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Abstract

This paper proposes a recursive method for analyzing repeated games with private monitoring. When private signals are observed in each period throughout the game, a repeated game is not recursive in the sense of APS [1] as more private information becomes available in later periods. The trick to make it recursive is to provide enough amount of private information to the players during the game by introducing a mediator. In another word, I relax the notion of equilibrium from sequential equilibrium to communication equilibrium. By doing so, the equilibrium payoff set expands by definition, but the recursiveness of the communication game allows me to obtain a bound on sustainable payoff set, which serves as a bound on sequential equilibrium payoffs.

1 Introduction

In long-term relationships, agents often receive a considerable amount of private information. However, once you try to write down a formal model which fully incorporates rich private information, you tend to find the model intractable and hopeless to solve. In such a model, there may be an incomplete information regarding to the players’ types. Then each player’s belief about the other player’s type, which is endogenous, needs to be tracked over time. This is already a complicated task. Things get even worse when private information is obtained during the game. This may happen because
the types of the players change over time or the players observe some private signals each period. As private information is accumulated over time, strategic complexity increases significantly over time. This is why most of the dynamic models assume that all (relevant) information is public, while most models with asymmetric information are “static” models. This paper tries to make a small step toward building a general approach to this type of problem. More specifically, I propose a method to obtain a nontrivial bound on sustainable payoffs in such environment.

In this paper, I focus on one particular class of models which has attracted researchers’ attention: repeated games with private monitoring. When monitoring is public, many important results such as Abreu, Pearce and Stacchetti [1](APS) or Fudenberg, Levine and Maskin [8](FLM) have been obtained. The former paper characterizes the set of pure strategy sequential equilibrium payoffs for a fixed discount factor and the latter proves a Folk Theorem in public strategies. When monitoring is private, however, such results are hard to obtain. All these results are based on a possibility of coordination among players via public information. This suggests that collusion/cooperation should be more difficult to sustain with only private information. Therefore you may naturally expect that private monitoring serves as a restriction on sustainable equilibrium payoffs and leads to a better prediction than repeated games with public monitoring, for which anything goes for patient players.

However it turned out that Folk Theorem property is far more robust than previously thought even in the absence of public information. Starting with a seminal paper by Sekiguchi [15], which shows that efficiency can be obtained in repeated prisoners’ dilemma with almost perfect private monitoring, many papers have proved Folk Theorems for gradually more general environments. Bhaskar and Obara[3] and Ely and Välimäki[4] proved a Folk Theorem for repeated prisoners’ dilemma with almost perfect private monitoring. Matsushima[12] proved a Folk Theorem for repeated prisoners’ dilemma with independent private monitoring, but not necessarily almost perfect. Finally, a recent paper by Hörner and Olszewski[9] proved a Folk Theorem for repeated games of general n-player stage game with almost perfect private monitoring.

On the other hand, no nontrivial example for which Folk Theorem fails has been found for this class of models so far. This paper approaches the same problem from the opposite side: finding an upper bound on sustainable payoffs which holds independent of players’ patience and identifying the environments where perfect collusion/cooperation is not possible.

Let me describe briefly my approach to obtain a nontrivial upper bound
for this class of models. As observed by Kandori[10], one of the technical
difficulties associated with private monitoring is lack of recursiveness. In
principle, each player’s \(t\)-period continuation strategy depends on all pri-
vate information up to period \(t-1\), from which each player infers the other
players’s continuation strategy. Thus a continuation game from \(t\)-period
is, in some sense, a repeated game with incomplete information, where each
player’s “type” is given by his or her private history. Since there is no private
information in the beginning of the game, the original game and each con-
tinuation game is not isomorphic. This makes it difficult to apply APS-type
recursive methods to characterize the set of equilibrium payoffs.

To circumvent this problem, I relax the notion of equilibrium from sequen-
tial equilibrium to communication equilibrium (Forges [5], Myerson
[13]). Then I show that an upper bound of communication equilibrium
can be recursively obtained. Since the equilibrium payoff set becomes larger
by definition, an upper bound of communication equilibrium payoffs serves
as an upper bound for sequential equilibrium payoffs. In communication
equilibrium, a mediator sends a message in the beginning of each stage and
players send messages in the end of the period after all private information
within the period is observed. To see why recursiveness is restored in a com-
munication equilibrium, take the second period of the game. Players have
some private information from the first period. Notice that the continua-
tion equilibrium is also a communication equilibrium of the original game
because the mediator can generate private information from the first period
artificially in the beginning of the game by enriching the message space.

Once games are made recursive by introducing communication, I ex-
tend Fudenberg and Levine[6]’s linear programming method to bound PPE
payoffs with public monitoring to communication equilibria.

As is clear from this brief sketch, private monitoring is not the only class
of models to which this method can be applied. This approach can be applied
more generally to other type of models with private information. One class
of such models is, maybe surprisingly, repeated games with imperfect public
monitoring. There are some private information even with imperfect public
monitoring: private action. The above results for the public monitoring case
is limited to public strategies. It is not yet known how much cooperation can
be achieved in private strategies for such games. Indeed Kandori and Obara
[11] showed that private strategies can support a Pareto improving payoff
which cannot be supported in public strategies. Such private strategies
have a feature that players randomly choose actions and their continuation
strategies depend on their realized actions. Hence, as in games with private
monitoring, you need to deal with private information, for which the above
recursive method is useful.

There are other models to which the current approach can be potentially applied. For the dynamic models of reputation, it is often the case that explicit characterization of equilibrium is difficult and only some lower bound of equilibrium payoffs can be obtained (Fudenberg and Levine [7]). This approach may succeed to obtain an upper bound of equilibrium payoffs to improve prediction of such models. In particular, it can be helpful to obtain Bad-reputation type result (Ely and Välimäki (2004)) for general environments.

Another, but related class of models is a model of repeated adverse selection, such as repeated auction. Recently there has been a notable progress (Aoyagi 2002), Athey, Bagwell and Sanchirico [2], Skrzypacz and Hopenhayn [16] in this area. However, all these models assume that players’ private types are i.i.d. over time and across players. Only one exception is Athey and Bagwell (2004). Clearly it would be more natural to assume that private types are correlated over time. Hopefully my approach will be fruitful in this unexplored area as well.

In the next section, I first present an algorithm to obtain an upper bound for PPE in repeated games with public monitoring. This bound is identical to the well known bound by Fudenberg and Levine [6]. Then this algorithm is generalized to repeated games with private monitoring. I provide some example of repeated partnership game with private monitoring, where Folk Theorem fails. However, it seems that this method has only a limited power in its current form. In Section 3, I focus on a special class of communication equilibrium, Belief-free communication equilibrium. For this class of equilibria, the obtained bound turns out to be tight, thus the limit equilibria can be characterized by the bounds as in Fudenberg and Levine [6]. Then I show that, for some class of partnership games, Folk theorem is obtained even for this class of communication equilibria. Thus there is indeed a cost of relaxing the equilibrium notion. Notably, Folk theorem is obtained for partnership games where folk theorem fails in public strategies. This suggests that a further refinement of the method may be necessary.

2 Recursive Bound

2.1 Public Monitoring and Public Strategy

Let \( a \in A \) be an action profile and \( y \in Y \) be a public signal. The distribution of the public signal given \( a \in A \) is \( \pi (y|a) \). Player \( i \)'s stage game payoff is \( g_i (a) \) and \( g (a) = (g_1 (a), ..., g_n (a)) \). For each weight vector \( \lambda \in \mathbb{R}^n / \{0\} \),
let $a^\lambda = \max_{a \in A} \lambda \cdot g(a)$ and $V^\lambda_0 = \lambda \cdot g(a^\lambda)$.

*Public strategy* is a strategy which depends only on past realization of public signals. A profile of public strategies is a perfect public equilibrium (PPE) if each player’s continuation strategy is a best response after every public history. I consider only perfect public equilibrium and try to obtain an upper bound of PPE payoffs.

Take any PPE $\sigma$ and let $\alpha \in \times_{i=1}^n \Delta A_i$ be the first period (possibly mixed) action profile. Abusing notation, let $\pi(y|\alpha)$ be the distribution of $y$ given mixed action $\alpha$. After each realization of public signal, the players expect a continuation payoff profile $w(y) = (w_1(y), ..., w_n(y))$. Thus player $i$’s average discounted equilibrium payoff is

$$(1 - \delta) g_i(\alpha) + \delta \sum_{y \in Y} \pi(y|\alpha) w_i(y)$$

and the following incentive constraints are satisfied for each $i$

$$(1 - \delta) g_i(\alpha) + \delta \sum_{y \in Y} \pi(y|\alpha) w_i(y) \geq (1 - \delta) g_i(a'_{i}, \alpha_{-i}) + \delta \sum_{y \in Y} \pi(y|a'_{i}, \alpha_{-i}) w_i(y) \quad \text{for all } a'_{i} \in A_i$$

Note that the continuation payoff profile $w(y)$ satisfies $\lambda \cdot w(y) \leq V^\lambda_0$ for all $y \in Y$ and $\lambda \in \mathbb{R}^n / \{0\}$. Given this, it is clear that the weighted social welfare level of any PPE in the direction of $\lambda$ is bounded by a solution to the following programming problem

$$\sup_{\alpha \in \times_{i=1}^n \Delta A_i, w:Y \rightarrow \mathbb{R}^n} (1 - \delta) \lambda \cdot g(\alpha) + \delta \lambda \cdot w(y)$$

$$\text{s.t. } (1 - \delta) g_i(\alpha) + \delta \sum_{y \in Y} \pi(y|\alpha) w_i(y) \geq (1 - \delta) g_i(a'_{i}, \alpha_{-i}) + \delta \sum_{y \in Y} \pi(y|a'_{i}, \alpha_{-i}) w_i(y) \quad \text{for all } a'_{i} \in A_i, \ i = 1, ..., n.$$ 

$$\lambda \cdot w(y) \leq V^\lambda_0 \quad \text{for all } y \in Y$$

This is because $w(y)$ can be chosen from the half space which includes the feasible set, while continuation payoffs have to be an equilibrium payoff profile for PPE.

Let $V^\lambda_1$ be a solution to this problem. A critical observation is that $w(y)$ itself is generated by PPE, thus also needs to be bounded by $V^\lambda_1$. Then I can
replace $V^\lambda_0$ with $V^\lambda_1$ in the above programming problem to obtain another bound $V^\lambda_2$ on PPE payoffs. This procedure can be repeated infinitely many times to obtain a decreasing sequence of bounds of PPE $\{V^\lambda_n\}_{n=0}^\infty$ for $\lambda$, which converges to some limit $\overline{V}^\lambda$. Quite nicely, it turns out that you need only one iteration to compute this limit.

**Theorem 1** $\{V^\lambda_n\}_{n=0}^\infty$ is a decreasing sequence and $\lim_{n \to \infty} V^\lambda_n = \overline{V}^\lambda$. Moreover, $\overline{V}^\lambda = V^\lambda_0 - \frac{1}{1-\delta} (V^\lambda_0 - V^\lambda_1)$

**Proof.** Let $\triangle = V^\lambda_0 - V^\lambda_1$. Then $V^\lambda_2 = V^\lambda_1 - \delta \triangle$ because simple translate $w_i - \frac{\triangle}{n}, i = 1, ..., n$ satisfy all the constraints and achieve the optimal solution. Thus $V^\lambda_1 - V^\lambda_2 = \delta \triangle$. In general, $V^\lambda_{n-1} - V^\lambda_n = \delta^{n-1} \triangle$. Then

$$V^\lambda_n = V^\lambda_0 - \left(V^\lambda_0 - V^\lambda_n\right)$$
$$= V^\lambda_0 - \sum_{k=1}^{n} \left(V^\lambda_{k-1} - V^\lambda_k\right)$$
$$= V^\lambda_0 - \sum_{k=1}^{n} \delta^{k-1} \triangle$$
$$= V^\lambda_0 - \frac{1 - \delta^n}{1 - \delta} \triangle$$

Thus $\lim_{n \to \infty} V^\lambda_n = \overline{V}^\lambda = V^\lambda_0 - \frac{1}{1-\delta} \triangle \blacksquare$

Note that this limit bound is identical to a solution to the linear programming problem studied by Fudenberg and Levine [?], that is, $\overline{V}^\lambda$ is the
maximized value of the following programming problem\(^1\)

\[
\sup_{\alpha \in \times_{i=1}^n \Delta A_i, \, v : Y \to \mathbb{R}^n} \lambda \cdot v
\]

\[
s.t. \quad v_i = (1 - \delta) g_i(\alpha) + \delta \sum_{y \in Y} \pi(y|\alpha) w_i(y) \quad \text{for} \quad i = 1, ..., n.
\]

\[
(1 - \delta) g_i(\alpha) + \delta \sum_{y \in Y} \pi(y|\alpha) w_i(y) \geq (1 - \delta) g_i(a_i, \alpha_{-i}) + \delta \sum_{y \in Y} \pi(y|a_i, \alpha_{-i}) w_i(y) \quad \text{for all} \quad a_i \in A_i, \quad i = 1, ..., n.
\]

\[
\lambda \cdot w(y) \leq \lambda \cdot v \quad \text{for all} \quad y \in Y.
\]

Since FL showed that the solution to this problem is independent of \(\delta \in (0, 1)\). We have the following corollary.

**Corollary 2** \(V_0^\lambda - \frac{1}{1-\delta} (V_0^\lambda - V_1^\lambda)\) is independent of \(\delta \in (0, 1)\).

Thus if \(\delta\) is large, then the difference between \(V_0\) and \(V_1\) gets smaller and these two effects cancel out.

This procedure to obtain an upper bound is based on the recursive property of PPE, thus does not work when recursiveness is lost. As already mentioned, Kandori and Obara [11] showed that public strategy is restrictive for this class of games and sometimes there is a Pareto-improving sequential equilibrium (private equilibrium) based on private strategies. The problem is that such private equilibrium is not recursive. In such equilibrium, players randomize over their actions and their continuation strategies depend on a realization of their actions. The same problem arises for repeated games with private monitoring, where each player observes a private signal of the other player’s action. Since private information is accumulated over time, the continuation game starting at period \(t\) is different from the continuation game starting at period \(t + 1\).

In the next section, I propose an approach which uses a slightly modified algorithm and overcomes this problem of non-recursiveness.

\(^1\)Let \(Q\) be the intersection of the half spaces \(\{v|\lambda \cdot v \leq \nabla^\lambda\}\) for all \(\lambda \neq 0\). Fudenberg and Levine [6] showed that this bound is tight in the sense that every payoff profile in \(\text{int}Q\) is achieved as \(\delta\) goes to 1 in PPE when \(Q\) is full dimensional.
2.2 Recursive Upper Bound, Private Information, and Communication Equilibria

2.2.1 Communication Equilibrium

Consider the following game. In period $t$, player $i$ receives a message $\omega_{i,t} \in \Omega_i$ from a mediator, chooses an action $a_{i,t}$, observes a private signal $s_{i,t} \in S_i$, and sends a message $m_{i,t} \in M_i$. The mediator’s message $\omega_t = (\omega_{1,t}, ..., \omega_{n,t}) \in \Omega$ depends on his past messages $\omega^t = (\omega_1, ..., \omega_{t-1})$ and the past message profiles $m^t = (m_1, ..., m_{t-1})$ from the players. Player $i$’s private history $h^t_i$ in the beginning of period $t$ is $(\omega^t_i, a^t_i, s^t_i, m^t_i)$. Player $i$’s period-$t$ action $a_{i,t}$ depends on $(h^t_i, \omega_{i,t})$ and player $i$’s message $m^t_i$ depends on $(h^t_i, \omega_{i,t}, a_{i,t}, s_{i,t})$.

Forges (1986) showed that there is no loss of generality to assume $\Omega_i = A_i$ and $M_i = S_i$. That is, a direct mechanism suffices for characterizing the communication equilibrium payoffs. Thus, in each period, the mediator just recommends each player which action to take and each player tells the mediator what he observed. To differentiate recommendations from actual actions or messages from true signals, I denote a recommendation by $\tilde{a}_i$ and a message by $\tilde{s}_i$. Let $\mu \in \Delta A$ be the correlated action the mediator chooses.

The joint distribution of the private signals depend on actions and given by $p(s|a)$. I assume that each private signal $s_i$ has the same support independent of $a$, otherwise I do not impose any assumption on the information structure. Note that imperfect public monitoring is a special case where $S_1 = ..., = S_n$ and all private signals are perfectly correlated.

The most important observation is that a certain recursiveness is recovered for a communication equilibrium. A continuation games after the first period is exactly like the original game (in $ex$ $ante$ sense) with additional private information from the first period. Since the mediator can embed such additional private information in the message space in the first period, the continuation equilibrium on the $equilibrium$ $path$ can be supported even in the original game with an appropriate message space: namely $\Omega' = \Omega \times A \times S$. This is because the mediator can mimic the distribution on $\Omega$ and the first period equilibrium outcome distribution on $A \times S$. On the other hand, this particular recursive property is weaker than the one for PPE. For PPE, a continuation strategy profile at any public history is an equilibrium profile of the original game. Here a continuation strategy profile after a deviation is not an equilibrium in the original communication game.
2.2.2 Recursive Programming

The weighted social surplus of a communication equilibrium can be represented as follows

\[ E^\mu [(1 - \delta) \lambda \cdot g(\bar{a}) + \delta \lambda \cdot w(\bar{a}, \bar{s})] \]

where \( w_i(a, s) \) is player \( i \)'s equilibrium continuation payoff when \( a \) is recommended and \( s \) is sent to the mediator. Since \( a = \bar{a} \) and \( s = \bar{s} \) in equilibrium, I use \((a, s)\) instead of \((\bar{a}, \bar{s})\) as long as it does not cause any confusion. The expectation is taken with respect to \( \mu(a)p(s|a) \), assuming that every player follows the mediator’s recommendation and sends a truthful message.

If player \( i \) is recommended to play \( a_i \), \( i \)'s incentive constraint is as follows

\[ E^\mu_i [(1 - \delta) g_i(a) + \delta w_i(a, s)|a_i] \geq E^\mu_i [(1 - \delta) g_i(a'_i, a_{-i}) + \delta w_i(a, \rho_i(s_i), s_{-i})|a'_i, a_i] \]

for all \( a'_i \in A \) and \( \rho: S_i \to S_i \).

LHS is the expected payoff when \( a_i \) is recommended and player \( i \) obeys, and RHS is the expected payoff when player \( i \) chooses \( a'_i \) in the current period even though \( a_i \) is recommended. \( w_i(a, \rho(s_i), s_{-i}) \) is player \( i \)'s continuation payoff when player \( i \) employs the continuation strategy which is supposed to be played after following the mediator’s recommendation \( a_i \) and observing \( \rho(s_i) \). Thus I am only considering one shot deviations. It is also possible to consider other type of deviations. Specifically, a player’s continuation strategy may be a continuation strategy after \( a'_i \neq a_i \) is played in the current period. I can allow that possibility by defining \( w_i \) as a function of both recommended action \( a_i \) and another action \( a'_i \) which could have been taken. However, such generality does not have any bite in the following programming problem, because such deviation can be easily deterred by setting \( w_i \) for which \( a_i \neq a'_i \) arbitrary small.

Next I introduce another constraint: recursive constraint. Let \( \Omega' \in A \times S \) be any common knowledge event in the current period given \( \mu \) and \( p \). For example, if the players are recommended to play a pure action profile \( a \), then \( a \times S \) is one such common knowledge event. Another example is when \( s \) is a public signal. Then, for each \( s = s_1 = \ldots = s_n, A \times s \) is a common knowledge event. Finally, \( A \times S \) is always a common knowledge event.

Note that the (ex-ante) continuation payoff from the second period given any such common knowledge event is also a communication equilibrium payoff. Thus, if there is any bound \( V^\lambda \) given \( \lambda \) on communication equilibrium, this bound also applies to continuation communication equilibria as well.
Hence I impose the following constraint for each common knowledge event $\Omega$:

$$E^\mu [\lambda \cdot w(a, s) | \Omega] = \sum_{(a, s) \in \Omega'} \frac{\mu(a) p(s|a)}{\Pr(\Omega)} \lambda \cdot w(a, s) = V^\lambda$$

Now consider the following recursive programming problem

$$V^\lambda_{t+1} = \sup_{\mu \in A, w: A \times S \to \mathbb{R}^n} E^\mu [(1 - \delta) g(a) + \delta \lambda \cdot w(a, s)]$$

s.t. $E^\mu [(1 - \delta) g_i(a) + \delta w_i(a, s)|a_i] \geq E^\mu [(1 - \delta) g(a') + \delta w(a, \rho(s_i), s_{-i})|a_{i}', a_i]$ for all $a_{i}', a_i \in A_i, \rho : S_i \to S_i$, $i = 1, ..., n$.

$$V^\lambda_t \geq E^\mu [\lambda \cdot w(a, s) | \Omega]$$

for any common knowledge event $\Omega \in A \times S$ given $\mu$ and for all $\lambda \in \mathbb{R}^n / \{0\}$.

Again I can set $V^\lambda_0$ to be a large number so that the recursive constraint is trivially satisfied for each $\lambda$. Then I can solve this problem for every $\lambda$ to obtain $V^\lambda_1$, which provides a new upper bound for continuation communication equilibria $E^\mu [\lambda \cdot w(a, s) | \Omega]$. Then $V^\lambda_2, V^\lambda_3, ...$ can be computed recursively. As before, this sequence is a decreasing sequence for each $\lambda$ and converging to some number $V^{\lambda,*}$, which is clearly a bound on communication equilibrium payoffs in the direction of $\lambda$. This $V^{\lambda,*}$ does depend on $\delta$ unlike $V^{\delta,*}$ for PPE. Since it increases in $\delta$, $\lim_{\delta \to 1} V^{\lambda,*} = V^{\lambda,*}$ provides a bound for communication equilibria independent of $\delta$. The following theorem summarizes these observations.

**Theorem 3** \( \left\{ V^\lambda_t \right\}_{t=0}^{\infty} \) is a decreasing sequence and $\lim_{n \to \infty} V^\lambda_t = V^{\lambda,*}$ for each $\lambda$. $V^{\lambda,*}$ is increasing in $\delta$. For any communication equilibrium payoff profile $w^* = (w_1^*, ..., w_n^*)$, $\lambda \cdot w^* \leq \lambda \cdot \lim_{\delta \to 1} V^{\lambda,*}$ holds for each $\lambda$ independent of $\delta$.

### 2.3 An Example

Consider the following partnership games with imperfect public monitoring. There are two players $i = 1, 2$. Each player takes an action $a_i$ from $\{C, D\}$.
and the payoff structure is assumed to be the one of Prisoners’ dilemma:

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>1, 1</td>
<td>−l, 1 + d</td>
</tr>
<tr>
<td>D</td>
<td>1 + d, −l</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

where \(d, l > 0\), and \(2 > 1 + d − l\) ((C, C) is efficient).

Each player receives a private signal \(s_i \in \{X, Y\}\). The distribution of \((s_1, s_2)\) satisfies,

\[
p(XX|a) = p(YY|a) = \frac{1}{2} \quad \text{for } a = (C, C) \text{ or } (D, D)
\]

and

\[
p(XY|a) = p(YX|a) = \frac{1}{2} \quad \text{for } a = (C, D) \text{ or } (D, C)
\]

Solving the programming problem recursively, I can obtain

\[
V^\delta = E\mu^\delta \left[ (1 - \delta) \sum_{i=1}^{2} g_i(a) + \delta \sum_{i=1}^{2} w^{\delta*}_i(a, s) \right]
\]

where \((\mu^{\delta*}, w^{\delta*}_i, i = 1, ..., n)\) is the solution in the limit.

Note first that, by the recursive constraint,

\[
V^\delta = E\mu^\delta \left[ (1 - \delta) \sum_{i=1}^{2} g_i(a) + \delta \sum_{i=1}^{2} w^{\delta*}_i(a, s) \right] \\
= E\mu^\delta \left[ (1 - \delta) \sum_{i=1}^{2} g_i(a) \right] + \delta E\mu^\delta \left[ \sum_{i=1}^{2} w^{\delta*}_i(a, s) \right] \\
\leq E\mu^\delta \left[ (1 - \delta) \sum_{i=1}^{2} g_i(a) \right] + V^{\delta*} \\
\Downarrow \\
V^{\delta*} \leq E\mu^{\delta*} \left[ \sum_{i=1}^{2} g_i(a) \right]
\]

Note also that \(\mu^{\delta*}(DD) = 1\) because there is no way to force players to play C. Each player can play D when recommended to play C while generating the identical private signal distribution independent of the other player’s action by always misrepresenting his private signal.

Therefore

\[V^{\delta*} \leq 0\]
3 Belief-Free Communication Equilibria

Now replace the recursive constraint

\[ V^\lambda_{t+1} \geq E_\mu \left[ (1-\delta) \lambda \cdot g(a) + \delta \lambda \cdot w(a, s) \right] \]

for any common knowledge event

\[ \Omega \in A \times S \text{ given } \mu \text{ and for all } \lambda \in \mathbb{R}^n / \{0\} \]

with a stronger condition:

\[ V^\lambda_{t+1} \geq \lambda \cdot w(a, s) \]

Thus each \( w(a, s) \) is subject to an upper bound.

I am implicitly considering a certain type of communication equilibrium where a continuation strategy profile is an equilibrium even if \((\vec{a}, \vec{s})\) is revealed to all the players in the end of each period. This communication equilibrium may be called belief-free communication equilibrium or ex-post communication equilibrium. This class of communication equilibrium is neither weaker than correlated equilibrium nor stronger than sequential equilibrium.

Now the programming problem is:

\[
V^\lambda_{t+1} = \sup_{\mu \in \Delta A} \sup_{w : A \times S \to \mathbb{R}^n} E_\mu \left[ (1-\delta) \lambda \cdot g(a) + \delta \lambda \cdot w(a, s) \right]
\]

s.t. \( E_\mu \left[ (1-\delta) g_i(a) + \delta w_i(a, s) \mid a_i \right] \geq E_\mu \left[ (1-\delta) g_i(a'_i, a_{-i}) + \delta w(a_i, \rho(s_i), s_{-i}) \mid a'_i, a_i \right] \)

for all \( a_i, a'_i \in A_i, \rho : S_i \to S_i, i = 1, ..., n \),

\[ V^\lambda_{t+1} \geq \lambda \cdot w(a, s) \]

\[ \Omega \in A \times S \text{ given } \mu \text{ and for all } \lambda \in \mathbb{R}^n / \{0\} \]

As before \( V^\lambda_{t+1} \downarrow V^\lambda \). This time \( V^\lambda \) is independent of \( \delta \). The bound obtained from this replaced constraint is exactly a bound on such belief-free communication equilibria.

Although a bound for communication equilibria is not tight, this bound for belief-free communication equilibria is tight in the following sense. Let \( W^\lambda = \{ w \in \mathbb{R}^n | V^\lambda \geq \lambda \cdot w \} \) be the half space, which consists of the payoff profiles bounded by \( V^\lambda \) in the direction of \( \lambda \). Then the following theorem is obtained.
Theorem 4  Any payoff profile in \( \text{int} \cap_{\lambda \in \mathbb{R}^n / \{0\}} W^\lambda \) is sustained by a belief-free communication equilibrium if players are enough patient and \( \cap_{\lambda \in \mathbb{R}^n / \{0\}} W^\lambda \) is full dimensional.

Proof. TBC. It is an application of Fudenberg and Levine [6]'s theorem.

I use this result to show that relaxing the notion of equilibrium is not without a cost. Even a restricted class of communication equilibria such as belief-free communication equilibria is enough to prove Folk theorem for many games, for which a folk theorem is not obtained in public strategies.

Example. Partnership Game

There are two players \( i = 1, 2 \). Each player takes an action \( a_i \) from \( \{C, D\} \) and the payoff structure is assumed to be the one of Prisoners’ dilemma:

\[
\begin{array}{c|cc}
 & C & D \\
\hline
C & 1, 1 & -l, 1 + d \\
D & 1 + d, -l & 0, 0 \\
\end{array}
\]

where \( d, l > 0 \), and \( 2 > 1 + d - l \) ((\( C, C \)) is efficient).

There is a public signal \( y \in \{g, b\} \). The distribution of \( y \) satisfies,

\[
\pi (b|CC) < \pi (b|DC) = \pi (b|CD) < \pi (b|DD)
\]

thus \( y = b \) is a bad signal.

Radner, Myerson and Maskin [14] showed that the efficient payoff profile \( (1, 1) \) is not sustainable with public strategies for this type of partnership games. However, a folk theorem can be obtained in belief-free communication equilibria.

Proposition 5  Folk Theorem holds for the above prisoner’s dilemma game.

This result follows from the previous theorem

Intuition. Here is how to support an almost efficient payoff profile. The mediator recommends \( (CC) \) with very high probability and \( (CD) \) or \( (DC) \) with a small probability. When \( (CC) \) is chosen, the continuation payoff does not depend on \( y \). When \( (DC) \) is chosen, player 1 serves as a monitor of player 2. In this case, player 2’s future payoff is transferred to player 1 when \( b \) is realized and vice versa for \( (DC) \). Player 2’s incentive constraint
is clearly satisfied. Player 1’s incentive constraint is also satisfied because player 1 is not sure whether player 2 is playing $C$ or $D$, and punished when $D$ is being played. Since the punishment is based on a transfer, it can be implemented without wasting any efficiency.

Note that when player 1 plays $C$ and observes $b$, he is not sure if he is punished (given $(CD)$) or forgiven (given $(CC)$). However, his continuation play is optimal independent of which have been the case.

References


