point $x$, which satisfies

$$p_1 + tx = p_2 + t(1-x) \Rightarrow x = \frac{t + p_2 - p_1}{2t}.$$  

Then firm 1 faces

$$D_1(p_1, p_2) = \frac{t + p_2 - p_1}{2t},$$

while firm 2 faces

$$D_2(p_1, p_2) = \frac{t + p_1 - p_2}{2t}.$$

Now, firm 1’s profit function is

$$\Pi_1 = \frac{t + p_2 - p_1}{2t} (p_1 - c_1).$$

Maximizing wrt $p_1$ yields reaction curve

$$R_1(p_2) = \frac{c_1 + t + p_2}{2}.$$  

Similar work for firm 2 produces

$$R_2(p_1) = \frac{c_2 + t + p_1}{2}.$$  

(ii) Solving for the resulting Nash equilibrium yields

$$p_1 = t + \frac{2c_1 + c_2}{3}, \quad p_2 = t + \frac{2c_2 + c_1}{3}.$$  

Each firm’s profit is then

$$\Pi_i = \frac{1}{2t} \left( t + \frac{c_j - c_i}{3} \right)^2.$$  

(iii) Straightforward computation yields

$$\frac{\partial^2 \Pi_i}{\partial c_i \partial c_j} = -\frac{1}{9t} < 0.$$  

This result shows that there is a strategic value in investing to lower one’s marginal cost. For note that

$$\Delta \left( \frac{\partial \Pi_i}{\partial c_i} \right) \simeq \frac{\partial^2 \Pi_i}{\partial c_i \partial c_j} \Delta c_j > 0$$

for $\Delta c_j < 0$ (i.e. an investment by the other firm to lowering its marginal cost). Then as

$$\frac{\partial \Pi_i}{\partial c_i} = -\frac{1}{3t} \left( t + \frac{c_j - c_i}{3} \right) < 0,$$

(the inequality following from the fact that we must have $0 < x < 1$ in equilibrium), an increase in $\partial \Pi_i / \partial c_i$ moves it closer to zero - in other words, the marginal effect on the profit of the firm
from lowering the marginal cost has decreased. Therefore an investment by one firm to lower its marginal cost makes a similar move by the other firm less desirable; being “tough” encourages the other firm to be “soft” in its investment decision.

(iv) Gross of the investment cost, firm i’s reduced-form profit function is

$$\Pi_i(p_i(c_i, c_j), p_j(c_i, c_j), c_i) = D_i(p_i(c_i, c_j), p_j(c_i, c_j))(p_i(c_i, c_j) - c_i),$$

where all prices are functions of the marginal cost in equilibrium. Taking the total derivative wrt $c_i$, we obtain

$$\frac{d\Pi_i}{dc_i} = \frac{\partial \Pi_i}{\partial c_i} + \frac{\partial \Pi_i}{\partial p_i} \frac{\partial p_i}{\partial c_i} + \frac{\partial \Pi_i}{\partial p_j} \frac{\partial p_j}{\partial c_j}.$$ 

The second term vanishes by the envelope theorem, leaving us with

$$\frac{d\Pi_i}{dc_i} = \frac{\partial \Pi_i}{\partial c_i} + \frac{\partial \Pi_i}{\partial p_j} \frac{\partial p_j}{\partial c_j}.$$ 

The first term is the direct effect of lowering marginal cost - lowering marginal cost makes each unit less costly to produce, increasing the profit margin on all units sold. The second term is the strategic effect - lowering marginal cost forces the other firm to lower his price to remain competitive, which decreases the first firm’s profits.
5 Exercise 6.4

(i) Consider the trigger strategy “offer product at monopoly price as long as all other firms also sell at the same price; otherwise sell at marginal cost forever.” We ask whether all firms playing this strategy is a Nash equilibrium. If one firm deviates, it is certainly an equilibrium - thereafter no individual firm has an incentive to deviate from marginal-cost pricing for fear of losing all market share or selling at a loss.

So we need only examine whether a firm is willing to avoid undercutting if all other firms do likewise. Such behavior is incentive-compatible only if the expected profits of undercutting (by some infinitesimal amount $\varepsilon$, say) and capturing the market for one period (but then deriving zero profit for all periods afterward) are less than the profits of cooperating and capturing a share of the monopoly profits in all periods:

$$\Pi^m \leq \frac{1}{n} \Pi^m \left(1 + \frac{\delta \mu}{1 - \delta \mu} + \cdots\right) \Rightarrow 1 \leq \frac{1}{n} \frac{1}{1 - \delta \mu} \Rightarrow \delta \geq \frac{1}{\mu} \left(1 - \frac{1}{n}\right).$$

(ii) Note that as $\mu$ increases, the minimum $\delta$ for which full collusion is sustainable decreases. So collusion is more easily sustainable in a more rapidly growing industry than in one that is stagnant or shrinking - intuitively, the expectation of higher future profits from a growing industry encourages collusion.
6 Exercise 6.6

(i) In this case we disregard behavior in market 1 entirely - firms can’t condition their behavior in that market on behavior in market 2, and vice versa, in the absence of multimarket contact. We consider the same trigger strategy described in the previous solution. In this case if a firm defects, it derives monopoly profits for two periods before the punishment triggers. So for collusion to be incentive-compatible, we must have

$$(1 + \delta)\Pi^m \leq \frac{\Pi^m/2}{1-\delta} \Rightarrow 1 - \delta^2 \leq \frac{1}{2} \Rightarrow \delta \geq \frac{1}{\sqrt{2}} \approx 0.71.$$ 

(ii) Now we consider the trigger strategy “offer products in both markets at monopoly prices as long as all other firms also sell at the same price; otherwise sell at marginal cost in both markets forever.” Again this is certainly an equilibrium if one firm triggers the punishment. We need only examine whether collusion is incentive-compatible. In this case the best defecting strategy is to undercut in market 2, then in market 1 one period later, so as to enjoy the maximum possible period of monopoly profit in both markets. For this to be less desirable than cooperating, we must have

$$(1 + \delta)\Pi^m + \left(\frac{1}{2} + \delta\right) \Pi^m \leq 2 \frac{\Pi^m/2}{1-\delta}$$

$$\Rightarrow \left(\frac{3}{2} + 2\delta\right) (1-\delta) \leq 1$$

$$\Rightarrow \frac{1}{2} + \frac{1}{2} \delta - 2\delta^2 \leq 0$$

$$\Rightarrow \left(\delta - \frac{1 - \sqrt{17}}{8}\right) \left(\delta - \frac{1 + \sqrt{17}}{8}\right) \geq 0$$

$$\Rightarrow \delta \geq \frac{1 + \sqrt{17}}{8} \approx 0.64.$$ 

So collusion is more easily sustainable with multimarket contact - the threat of punishment in both markets makes defection less desirable.