The Multiple-partners Assignment Game with Heterogeneous Sells and Multi-unit Demands: Competitive Equilibria

DANIEL JAUME\textsuperscript{†}, JORDI MASSÓ\textsuperscript{‡}, and ALEJANDRO NEME\textsuperscript{†}

December 2007

Abstract: A multiple-partners assignment game with heterogeneous sells and multi-unit demands consists of a set of sellers that own a given number of indivisible units of (potentially many different) goods and a set of buyers who value those units and want to buy at most an exogenously fixed number of units. We define a competitive equilibrium for this generalized assignment game and prove its existence using linear programming. We show that the set of competitive equilibria (pairs of price vectors and assignments) has a Cartesian product structure: each equilibrium price vector is part of a competitive equilibrium with all equilibrium assignments, and vice versa.

\textsuperscript{†}We thank Alejandro Manelli for his very helpful comments. The work of D. Jaume and A. Neme is partially supported by the Universidad Nacional de San Luis through grant 319502, by the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) through grant PICT-02114, and by the Agencia Nacional de Promoción Científica y Técnica through grants 03-10814 and PAV-008. The work of J. Massó is partially supported by the Spanish Ministry of Education and Science through grant EJ2005-01481/ECON and FEDER and project CONSOLIDER-INGENIO 2010 (CDS2006-00016), and by the Generalitat de Catalunya through grant SGR2005-00454 and the Barcelona Economics Program (CREA). All authors acknowledge financial support from the grant PCI España-Iberoamérica 2007 (Programa de Cooperación Interuniversitaria de la Agencia Española de Cooperación Internacional-AECI).

\textsuperscript{‡}Instituto de Matemática Aplicada de San Luis. Universidad Nacional de San Luis and CONICET. Ejército de los Andes 950. 5700, San Luis, Argentina. E-mails: djaume@unsl.edu.ar, and aneme@unsl.edu.ar

\textsuperscript{‡‡}Departament d’Economia i d’Història Econòmica and CODE. Universitat Autònoma de Barcelona. 08193, Bellaterra (Barcelona), Spain. E-mail: jordi.masso@uab.es
We also show that the set of equilibrium price vectors has a natural lattice structure and we study how this structure is translated into the set of agents’ utilities that are attainable at equilibrium.

Journal of Economic Literature Classification Numbers: C78; D78.

Keywords: Matching; Assignment game; Indivisible goods; Competitive equilibrium; Lattice.

1 Introduction

We study competitive equilibria of markets with indivisible goods. The multiple-partners assignment game with heterogeneous sells and multi-unit demands (a market) is a many-to-many assignment problem with transferable utility in which agents can be partitioned into two disjoint sets: the set of buyers and the set of sellers. In this market sellers deliver indivisible units of (potentially different) goods to buyers who pay a given amount of money for every unit of each good. Each seller owns a given number of indivisible units of each good and each buyer may buy different units of the goods up to an exogenously fixed number which comes from constraints on his capacity to transport, storage, etc. Each seller assigns a per-unit value (or reservation price) to each of the different goods that he owns. Each buyer assigns a valuation (or maximal willingness to pay) to each unit of the different goods; this means that his marginal utility of each of the goods is constant.

The primary market of blood (whole blood, red blood cells, white blood cells, platelets, plasma, and its products) is a prototypical example of these type of markets. Only in the USA this market moved around $8 billion in 2005 and it is expected to approach $9.9 billion in 2010, raising at an average annual growth rate of 8% (see Business Communications Company, 2005). There has been a constant increase on the need of blood all over the world due to an aging population and the changes in the incidence of diseases, surgical procedures and catastrophes that require transfusions. Since there are no (artificial) substitutes of the blood, it can only be obtained through human donation. To satisfy optimally the need of blood requires that around 3.5% of the population donates once a year. The role of the Blood Services in each country is not only to keep donation up but also to collect, prepare, and distribute blood products in a safety, efficient, and appropriate manner. Hospitals (buyers in our model) use the different types of blood which are collected by blood banks (sellers in our model). The red blood cells is the most important component of the blood. It is storage in plastic bags of one unit of red blood cells (approximately half liter); thus,
units to be interchanged are indivisible. Each blood bank has available a certain quantity of units of red blood cells of each of the blood types A+, A−, B+, B−, O+, O−, AB+, and AB− (goods in our model). Each hospital can only maintain cooled a maximum number of units of red blood cells, independently of the blood type. Reservation prices may differ across blood banks and blood types, and values may differ across hospitals and blood types as well; geographic and demographic reasons justify these heterogeneities. For instance, according to Engber (2006), the same unit of red blood cells might cost $220 in Los Angeles (California) but only $150 in Des Moines (Iowa).1

There are many other assignment problems with these characteristics. They are many-to-many because each agent can be assigned to (i.e., perform a transaction with) many agents of the other side of the market. They have transferable utility because money may be used as a means of exchange. They are heterogeneous because a unit of a particular good may be different from a unit of another good. They are multi-demanded because buyers may be willing to buy several units of different goods. The main two questions to be answered are the following: Given an initial distribution of units of the goods among all sellers, what is their optimal assignment to buyers?, and what are the prices (if any) that would clear the market?

Given a market, an assignment is a description of how many units of each of the goods are interchanged between every pair formed by a buyer and a seller. An assignment is feasible if it satisfies the quantity and capacity constraints of all agents. A feasible assignment is optimal if it maximizes the total net value (the sum of the valuations minus the reserve price of all interchanged units). It turns out that the set of optimal assignments of a market can be identified with the set of solutions of a Primal Linear Problem where the objective function (to be maximized) is the total net value, which depends linearly on the assignment, subject to non-negativity constraints and to feasibility constraints. Results on linear programming (see Schrijer, 1996) guarantee that the Primal Linear Program has at least one solution, since the set of all real-valued solutions is a polytope whose vertices have all integer-valued coordinates.

To choose an optimal assignment requires information about valuations, reservation prices, and quantity and capacity constraints. Hence, competitive markets may emerge (or be used) as a way of selecting an optimal assignment with low informational requirements. We will assume that buyers and sellers interchange units of the goods with money through competitive markets in which a price vector (a non-negative price for each good)

1 In Iowa, 12% of the population donated in 2006 compared to a national average of 3%.
is announced. Given the price vector, each seller determines the optimal number of units he wants to sell of each of the goods he owns and each buyer determines the optimal number of units he wants to buy of each good, without exceeding his capacity constraint. A price vector $p$ is an equilibrium price vector of the market if the plans of all sellers and buyers are compatible at $p$; namely, the market of each good clears in the sense that all optimal plans constitute a feasible and compatible set of interchanges (they are a feasible assignment). In this case we say that the equilibrium price vector and the feasible assignment are compatible. A competitive equilibrium of the market is a pair formed by an equilibrium price vector and a compatible assignment. We show using well-known duality theorems that each market has at least a competitive equilibrium. First, we observe that the Dual Linear Problem associated to the Primal Linear Program has a non-empty set of solutions; second, we give a procedure to construct an equilibrium price vector from a given solution of the Dual Linear Program; and third, we show that any optimal solution of the Primal Linear Program is compatible with this equilibrium price vector. Thus, the set of competitive equilibria of a market is intimately related to the set of solutions of the Primal Linear Program (compatible optimal assignments) and the Dual Linear Program (equilibrium price vectors).\[^2\]

We show that the set of competitive equilibria of a market has a Cartesian product structure: each equilibrium price vector is compatible with all optimal assignments and each optimal assignment is compatible with all equilibrium price vectors. We also show that the set of equilibrium price vectors has a lattice structure with the natural order of vectors $\geq$ (a reflexive, transitive, antisymmetric, and incomplete binary relation) on the $n$-dimensional Euclidian space, where $n$ is the number of goods and given two vectors $x, y \in \mathbb{R}^n$, $x \geq y$ if and only if $x_j \geq y_j$ for all $j = 1, \ldots, n$. Moreover, and as a consequence of this lattice structure, the set of equilibrium price vectors contains two extreme elements: the sellers-optimal equilibrium price vector with each component being larger or equal to the corresponding component of all other equilibrium price vectors and the buyers-optimal equilibrium price vector with each component being smaller or equal to the corresponding component of all other equilibrium price vectors. We observe that, in contrast to the Shapley and Shubik (1972)’s assignment game, this natural order $\geq$ does not translate into the set of utilities of buyers (nor the set of utilities of sellers) that can be attainable.

\[^2\]Shapley and Shubik (1972) already pointed out the relationships among the set of competitive equilibria of an assignment game, the core of its associated TU-game, and the solutions of the corresponding primal and dual linear problems.
at equilibrium. Partly, this is because there is a insubstantial multiplicity of equilibrium prices of the goods that are not interchanged in any equilibrium assignment. We solve this multiplicity by defining the set of restricted equilibrium price vectors as those equilibrium price vectors for which the price of the goods that are never interchanged in equilibrium is equal to their maximal one without altering the equilibrium property of the full price vector. Then, we show that the set of utilities of buyers that are attainable at equilibrium embeds the lattice structure of the set of restricted equilibrium price vectors. However, we also show that the set of utilities of the sellers that are attainable at equilibrium does not inherit this structure.

There are several papers that have studied generalized versions of Shapley and Shubik (1972)'s one-to-one assignment game. Camiña (2006) and Sotomayor (1999, 2002, 2003, and 2007) are some of them. However, most of the emphasis of this literature has been put on the study of alternative cooperative solutions of the associated TU-game, although Camiña (2006) and Sotomayor (2007) also study the competitive equilibria of their generalized assignment games. See the beginning of Section 4 for a description of this very related literature as well as its connections with our model and results.

The paper is organized as follows. In Section 2 we define the multiple-partners assignment game with heterogeneous sells and multi-unit demands, optimal assignments, and its associated Primal Linear Program. In Section 3 we present the notion of a competitive equilibrium and show its existence by using duality theorems of Linear Programming. In Section 4 we study the structure of the set of competitive equilibria by showing that it is a Cartesian product of the set of equilibrium price vectors times the set of optimal assignments and that the set of equilibrium price vectors has a lattice structure with the natural partial order $\geq$; we also show how this partial order (on the set of restricted equilibrium price vectors) gives a lattice structure to the set of utilities of the buyers that are attainable at equilibrium. In Section 5 we conclude with some final remarks.

2 Preliminaries

The multiple-partners assignment game with heterogeneous sells and multi-unit demands (a market) consists of seven objects. The first three are three finite and disjoint sets. The set of $m$ buyers $B = \{b_1, ..., b_m\}$, the set of $n$ type of goods $G = \{g_1, ..., g_n\}$, and the set of $t$ sellers $S = \{s_1, ..., s_t\}$. We identify buyer $b_i$ with $i$, good $g_j$ with $j$, and seller $s_k$ with $k$.

For each buyer $i \in B$ and each good $j \in G$, let $v_{ij} \geq 0$ be the monetary valuation that
buyer $i$ assigns to good $j$; namely, $v_{ij}$ is the maximum price that buyer $i$ is willing to pay for each unit of good $j$. We denote by $V = (v_{ij})_{(i,j)\in B\times G}$ the matrix of valuations. Each buyer $i \in B$ can buy at most $d_i > 0$ units in total. We are assuming that buyers have a constant marginal valuation of each unit of each good and that they are constrained on their total demand. The amount $d_i$ should be interpreted as a capacity constraint of buyer $i$ due to limits on his ability to storage, transport, etc. We denote by $d = (d_i)_{i \in B}$ the vector of maximal demands.

For each good $j \in G$ and each seller $k \in S$, let $r_{jk} \geq 0$ be the monetary valuation that seller $k$ assigns to good $j$; namely, $r_{jk}$ is the reservation (or minimum) price that seller $k$ is willing to accept for each unit of good $j$. We denote by $R = (r_{jk})_{(j,k)\in G\times S}$ the matrix of reservation prices. Each seller $k \in S$ has a given number $q_{jk} \in \mathbb{Z}_+$ of indivisible units of each good $j \in G$, where $\mathbb{Z}_+$ is the set of non-negative integers. We denote by $Q = (q_{jk})_{(j,k)\in G\times S}$ the capacity matrix. Observe that we are admitting the possibility that seller $k$ may have zero units of some of the goods. However, we require that the reservation price for buyer $k$ of a good that he has no units to sell has to be equal to zero; namely, for all $k \in S$ and all $j \in G$,

$$\text{if } q_{jk} = 0 \text{ then } r_{jk} = 0.$$  

(1)

Moreover, we assume that there is a strictly amount of each good; namely,

$$\text{for each } j \in G \text{ there exists } k \in S \text{ such that } q_{jk} > 0.$$  

(2)

A market $M$ is a 7-tuple $(B, G, S, V, d, R, Q)$. An assignment for market $M$ is a matrix $A = (A_{ijk})_{(i,j,k)\in B\times G\times S} \in \mathbb{Z}_+^{m \times n \times t}$. Given an assignment $A$, each $A_{ijk}$ should be interpreted as follows: buyer $i$ receives $A_{ijk}$ units of good $j$ from seller $k$. When no confusion can arise, we omit the sets to which the subscripts belong to and write, for instance, $\sum_{i,j,k} A_{ijk}$ and $\sum_i A_{ijk}$ instead of $\sum_{(i,j,k)\in B\times G\times S} A_{ijk}$ and $\sum_{i\in B} A_{ijk}$, respectively. We are only interested on assignments satisfying all demand and supply restrictions of feasibility.

**Definition 1** The assignment $A$ is feasible for market $M$ if the following two sets of inequalities hold:

(Demand Feasibility) For all $i \in B$, $\sum_j A_{ijk} \leq d_i$.

(Supply Feasibility) For all $(j, k) \in G \times S$, $\sum_i A_{ijk} \leq q_{jk}$.

The inequality in (Demand Feasibility) says that each buyer $i$ buys, in total, at most $d_i$ units of all goods while the inequality in (Supply Feasibility) says that each seller $k$ sells at
most $q_{jk}$ units of each good $j$. We denote by $F$ the set of all feasible assignments of market $M$.

For each $(i, j, k) \in B \times G \times S$, let

$$
\tau_{ijk} = \begin{cases} 
    v_{ij} - r_{jk} & \text{if } q_{jk} > 0 \\
    0 & \text{if } q_{jk} = 0 
\end{cases}
$$

be the per unit gain from the trade of good $j$ between buyer $i$ and seller $k$; observe that if seller $k$ does not have any unit of good $j$ the per unit gain from trade of good $j$ with all buyers is equal to zero. Let $M$ be a market and $A \in F$ be a feasible assignment. We define the total gain from trade of market $M$ at assignment $A$ as

$$
T(A) = \sum_{ijk} \tau_{ijk} \cdot A_{ijk}.
$$

**Definition 2** A feasible assignment $A^*$ is optimal for market $M$ if $T(A^*) \geq T(A)$ holds for any feasible assignment $A \in F$.

We denote by $F^*$ the set of all optimal assignments for market $M$. Before proceeding it seems useful to consider an example.

**Example 1** Let $M = (B, S, G, V, d, R, Q)$ be a market where $B = \{b_1, b_2\}$, $S = \{s_1, s_2\}$, $G = \{g_1, g_2\}$, $V = \begin{pmatrix} 30 & 15 \\ 10 & 20 \end{pmatrix}$, $d = (20, 15)$, $R = \begin{pmatrix} 5 & 4 \\ 3 & 6 \end{pmatrix}$, and $Q = \begin{pmatrix} 10 & 8 \\ 7 & 9 \end{pmatrix}$.

Consider the feasible assignment $A = (A_{ijk})_{(i,j,k) \in B \times G \times S}$ for market $M$, where

- $A_{111} = 10$: buyer 1 receives 10 units of good 1 from seller 1,
- $A_{112} = 8$: buyer 1 receives 8 units of good 1 from seller 2,
- $A_{121} = 1$: buyer 1 receives 1 unit of good 2 from seller 1,
- $A_{122} = 0$: buyer 1 does not receive any unit of good 2 from seller 2,
- $A_{211} = 0$: buyer 2 does not receive any unit of good 1 from seller 1,
- $A_{212} = 0$: buyer 2 does not receive any unit of good 1 from seller 2,
- $A_{221} = 6$: buyer 2 receives 6 units of good 2 from seller 1,
- $A_{222} = 9$: buyer 2 receives 9 units of good 2 from seller 2.

The total gain from trade at assignment $A$ is $T(A) = (30 - 5) \cdot 10 + (30 - 4) \cdot 8 + (15 - 3) \cdot 1 + (20 - 3) \cdot 6 + (20 - 6) \cdot 9 = 698$.

In order to find the set of optimal assignments for market $M$ we consider the following Primal Linear Program (PLP).
Primal linear Program (PLP): Given the market $M = (B, S, G, V, d, R, Q)$, choose $A \in F$ in order to

$$
\max_{A \in F} T(A).
$$

Note that (4) is a compact way of writing

$$
\max_{(A_{i,j,k})_{(i,j,k) \in B \times G \times S \in Z^{m \times n \times t}} \sum_{i,j,k} \tau_{i,j,k} \cdot A_{i,j,k}
\text{ s. t.} \ (P.1) \ \sum_{j,k} A_{i,j,k} \leq d_i \ \text{ for all } i \in B,
(P.2) \ \sum_{i} A_{i,j,k} \leq q_{j,k} \ \text{ for all } (j, k) \in G \times S,
(P.3) \ A_{i,j,k} \geq 0 \ \text{ for all } (i, j, k) \in B \times G \times S.
$$

Remark 1 Results in integer programming guarantee that the set of solutions of the (PLP) is non-empty (see Schrijer, 1996); namely, $F^* \neq \emptyset$.

Observe that the assignment $A$ of Example 1 is optimal; i.e., $A \in F^*$.

3 Competitive Equilibria

3.1 Definition and Existence

We consider the situation where buyers and sellers trade through competitive markets. That is, there is a unique market (and its corresponding unique price) for each of the goods. Hence, a price vector is an $n$–dimensional vector of non-negative real numbers. Buyers and sellers are price-takers: given a price vector $p = (p_j)_{j \in G} \in \mathbb{R}^n_+$, sellers supply units of the goods (up to their capacity) in order to maximize revenues at $p$ and buyers demand units of the goods (up to their maximal demands) in order to maximize the total net valuation at $p$.

Supply of seller $k$: For each price vector $p = (p_j)_{j \in G} \in \mathbb{R}^n_+$, seller $k$ supplies of every good $j$ any feasible amount that maximizes revenues; namely,

$$
S_{j,k}(p_j) = \begin{cases} 
\{q_j\} & \text{if } p_j > r_{j,k} \\
\{0, 1, ..., q_j\} & \text{if } p_j = r_{j,k} \\
\{0\} & \text{if } p_j < r_{j,k}.
\end{cases}
$$

(5)
To define the demands of buyers we need the following notation. Let \( p \in \mathbb{R}^n_+ \) be given and consider buyer \( i \). Let

\[
\nabla_i^>(p) = \{ j \in G \mid v_{ij} - p_j = \max_{j' \in G} \{ v_{ij'} - p_{j'} \} > 0 \}
\]

be the set of goods that give to buyer \( i \) the maximum (and strictly positive) net valuation at \( p \). Obviously, for some \( p \), the set \( \nabla_i^>(p) \) may be empty. Let

\[
\nabla_i^\geq(p) = \{ j \in G \mid v_{ij} - p_j = \max\{0, \max_{j' \in G} \{ v_{ij'} - p_{j'} \} \} \}
\]

be the set of goods that give to buyer \( i \) the maximum (and non-negative) net valuation at \( p \). Obviously, for some \( p \), the set \( \nabla_i^\geq(p) \) may also be empty. Moreover, it is immediate to see that, for all \( i \in B \), the following remark holds.

**Remark 2** For all \( p \in \mathbb{R}^n_+ \), \( \nabla_i^>(p) \subseteq \nabla_i^\geq(p) \).

**Demand of buyer \( i \):** For each price vector \( p = (p_j)_{j \in G} \in \mathbb{R}^n_+ \) buyer \( i \) demands any feasible amounts of the goods that maximize the net valuations at \( p \); namely,

\[
D_i(p) = \{ \alpha = (\alpha_{jk})_{(j,k) \in G \times S} \in \mathbb{Z}^{n \times t} \mid \ 
\begin{align*}
(D.a) & \forall (j, k) \in G \times S, \ \alpha_{jk} \geq 0, \\
(D.b) & \sum_{jk} \alpha_{jk} \leq d_i, \\
(D.c) & \nabla_i^>(p) \neq \emptyset \implies \sum_{jk} \alpha_{jk} = d_i, \text{ and} \\
(D.d) & \sum_{k} \alpha_{jk} > 0 \implies j \in \nabla_i^\geq(p) \}\}
\]

Thus, \( D_i(p) \) describes the set of all trades that maximize the net valuation of buyer \( i \) at \( p \). Observe that the set of trades described by each element in the set \( D_i(p) \) give the same net valuation to buyer \( i \); i.e., \( i \) is indifferent among all trade plans specified by each \( \alpha \in D_i(p) \).

Let \( A \) be an assignment and let \( i \) be a buyer. We denote by \( A(i) = (A(i)_{jk})_{(j,k) \in G \times S} \) the element in \( \mathbb{Z}^{n \times t} \) such that, for all \( (j,k) \in G \times S \), \( A(i)_{jk} = A_{ijk} \).

**Definition 3** A price vector \( p = (p_j)_{j \in G} \in \mathbb{R}^n_+ \) is an **equilibrium price vector** of market \( M \) if there exists a feasible assignment \( A \in F \) such that:

(E.D) For each buyer \( i \in B \), \( A(i) \in D_i(p) \).

(E.S) For each good \( j \in G \) and each seller \( k \in S \), \( \sum_i A_{ijk} \in S_{jk}(p_j) \).

We say that an equilibrium price vector \( p \) and a feasible assignment \( A \) are **compatible** if they satisfy conditions (E.D) and (E.S). An **equilibrium** of market \( M \) is a pair \( (p, A) \in \mathbb{R}^n_+ \times \mathbb{Z}^{n \times n \times t}_+ \) where \( p \) is an equilibrium price vector of market \( M \) and \( A \) is one of its compatible assignments.
It is immediate to see that the following remark holds.

**Remark 3** Let $A \in F$ be a feasible assignment. Then, for all $i \in N$ and all $p \in \mathbb{R}_+^n$, $A(i)$ satisfies conditions (D.a) and (D.b) in the definition of the set $D_i(p)$.

Let $P^*$ be the set of equilibrium price vectors of market $M$. Theorem 1 below states that the set $P^*$ is always non-empty.

**Theorem 1** For every market $M$, $P^* \neq \emptyset$.

The proof of Theorem 1 is based on the fact that the Dual Linear Program (DLP) associated to the (PLP) has at least one solution. Thus, before proving Theorem 1 we present the (DLP).

### 3.2 The Dual Linear Program

In this subsection we present the Dual Linear Program (DLP) and state two well-known theorems of linear programming: the Strong Duality Theorem (Theorem 2) and the Complementary Slackness Theorem (Theorem 3). Using these two theorems we will show in Proposition 1 that there exists a strong link between the set of equilibria and the set of solutions of the (PLP) and the (DLP). Finally, in Subsection 3.3 we will prove Theorem 1.

Let $\gamma = (\gamma_i)_{i \in B} \in \mathbb{R}^m$ be an $m$–dimensional vector and $\pi = (\pi_{jk})_{(j,k) \in G \times S} \in \mathbb{R}^{n \times t}$ be a $(n \times t)$–matrix.

**Dual Linear Program (DLP):** Given the market $M = (B,S,G,V,d,R,Q)$, choose $(\gamma, \pi) \in \mathbb{R}^m \times \mathbb{R}^{n \times t}$ in order to

$$
\min_{(\gamma,\pi) \in \mathbb{R}^m \times \mathbb{R}^{n \times t}} \sum_i d_i \cdot \gamma_i + \sum_{jk} q_{jk} \cdot \pi_{jk}
$$

**s. t.**

\[ (D.1) \quad \gamma_i + \pi_{jk} \geq \tau_{ijk} \quad \text{for all } (i,j,k) \in B \times G \times S, \]

\[ (D.2) \quad \gamma_i \geq 0 \quad \text{for all } i \in B, \]

\[ (D.3) \quad \pi_{jk} \geq 0 \quad \text{for all } (j,k) \in G \times S. \]

Let $D$ be the set of dual feasible solutions (i.e., the set of vectors $\gamma \in \mathbb{R}^m$ and matrices $\pi \in \mathbb{R}^{n \times t}$ that satisfy conditions (D.1), (D.2), and (D.3)), and let $D^*$ be the set of solutions of the (DLP).

**Remark 4** Results in linear programming guarantee that the (DLP) has at least a solution (see Schrijer, 1996); namely, $D^* \neq \emptyset$. 

9
Remark 5  Let \((\gamma, \pi) \in D^*\) and assume that \(q_{jk} = 0\) for some \((j, k) \in G \times S\). Let \(\pi'_{jk} \geq 0\) be arbitrary. Define \((\pi_{-jk}, \pi'_{jk})\) as the \((n \times t)\)-matrix obtained from \(\pi\) after replacing \(\pi_{jk}\) by \(\pi'_{jk}\). Then, \((\gamma, (\pi_{-jk}, \pi'_{jk})) \in D^*\); that is, the value of the \(jk\)-th entry of \(\pi\) is irrelevant. Hence, we assume without loss of generality that \(\pi_{jk} = 0\) whenever \(q_{jk} = 0\).

Let \(M\) be a market and \((\gamma, \pi) \in D\) be a dual feasible solution. We write \(T^d(\gamma, \pi)\) to denote the value of the objective function of the (DLP) at \((\gamma, \pi)\); that is,

\[ T^d(\gamma, \pi) = \sum_i d_i \cdot \gamma_i + \sum_{jk} q_{jk} \cdot \pi_{jk}. \]

The Strong Duality Theorem (SDT) of Linear Programming applied to our setting says the following (see Dantzig, 1963).

**Theorem 2 (SDT)**  Let \(M\) be a market and assume \(A \in F\) and \((\gamma, \pi) \in D\). Then, \(A \in F^*\) and \((\gamma, \pi) \in D^*\) if and only if \(T(A) = T^d(\gamma, \pi)\). (8)

The Complementary Slackness Theorem (CST) of Linear Programming (see Schrijer, 1996) says that if a restriction is not binding then the corresponding variable has to be equal to zero and if a variable is not equal to zero then its corresponding restriction has to be binding. Hence, applied to our setting the Complementary Slackness Theorem says the following.

**Theorem 3 (CST)**  Let \(M\) be a market. Then, for all \(A \in F^*\) and \((\gamma, \pi) \in D^*\), the following properties hold:

(CS.1)  For all \((i, j, k) \in B \times G \times S\), \(A_{ijk} \cdot (\gamma_i + \pi_{jk} - \tau_{ijk}) = 0\).

(CS.2)  For all \(i \in B\), \((\sum_{jk} A_{ijk} - d_i) \cdot \gamma_i = 0\).

(CS.3)  For all \((j, k) \in G \times S\), \((\sum_i A_{ijk} - q_{jk}) \cdot \pi_{jk} = 0\).

Proposition 1 below says that the set of equilibria (pairs of equilibrium price vectors and compatible assignments) is strongly related to the set of solutions of the two Linear Programs. In order to state and prove it, we need to relate price vectors with dual solutions.

Define the mappings \(\gamma(\cdot) : \mathbb{R}^n_+ \to \mathbb{R}^n_+\) and \(\pi(\cdot) : \mathbb{R}^n_+ \to \mathbb{R}^{n \times t}_+\) as follows. Let \(p \in \mathbb{R}^n_+\) be given. For each \(i \in B\), define

\[ \gamma_i(p) = \begin{cases} v_{ij} - p_j & \text{if there exists } j \in \nabla_i^+(p) \\ 0 & \text{otherwise,} \end{cases} \] (9)
and for each \((j, k) \in G \times S\), define

\[
\pi_{jk}(p) = \begin{cases} 
  p_j - r_{jk} & \text{if } p_j - r_{jk} > 0 \\
  0 & \text{otherwise.}
\end{cases}
\]  

(10)

The number \(\gamma_i(p)\) is the gain obtained by buyer \(i\) from each unit that he wants to buy at \(p\) (if any) and the number \(\pi_{jk}(p)\) is the profit obtained by seller \(k\) from each unit of good \(j\) that he wants to sell at \(p\) (if any).

**Proposition 1** Let \(M\) be a market and let \(p \in \mathbb{R}_+^n\) be a price vector. The following two statements hold.

1. Assume \(p \in P^*\). Then, \(A \in F^*\) if and only if \(p\) and \(A\) are compatible.
2. \(p \in P^*\) if and only if \((\gamma(p), \pi(p)) \in D^*\).

**Proof** The statements of Proposition 1 will follow from Lemmata 2, 3, 4, and 5 below. We start by stating and proving a lemma that will be used in the proofs of Lemmata 4 and 5.

**Lemma 1** Assume \((\gamma(p), \pi(p)) \in D^*\) and \(A \in F^*\). Then, \(p\) and \(A\) are compatible.

**Proof of Lemma 1** Assume \(p \in \mathbb{R}_+^n\) is such that \((\gamma(p), \pi(p)) \in D^*\) and \(A \in F^*\). To show that \(p\) and \(A\) are compatible, we first show that for all \(i \in B\), \(A(i) = D_i(p)\). Since \(A\) is feasible, \((D.a)\) and \((D.b)\) hold by Remark 3. To show that \((D.c)\) holds, assume \(\nabla^2_i(p) \neq 0\). Then, \(v_{ij} - p_j > 0\) for some \(j \in G\). By definition, \(\gamma_i(p) > 0\). By condition \((CS.2)\) in Theorem 3, \(\sum_{jk} A_{ijk} = d_i\); namely, condition \((D.c)\) in the definition of \(D_i(p)\) holds.

To show that \((D.d)\) holds, fix \((i, j) \in B \times G\) and assume \(\sum_k A_{ijk} > 0\). We want to show that \(j \in \nabla^2_i(p)\). Since \(\sum_k A_{ijk} > 0\), there exists a seller \(k \in S\) such that \(A_{ijk} > 0\). Thus, \(q_{jk} > 0\) holds. Moreover, by condition \((CS.1)\) in Theorem 3, \(\gamma_i(p) + \pi_{jk}(p) = \tau_{ijk}\). Since, by \((3)\), \(q_{jk} > 0\) implies \(\tau_{ijk} = v_{ij} - r_{jk}\), we have

\[
\gamma_i(p) + \pi_{jk}(p) + r_{jk} = v_{ij}.
\]  

(11)

We distinguish between the following two cases.

**Case 1:** \(p_j - r_{jk} \geq 0\). Then, \(\pi_{jk}(p) = p_j - r_{jk} \geq 0\). By \((11)\), \(\gamma_i(p) = v_{ij} - p_j\). If \(\gamma_i(p) = v_{ij} - p_j > 0\) then \(j \in \nabla^2_i(p)\). By Remark 2, \(j \in \nabla^2_i(p)\). If \(\gamma_i(p) = v_{ij} - p_j = 0\) then \(\nabla^2_i(p) = \emptyset\). Hence, for all \((j', k') \in G \times S\), \(0 \geq v_{ij'} - p_{j'}\). Thus, \(j \in \nabla^2_i(p)\).

**Case 2:** \(p_j - r_{jk} < 0\). Then, \(\pi_{jk}(p) = 0\). By \((11)\), \(\gamma_i(p) + r_{jk} = v_{ij}\). Hence, \(\gamma_i(p) + p_j < v_{ij}\). Thus, \(\gamma_i(p) < v_{ij} - p_j\). Hence, by definition of \(\gamma_i(p)\), there exists \(j' \in \nabla^2_i(p)\) such that \(\gamma_i(p) = v_{ij'} - p_{j'} < v_{ij} - p_j\), but this is impossible (i.e., Case 2 never occurs).
Hence, condition (D.d) holds for $i \in B$. Thus, $A(i) \in D_i(p)$ for all $i \in B$.

We want to show now that, for all $(j, k) \in G \times S$, $\sum_i A_{ijk} \in S_{jk}(p_j)$ holds. Fix $(j, k) \in G \times S$. Since $A$ is feasible, $0 \leq \sum_i A_{ijk} \leq q_{jk}$. Assume $p_j = r_{jk}$. Then, $\sum_i A_{ijk} \in S_{jk}(p_j)$ holds trivially. Assume $p_j > r_{jk}$. Then, $\pi_{jk}(p) = p_j - r_{jk} > 0$. By condition (CS.3) in Theorem 3, $\sum_i A_{ijk} = q_{jk}$. Thus, $\sum_i A_{ijk} \in S_{jk}(p_j) = \{q_{jk}\}$. Finally, assume $p_j < r_{jk}$. Then, $\pi_{jk}(p) = 0$ and $S_{jk}(p_j) = \{0\}$. Suppose $A_{ijk} > 0$. Then, $q_{jk} > 0$. By condition (CS.1) in Theorem 3, $\gamma_i(p) + \pi_{jk}(p) = \tau_{ijk} = v_{ij} - r_{jk} \geq 0$. Since $p_j < r_{jk}$,

$$v_{ij} - p_j > v_{ij} - r_{jk} = \gamma_i(p) \geq 0,$$

a contradiction with the definition of $\gamma_i(p)$. Thus, for all $i \in B$, $A_{ijk} = 0$ and $\sum_i A_{ijk} = 0 \in S_{jk}(p_{jk}) = \{0\}$.

**LEMMA 2  [\iff of (1.1)]** Assume $p \in P^*$ and $A \in F$ are compatible. Then, $A \in F^*$.

**PROOF OF LEMMA 2** Let $p \in P^*$ and $A \in F$ be compatible. We show that $A \in F^*$ in two steps. We first show in Claim 1 that $(\gamma(p), \pi(p)) \in D$. Then, we show in Claim 2 that $T(A) = T^d(\gamma(p), \pi(p))$, and hence, by Theorem 2, $A$ is a solution of the (PLP).

**CLAIM 1** $(\gamma(p), \pi(p)) \in D$.

**PROOF OF CLAIM 1** By their definitions, $\gamma_i(p) \geq 0$ for all $i \in B$ and $\pi_{jk}(p) \geq 0$ for all $(j, k) \in G \times S$; namely, restrictions (D.2) and (D.3) of the (DLP) hold. To show that, for all $(i, j, k) \in B \times G \times S$,

$$\gamma_i(p) + \pi_{jk}(p) \geq \tau_{ijk} \tag{12}$$

holds, fix $i \in B$ and assume first that $\gamma_i(p) = 0$. Then, $v_{ij} - p_j \leq 0$ for all $j \in G$. If $q_{jk} > 0$ then, by (3), $\tau_{ijk} = v_{ij} - r_{jk} \leq p_j - r_{jk} \leq \pi_{jk}(p)$. Thus, since $\gamma_i(p) = 0$, (12) holds. If $q_{jk} = 0$ then, by (3), $\tau_{ijk} = 0$. Thus, by definition of $\pi_{jk}(p)$ and since $\gamma_i(p) = 0$, (12) holds. Hence, if $\gamma_i(p) = 0$ then (12) holds.

Assume now $\gamma_i(p) > 0$. Then, there exists $j \in \nabla_i^>(p)$ such that $\gamma_i(p) = v_{ij} - p_j > 0$. By definition of $\nabla_i^>(p)$, for all $(j', k') \in G \times S$,

$$v_{ij} - p_j + \pi_{j'k'}(p) \geq v_{ij'} - p_{j'} + \pi_{j'k'}(p) \geq v_{ij'} - p_{j'} + p_j - r_{j'k'} = v_{ij'} - r_{j'k'}.$$

If $q_{j'k'} > 0$ then, by (3), $\tau_{ij'k'} = v_{ij'} - r_{j'k'}$ and hence, $v_{ij} - p_j + \pi_{j'k'}(p) \geq \tau_{ij'k'}$. If $q_{j'k'} = 0$ then $\tau_{ij'k'} = 0$, and since $v_{ij} - p_j > 0$ and $\pi_{j'k'}(p) \geq 0$, $v_{ij} - p_j + \pi_{j'k'}(p) \geq \tau_{ij'k'}$ holds as
well. Thus, for all \((i, j', k') \in B \times G \times S\), \(\gamma_i(p) + \pi_{j'k'}(p) \geq \tau_{ij'k'}\). Hence, (12) holds as well when \(\gamma_i(p) > 0\). Thus, \((\gamma(p), \pi(p)) \in D\). This ends the proof of Claim 1. \hfill \Box

**Claim 2:** \(T(A) = T_d(\gamma(p), \pi(p))\).

**Proof of Claim 2:** By definition,

\[
T(A) = \sum_{ijk} \tau_{ijk} \cdot A_{ijk}. \tag{13}
\]

Condition (E.D) in the definition of an equilibrium price vector implies that, for every \(i \in B, A(i) \in D_i(p)\). Fix \((i, j, k) \in B \times G \times S\) and assume \(A_{ijk} > 0\). By condition (D.d) in the definition of \(D_i(p)\), \(j \in \nabla_i^z(p)\). Observe that \(A_{ijk} > 0\) implies \(q_{jk} > 0\). Thus, by (3),

\[
\tau_{ijk} = v_{ij} - r_{jk}. \tag{14}
\]

If \(\nabla_i^z(p) \neq \emptyset\) then \(j \in \nabla_i^z(p)\) implies \(v_{ij} - p_j \geq v_{ij'} - p_{j'}\) for all \(j' \in G\). Thus, \(j \in \nabla_i^z(p)\). Hence, by definitions of \(\gamma_i(p)\) and \(\nabla_i^z(p)\), and condition (14),

\[
\gamma_i(p) + p_j - r_{jk} = v_{ij} - p_j + p_j - r_{jk} = v_{ij} - r_{jk} = \tau_{ijk}.
\]

If \(\nabla_i^z(p) = \emptyset\) then, since \(j \in \nabla_i^z(p)\), \(v_{ij} - p_j = 0\) and \(\gamma_i(p) = 0\). Hence, by (14),

\[
\gamma_i(p) + p_j - r_{jk} = p_j - r_{jk} = v_{ij} - r_{jk} = \tau_{ijk}.
\]

Thus, \(\gamma_i(p) + p_j - r_{jk} = \tau_{ijk}\). Hence, for all \((i, j, k) \in B \times G \times S\) such that \(A_{ijk} > 0\), \(\tau_{ijk} \cdot A_{ijk} = (\gamma_i(p) + p_j - r_{jk}) \cdot A_{ijk}\). From (13),

\[
T(A) = \sum_{ijk} \tau_{ijk} \cdot A_{ijk} = \sum_{ijk} (\gamma_i(p) + p_j - r_{jk}) \cdot A_{ijk} = \sum_{ijk} \gamma_i(p) \cdot A_{ijk} + \sum_{ijk} (p_j - r_{jk}) \cdot A_{ijk}.
\]

Thus,

\[
T(A) = \sum_i (\sum_{jk} A_{ijk}) \cdot \gamma_i(p) + \sum_{jk} (\sum_i A_{ijk}) \cdot (p_j - r_{jk}) \cdot \gamma_i(p). \tag{15}
\]

Fix \(i \in B\). By condition (D.c) in the definition of \(D_i(p)\), if \(\sum_{jk} A_{ijk} < d_i\) then \(\nabla_i^z(p) = \emptyset\), and by the definition of \(\gamma_i(p), \gamma_i(p) = 0\). Hence, by (15),

\[
T(A) = \sum_i d_i \cdot \gamma_i(p) + \sum_{jk} (\sum_i A_{ijk}) \cdot (p_j - r_{jk}). \tag{16}
\]
Condition (E.S) in the definition of an equilibrium price vector implies that, for every $(j, k) \in G \times S$, $\sum_i A_{ijk} \in S_k(p_j)$. To show that, for all $(j, k) \in G \times S$,

$$(\sum_i A_{ijk}) \cdot (p_j - r_{jk}) = q_{jk} \cdot \pi_{jk}(p)$$

(17)

holds, we distinguish among several cases.

**Case 1:** $q_{jk} = 0$. Then, by supply feasibility, $\sum_i A_{ijk} = 0$. Thus, (17) holds.

**Case 2:** $q_{jk} > 0$.

**Case 2.1:** $\sum_i A_{ijk} = q_{jk}$. Then, by (E.S), $p_j - r_{jk} \geq 0$. Hence, $p_j - r_{jk} = \pi_{jk}(p)$. Thus, (17) holds.

**Case 2.2:** $0 < \sum_i A_{ijk} < q_{jk}$. Then, by (E.S), $p_j = r_{jk}$. Hence, $\pi_{jk}(p) = p_j - r_{jk} = 0$. Thus, (17) holds.

**Case 2.3:** $\sum_i A_{ijk} = 0$. Then, by (E.S), $p_j \leq r_{jk}$. Hence, $\pi_{jk}(p) = 0$. Thus, (17) holds.

Hence, for all $(j, k) \in G \times S$, (17) holds. Thus, by (16), $T(A) = \sum_i d_i \cdot \gamma_i(p) + \sum_{jk} q_{jk} \cdot \pi_{jk}(p)$. Therefore, $T(A) = T^d(\gamma(p), \pi(p))$. ◦

The statement of Lemma 2 follows from Claims 1 and 2.

**Lemma 3** \([\iff of (1.2)]\) Assume $p \in P^*$. Then, $(\gamma(p), \pi(p)) \in D^*$.

**Proof of Lemma 3** Assume $p \in P^*$ and let $A \in F$ be any assignment compatible with $p$. Thus, the hypothesis of Lemma 2 hold. By Claims 1 and 2 in the proof of Lemma 2, $(\gamma(p), \pi(p)) \in D$ and $T(A) = T^d(\gamma(p), \pi(p))$. By Theorem 2, $(\gamma(p), \pi(p)) \in D^*$.

**Lemma 4** \([\iff of (1.1)]\) Assume $p \in P^*$ and $A \in F^*$. Then, $p$ and $A$ are compatible.

**Proof of Lemma 4** Follows from Lemmata 1 and 3.

**Lemma 5** \([\iff of (1.2)]\) Assume $(\gamma(p), \pi(p)) \in D^*$. Then, $p \in P^*$.

**Proof of Lemma 5** Let $p \in \mathbb{R}^n_+$ be such that $(\gamma(p), \pi(p)) \in D^*$. We want to show that $p$ is an equilibrium price vector of $M$. Let $A \in F^*$ be arbitrary. By Lemma 1, $p$ and $A$ are compatible. Hence, by definition, $p \in P^*$.

Proposition 1 holds since condition (1.1) follows from Lemmata 2 and 4, and condition (1.2) follows from Lemmata 3 and 5.
3.3 Proof of Theorem 1

Before proving Theorem 1 we define for each solution \((\gamma^*, \pi^*) \in D^*\) of the (DLP) its associated price vector \(p^{(\gamma^*, \pi^*)} = (p_j^{(\gamma^*, \pi^*)})_{j \in G}\) as follows. For each \(j \in G\),

\[
p_j^{(\gamma^*, \pi^*)} = \min_{\{k \in K|q_{jk} > 0\}} \{\pi_{jk}^* + r_{jk}\}. \tag{18}
\]

**Proof of Theorem 1** Let \(A^* \in F^*\) and \((\gamma^*, \pi^*) \in D^*\) be solutions of the (PLP) and (DLP), respectively. By Remarks 1 and 4, they exist. To show that \(P^* \neq \emptyset\), we will show that \((p^{(\gamma^*, \pi^*)}, A^*)\) is an equilibrium of \(M\). We first show that for all \(i \in B\), \(A^*(i) \in D_i(p^{(\gamma^*, \pi^*)})\).

Fix \(i \in B\). Since \(A^*\) is feasible, (D.a) and (D.b) hold by Remark 3.

Before proceeding, observe that by restriction (D.1) in the (DLP), for all \((j', k') \in G \times S\),

\[
\gamma_i^* + \pi_{j'k'}^* \geq \tau_{ijk'}.
\]

Thus, \(\gamma_i^* \geq \tau_{ijk'} - \pi_{j'k'}^*\). If \((j', k') \in G \times S\) is such that \(q_{jk'} > 0\) then, by (3), \(\gamma_i^* \geq v_{ij'} - (\pi_{j'k'}^* + r_{jk'})\). Thus, for all \(j' \in G\),

\[
\gamma_i^* \geq v_{ij'} - \min_{\{k \in K|q_{jk} > 0\}} \{\pi_{jk}^* + r_{jk}\}. \tag{19}
\]

To show that (D.c) holds assume that \(\sum_{jk} A_{ijk}^* < d_i\). By (CS.2) of Theorem 3,

\[
\gamma_i^* = 0. \tag{20}
\]

By (18) and (19), \(\gamma_i^* \geq v_{ij} - p_j^{(\gamma^*, \pi^*)}\) for all \(j \in G\). By (20), \(0 \geq v_{ij} - p_j^{(\gamma^*, \pi^*)}\) for all \(j \in G\). Hence, \(\triangledown_i^2(p^{(\gamma^*, \pi^*)}) = \emptyset\).

To show that (D.d) holds, fix \(j \in G\) and assume that \(\sum_k A_{ijk}^* > 0\). We want to show that \(j \in \triangledown_i^2(p^{(\gamma^*, \pi^*)})\). By assumption, there exists \(k' \in S\) such that \(A_{ijk'}^* > 0\). Thus, \(q_{jk'} > 0\). By (CS.1) of Theorem 3, \(\gamma_i^* + \pi_{j'k'}^* = \tau_{ijk'} = v_{ij} - r_{jk'}\). Thus, \(\gamma_i^* = v_{ij} - (\pi_{j'k'}^* + r_{jk'})\). Hence,

\[
\gamma_i^* \leq v_{ij} - \min_{\{k \in K|q_{jk} > 0\}} \{\pi_{jk}^* + r_{jk}\}. \tag{19'}
\]

By (19), \(\gamma_i^* = v_{ij} - \min_{\{k \in K|q_{jk} > 0\}} \{\pi_{jk}^* + r_{jk}\}\). By (18),

\[
\gamma_i^* = v_{ij} - p_j^{(\gamma^*, \pi^*)}. \tag{21}
\]

By (18) and (19), \(\gamma_i^* \geq v_{ij'} - p_j^{(\gamma^*, \pi^*)}\) for all \(j' \in G\). By (21), \(v_{ij} - p_j^{(\gamma^*, \pi^*)} \geq v_{ij'} - p_j^{(\gamma^*, \pi^*)}\) for all \(j' \in G\). By restriction (D.2) in the (DLP), \(\gamma_i^* \geq 0\). Hence, \(j \in \triangledown_i^2(p^{(\gamma^*, \pi^*)})\).

To show that (E.S) holds fix \((j, k) \in G \times S\). We want to show that \(\sum_i A_{ijk}^* \in S_{jk}(p_j^{(\gamma^*, \pi^*)})\). Assume first that \(\sum_i A_{ijk}^* < q_{jk}\). Then, by the (CS.2) in Theorem 3, \(\pi_{jk}^* = 0\). Since, by definition, \(p_j^{(\gamma^*, \pi^*)} = \min_{\{k \in K|q_{jk} > 0\}} \{\pi_{jk}^* + r_{jk}\}\), \(p_j \leq \pi_{jk}^* + r_{jk}\) for all \(k'\) such that \(q_{jk'} > 0\). But since \(0 \leq \sum_i A_{ijk}^* < q_{jk}\) and \(\pi_{jk}^* = 0\), \(p_j^{(\gamma^*, \pi^*)} \leq r_{jk}\). Hence, if \(p_j > r_{jk}\),
then \( \sum_i A^*_{ijk} \in \{q_{jk}\} = S_{jk}(p^{(\gamma,\pi^*)}) \). If \( p_j^{(\gamma,\pi^*)} = r_{jk} \) (E.S) holds trivially since \( \sum_i A^*_{ijk} \in \{0,...,q_{jk}\} \). Assume \( p_j^{(\gamma,\pi^*)} < r_{jk} \). By (1), \( q_{jk} > 0 \). To get a contradiction, assume there exists \( i \in B \) such that \( A^*_{ijk} > 0 \). By (CS.1) of Theorem 3, and since, by (3), \( \tau_{ijk} = v_{ij} - r_{jk} \), \( \gamma^*_i + \pi^*_j = \tau_{ijk} = v_{ij} - r_{jk} \). By hypothesis, and since by restriction (D.3) of the (DLP), \( \pi^*_j \geq 0 \), \( \gamma^*_i \leq \gamma^*_i + \pi^*_j < v_{ij} - p_j^{(\gamma,\pi^*)} \). Thus, \( \gamma^*_i < v_{ij} - p_j^{(\gamma,\pi^*)} \), contradicting condition (19). Thus, for all \( i \in B \), \( A^*_{ijk} = 0 \). Hence, \( \sum_i A^*_{ijk} = 0 \in \{0\} = S_{jk}(p^{(\gamma,\pi^*)}) \). Thus \( p^{(\gamma,\pi^*)} \in P^* \).

The proof of Theorem 1 (which proves that \( P^* \) is non-empty by showing that for all \( (\gamma^*,\pi^*) \in D^* \), \( p^{(\gamma,\pi^*)} \in P^* \)) implies that the following corollary holds.

**Corollary 1** Let \( (\gamma, \pi) \in D^* \). Then, \( p^{(\gamma, \pi)} \in P^* \).

### 4 Structure of the Set of Competitive Equilibria

The assignment game of Shapley and Shubik (1972) is a particular instance of our model where each seller owns one indivisible object and each buyer wants to buy at most one object. Since objects owned by different sellers may be perceived differently by different buyers (or they may, indeed, be different), we can identify the set of goods \( G \) with the set of sellers \( S \). Namely, a market \( M \) is an assignment game if \( d_i = 1 \) for all \( i \in B \), \( n = t \) and for all \( (j,k) \in G \times S \),

\[
q_{jk} = \begin{cases} 
1 & \text{if } j = k \\
0 & \text{if } j \neq k. 
\end{cases}
\]

Hence, each seller \( j \in S \) has a reservation value \( r_j \geq 0 \) of the indivisible object \( j \in G \) that he owns. Thus, an assignment game can be identified as an \((m \times t)\)-matrix \( a \), where for all \((i,j) \in B \times S \), \( a_{ij} = \max\{0, v_{ij} - r_j\} \).

The set of competitive equilibria of an assignment game has the following three properties. (1) The set of equilibria is the Cartesian product of the set of equilibrium price vectors times the set of optimal assignments. (2) The set of equilibrium price vectors \( P^* \) endowed with the partial order \( \geq \) on \( \mathbb{R}_+^n \) (where \( p \geq p' \) if and only if \( p_j \geq p'_j \) for all \( j \in G \)) is a complete lattice.\(^3\) In particular, given \( p, p' \in P^* \), \( (\max\{p_j, p'_j\})_{j \in G} \in P^* \) and \( (\min\{p_j, p'_j\})_{j \in G} \in P^* \). Moreover, the set of equilibrium price vectors contains two extreme vectors \( p^B \) and \( p^S \) with the property that for any equilibrium price vector \( p \in P^* \), \( p^S \geq p \geq p^B \). (3) This structure

\(^3\)See subsection 4.5 for a self-contained definition of a lattice and a complete lattice.
is translated into the set of utilities that are attainable at equilibrium as follows. Given $p \in P^*$ and an optimal assignment $\mu = (\mu_{ij})_{(i,j) \in B \times S}$, define for each $i \in B$,

$$u_i(p) = \begin{cases} v_{ij} - p_j & \text{if } \mu_{ij} = 1 \text{ for some } j \in S \\ 0 & \text{otherwise,} \end{cases}$$

and for each $j \in S$,

$$w_j(p) = \begin{cases} p_j - r_j & \text{if } \mu_{ij} = 1 \text{ for some } i \in B \\ 0 & \text{otherwise} \end{cases}$$

(it turns out that these utilities are independent of the chosen optimal assignment $\mu$; thus, we can write them as depending only on the equilibrium price vector $p$). Then, for all $p, p' \in P^*$, the following three statements are equivalent:

(a) $p_j \geq p'_j$ for all $j \in G$.
(b) $u_i(p') \geq u_i(p)$ for all $i \in B$.
(c) $w_j(p) \geq w_j(p')$ for all $j \in S$.

Hence, we can define two binary relations $\succeq_B$ and $\succeq_S$ on the set of utilities that are attainable at equilibrium (i.e., on the sets $U^* = \{u \in \mathbb{R}^m \mid \text{there exists } p \in P^* \text{ such that, for all } i \in B, u_i = u_i(p)\}$ and $W^* = \{w \in \mathbb{R}^l \mid \text{there exists } p \in P^* \text{ such that, for all } j \in S, w_j = w_j(p)\}$, respectively) by using the partial order $\geq$ on an Euclidean space where, for $x, y \in \mathbb{R}^s$, $x \geq y$ if and only if $x_i \geq y_i$ for all $i = 1, \ldots, l$. Namely, for all $p, p' \in P^*$, the following equivalences hold:

$$(u_i(p'))_{i \in B} \succeq_B (u_i(p))_{i \in B} \iff u_i(p') \geq u_i(p) \text{ for all } i \in B$$
$$\iff (p_j)_{j \in G} \succeq (p'_j)_{j \in G}$$
$$\iff w_j(p) \geq w_j(p') \text{ for all } j \in S$$
$$\iff (w_j(p))_{j \in S} \succeq_S (w_j(p'))_{j \in S}.$$
possible one that keeps the equilibrium properties of the full price vector. We call this set the set of restricted equilibrium price vectors and denote it by $P^{**}$. Then we show that the equivalence of (a) and (b) above holds on $P^{**}$ and that property (c) above is not anymore equivalent to properties (a) and (b) on the set $P^{**}$; i.e., for all $p, p' \in P^{**}$, (a) and (b) are equivalent and each implies (c) but (c) neither implies (a) nor (b).

Camiña (2006) studies an instance of our model in which there is a unique seller that owns $n$ different indivisible objects and each buyer wants to buy at most one object; i.e., $t = 1$, $q_{j1} = 1$ for all $j = 1, \ldots, n$, and $d_i = 1$ for all $i \in B$. She shows that the set of core utilities has the following properties: (i) it is non-empty, (ii) it may not coincide with the set of utilities that are attainable at equilibrium, and (iii) it forms a complete lattice with the partial order $\succeq_B$.

Sotomayor (2007) studies another extension of the assignment game in which buyers may want to buy more than one indivisible object although they are not interested in acquiring more than one object from a given seller, and each seller owns a number of identical and indivisible objects (and thus, we can also identify the set of goods with the set of sellers); i.e., $n = t$, $d_i \in \mathbb{Z}_+ \setminus \{0\}$ for all $i \in B$, $q_{jk} \in \mathbb{Z}_+ \setminus \{0\}$ if $j = k$ and $q_{jk} = 0$ if $j \neq k$, and $A_{ijk} \in \{0, 1\}$ for all $i \in B$ and all $j = k$. Observe that this last condition is not a restriction on the data of the problem but on the set of feasible assignments. Sotomayor (2007) shows that the set of utilities that are attainable at equilibrium forms a dual lattice with the partial orders $\succeq_B$ and $\succeq_S^*$ after an appropriate distortion of the set of stable utilities; this distortion is done by an order-preserving and non-identical map whose fixed points are the set of utilities that are attainable at equilibrium. Sotomayor (2007)’s results use Tarski’s algebraic fixed point theorem while our results are based on duality theorems of linear programming.

### 4.1 Cartesian Product Structure of the Set of Equilibria

The first property of the set of equilibria is that it has a Cartesian product structure; namely, if $(p, A)$ and $(p', A')$ are two equilibria of $M$ then, $(p, A')$ and $(p', A)$ are also two equilibria of $M$. This follows immediately from Lemmata 2 and 4 used to prove Proposition 1. We state it as Proposition 2 below.

**Proposition 2** Let $M$ be a market. Then, $(p, A)$ is an equilibrium of $M$ if and only if $p \in P^*$ and $A \in F^*$.

**Proof** Assume $(p, A)$ is an equilibrium of $M$. By definition, $p \in P^*$. Moreover, $p$ and $A$
are compatible. By Lemma 2, \( A \in F^* \). Assume \( p \in P^* \) and \( A \in F^* \). By Lemma 4, \( p \) and \( A \) are compatible. Thus, \((p,A)\) is an equilibrium of \( M \).

### 4.2 Extreme Equilibrium Price Vectors

In this subsection we show that the set of equilibrium price vectors \( P^* \) of \( M \) has the property that each of its non-empty subsets has a supremum and an infimum. This will imply that the set \( P^* \) has a natural lattice structure. Let \( Z \subseteq P^* \) be a non-empty subset of equilibrium price vectors of \( M \). Define the price vector \( p^B(Z) \in \mathbb{R}^n_+ \) by taking, for each \( j \in G \), the infimum among all \( j \)-components of the vectors in \( Z \). Similarly, define the price vector \( p^S(Z) \in \mathbb{R}^n_+ \) by taking, for each \( j \in G \), the supremum among all \( j \)-components of the vectors in \( Z \). Then, it turns out that \( p^B(Z) \) and \( p^S(Z) \) are also equilibrium price vectors of \( M \). Formally, given a market \( M \), define \( p^B(Z) = (p^B_j(Z))_{j \in G} \in \mathbb{R}^n_+ \) and \( p^S(Z) = (p^S_j(Z))_{j \in G} \in \mathbb{R}^n_+ \) as follows: for each \( j \in G \), let

\[
p^B_j(Z) = \inf_{p \in Z} p_j \quad \text{and} \quad p^S_j(Z) = \sup_{p \in Z} p_j.
\]

We write \( p^B \) and \( p^S \) instead of \( p^B(P^*) \) and \( p^S(P^*) \).

**Proposition 3** Let \( M \) be a market. Then, \( p^B(Z), p^S(Z) \in P^* \) for all \( \emptyset \neq Z \subseteq P^* \).

**Proof** Let \( A \in F^* \) be an optimal assignment of \( M \). Given a non empty subset \( Z \subseteq P^* \), we will prove that \( p^B(Z) \) and \( p^S(Z) \) are equilibrium price vectors of \( M \) by showing that conditions (E.D) and (E.S) of Definition 3 are satisfied by \( p^B(Z) \) and \( p^S(Z) \) with respect to \( A \).

(E.D) for \( p^S(Z) \): For every \( i \in B \), \( A(i) \in D_i(p^S(Z)) \). Fix \( i \in B \). Since \( A \) is feasible, conditions (D.a) and (D.b) hold by Remark 3.

To show that condition (D.c) holds, assume \( \nabla^>_i(p^S(Z)) \neq \emptyset \). Then, there exists \( j \in \nabla^>_i(p^S(Z)) \) such that \( v_{ij} - p^S_j(Z) > 0 \). Since \( p^S_j(Z) = \sup_{p \in Z} p_j \), we have that for every \( p \in Z \), \( 0 < v_{ij} - p^S_j(Z) \leq v_{ij} - p_j \), which implies that \( \nabla^>_i(p) \neq \emptyset \). Because \( p \) and \( A \) are compatible, \( \sum_{jk} A_{ijk} = d_i \). Thus condition (D.c) holds for \( p^S(Z) \).

To show that condition (D.d) holds, let \( j \in G \) be such that \( \sum_k A_{ijk} > 0 \). We have to show that \( j \in \nabla^>_i(p^S(Z)) \). Since for all \( p \in Z \), \( p \) and \( A \) are compatible, \( j \in \nabla^>_i(p) \) for every \( p \in Z \). By definition of \( \nabla^>_i(p) \), \( v_{ij} - p_j \geq 0 \) and \( v_{ij} - p_j \geq v_{ij'} - p_{j'} \) for every \( j' \in G \). For every \( j' \in G \),

\[
v_{ij'} - p_{j'} \geq v_{ij'} - \sup_{p \in Z} p_{j'}
\]

19
holds for all $p \in Z$. Let $\{p^m\}_{m \in \mathbb{N}}$ be a sequence such that, for all $m \in \mathbb{N}$, $p^m \in Z$ and $\{p^m_j\}_{m \in \mathbb{N}} \rightarrow \sup_{p \in Z} p_j$. By (23), $v_{ij} - p^m_j \geq v_{ij'} - \sup_{p \in Z} p_j'$ for all $m \in \mathbb{N}$. Since $j \in \nabla^0_i(p)$ for every $p \in Z$, $j \in \nabla^0_i(p^m)$ for every $m \in \mathbb{N}$. Thus, $v_{ij} - p^m_j \geq v_{ij'} - p^m_j$ for all $m \in \mathbb{N}$. Thus, $v_{ij} - p^m_j \geq v_{ij'} - \sup_{p \in Z} p_j'$ for all $m \in \mathbb{N}$. Hence, $v_{ij} - \sup_{p \in Z} p_j \geq v_{ij'} - \sup_{p \in Z} p_j'$. Thus, $j \in \nabla^0_i(p^S(Z))$.

(E.S) for $p^S(Z)$: For every $(j, k) \in G \times S$, $\sum_i A_{ijk} \in S_{jk}(p^S_j(Z))$.

Fix $(j, k) \in G \times S$. If $p^S_j(Z) < r_{jk}$ then, for all $p \in Z$, $p_j \leq p^S_j(Z) < r_{jk}$. Thus, $\sum_i A_{ijk} = 0 \in S_{jk}(p_j)$. Thus, $\sum_i A_{ijk} \in \{0\} = S_{jk}(p^S_j(Z))$. If $p^S_j(Z) > r_{jk}$, let $\{p^m\}_{m \in \mathbb{N}}$ be a sequence such that, for all $m \in \mathbb{N}$, $p^m \in Z$ and $\{p^m_j\}_{m \in \mathbb{N}} \rightarrow \sup_{p \in Z} p_j$. Then, there exists $m \in \mathbb{N}$ such that, for all $m > \tilde{m}$, $p^m_j > r_{jk}$. Thus, $\sum_i A_{ijk} \in \{q_{jk}\} = S_{jk}(p^S_j(Z))$ for all $m > \tilde{m}$. Hence, $\sum_i A_{ijk} \in \{q_{jk}\} = S_{jk}(p^S_j(Z))$.

(E.D) for $p^B(Z)$: For every $i \in B$, $A(i) \in D_i(p^B(Z))$.

Fix $i \in B$. Since $A$ is feasible, conditions (D.a) and (D.b) hold by Remark 3.

To show that condition (D.c) holds, assume $\nabla^0_i(p^B(Z)) \neq \emptyset$. Then, there exists $j \in \nabla^0_i(p^B(Z))$ such that $v_{ij} - p^B_j(Z) > 0$. Let $\{p^m\}_{m \in \mathbb{N}}$ be a sequence such that, for all $m \in \mathbb{N}$, $p^m \in Z$ and $\{p^m_j\}_{m \in \mathbb{N}} \rightarrow \inf_{p \in Z} p_j$. Then, there exist $m \in \mathbb{N}$ such that, for all $m > \tilde{m}$, $v_{ij} - p^m_j > 0$, which implies that $\nabla^0_i(p^m) \neq \emptyset$. Because $p^m$ and $A$ are compatible, $\sum_{jk} A_{ijk} = d_i$. Thus, condition (D.c) holds for $p^B(Z)$.

To show that condition (D.d) holds, let $j \in G$ be such that $\sum_k A_{ijk} > 0$. We have to show that $j \in \nabla^0_i(p^B(Z))$. Since for all $p \in Z$, $p$ and $A$ are compatible, $j \in \nabla^0_i(p)$ for every $p \in Z$. By definition of $\nabla^0_i(p)$, $v_{ij} - p_j \geq 0$ and $v_{ij} - p_j \geq v_{ij'} - p_j'$ for every $j' \in G$. By definition of $p^B(Z)$, $v_{ij} - p^B_j(Z) = v_{ij} - \inf_{p \in Z} p_j \geq v_{ij} - p_j$ for all $p \in Z$. Fix $j' \in G$ and let $\{p^m\}_{m \in \mathbb{N}}$ be a sequence such that, for all $m \in \mathbb{N}$, $p^m \in Z$ and $\{p^m_j\}_{m \in \mathbb{N}} \rightarrow \inf_{p \in Z} p_{j'}$. Then, by definition of $p^B(Z)$, $v_{ij} - p^B_j(Z) \geq v_{ij'} - p^m_{j'}$ for every $m \in \mathbb{N}$. Since $j \in \nabla^0_i(p^m)$, $v_{ij} - p^m_j \geq v_{ij'} - p^m_{j'}$ for every $m \in \mathbb{N}$. Thus, $v_{ij} - p^B_j(Z) \geq v_{ij'} - p^B_{j'}$ for every $m \in \mathbb{N}$. Hence, $v_{ij} - p^B_j(Z) \geq v_{ij'} - p^B_{j'}(Z)$. Since this holds for all $j' \in G$, $j \in \nabla^0_i(p^B(Z))$.

(E.S) for $p^B(Z)$: For every $(j, k) \in G \times S$, $\sum_i A_{ijk} \in S_{jk}(p^B_j(Z))$.

Fix $(j, k) \in G \times S$. If $p^B_j(Z) > r_{jk}$ then, $p_j > r_{jk}$ for all $p \in Z$. Thus, $\sum_i A_{ijk} \in \{q_{jk}\} = S_{jk}(p^B_j(Z))$. Thus, $\sum_i A_{ijk} \in S_{jk}(p^B_j(Z))$. If $p^B_j(Z) < r_{jk}$, let $\{p^m\}_{m \in \mathbb{N}}$ be a sequence such that, for all $m \in \mathbb{N}$, $p^m \in Z$ and $\{p^m_j\}_{m \in \mathbb{N}} \rightarrow \inf_{p \in Z} p_j$. Then, there exists $m \in \mathbb{N}$ such that, for all $m > \tilde{m}$, $p^m_j < r_{jk}$. Thus, $\sum_i A_{ijk} \in \{0\} = S_{jk}(p^B_j(Z))$ for all $m > \tilde{m}$. Hence, $\sum_i A_{ijk} \in \{0\} = S_{jk}(p^B_j(Z))$.

Remark 6 By Proposition 3 we can write for each $\emptyset \neq Z \subseteq P^*$ and $j \in G$, $p^S_j(Z) =$
max_{p \in Z} p_j and p_j^B(Z) = \min_{p \in Z} p_j. In particular, p_j^S = \max_{p \in P^*} p_j and p_j^B = \min_{p \in P^*} p_j.

4.3 Utilities

Let $p \in \mathbb{R}^n_+$ be a price vector and $A \in F$ a feasible assignment of market $M$. We define the utility of buyer $i \in B$ at the pair $(p, A)$ as the total net gain obtained by $i$ from his exchanges specified by $A$ at price $p$. We denote it by $u_i(p, A)$; namely,

$$u_i(p, A) = \sum_{j \in G} (v_{ij} - p_j) \cdot A_{ijk}.$$ 

We define the utility of seller $k \in S$ at the pair $(p, A)$ as the total net gain obtained by $k$ from his exchanges specified by $A$ at price $p$. We denote it by $w_k(p, A)$; namely,

$$w_k(p, A) = \sum_{j \in G} (p_j - r_{jk}) \cdot A_{ijk}.$$ 

Define

$$G^> = \{j \in G \mid \text{there exists } A \in F^* \text{ such that } A_{ijk} > 0 \text{ for some } (i, k) \in B \times S\}$$

as the set of goods that are exchanged at some optimal assignment. For each seller $k \in S$, define

$$G^>_k = \{j \in G \mid \text{there exists } A \in F^* \text{ such that } A_{ijk} > 0 \text{ for some } i \in B\}$$

as the set of goods that $k$ sells strictly positive amounts at some optimal assignment. Obviously, $G^> = \bigcup_{k \in S} G^>_k$.

Next lemma states that at equilibrium utilities are independent of the particular optimal assignment chosen since they only depend on the equilibrium price vector (which determines the associated solution of the (DLP)).

**Lemma 6** Let $p \in P^*$ be an equilibrium price vector of $M$ and let $A \in F^*$ be an optimal assignment of $M$. Then, the following two conditions hold:

(L6.1) For each buyer $i \in B$, $u_i(p, A) = \gamma_i(p) \cdot d_i$.

(L6.2) For each seller $k \in S$, $w_k(p, A) = \sum_{j \in G^>_k} (p_j - r_{jk}) \cdot q_{jk}$.

**Proof of Lemma 6** Let $(p, A) \in P^* \times F^*$. Note that $p$ and $A$ are compatible. To prove (L6.1), fix $i \in B$. By definition, $u_i(p, A) = \sum_{j \in G} (v_{ij} - p_j) \cdot A_{ijk}$. Let $(j, k) \in G \times S$ be given.

---

4Observe that $w_k(p, A)$ can also be written as $\sum_{j \in G} \pi_{jk}(p) \cdot q_{jk}$. 21
If $A_{ijk} = 0$ then, $(v_{ij} - p_j) \cdot A_{ijk}$ can trivially be written as $\gamma_i(p) \cdot A_{ijk}$. If $A_{ijk} \neq 0$ then, by condition (D.d) in the definition of $D_i(p)$, $j \in \nabla_i^2(p)$, which implies that $(v_{ij} - p_j) = \gamma_i(p)$, and

$$u_i(p, A) = \gamma_i(p) \cdot (\sum_{jk} A_{ijk}).$$

If $\gamma_i(p) = 0$ then, the statement holds because $\gamma_i(p) \cdot (\sum_{jk} A_{ijk}) = \gamma_i(p) \cdot d_i = 0$. By condition (CS.2) in Theorem 3, if $\gamma_i(p) \neq 0$ then $\sum_{jk} A_{ijk} = d_i$. Thus,

$$u_i(p, A) = \gamma_i(p) \cdot d_i,$$

To prove (L6.2), fix $k \in S$. By definition, $w_k(p, A) = \sum_{ij}(p_j - r_{jk}) \cdot A_{ijk}$. Then,

$$\sum_{ij}(p_j - r_{jk}) \cdot A_{ijk} = \sum_{j}(p_j - r_{jk}) \cdot (\sum_{i} A_{ijk}).$$

Since $p \in P^*$, by (E.S), if $(p_j - r_{jk}) > 0$ then $\sum_{i} A_{ijk} = q_{jk}$. If $(p_j - r_{jk}) < 0$ then, $S_{jk}(p) = \{0\}$. Hence, since $p$ and $A$ are compatible, $\sum_{i} A_{ijk} = 0$. Therefore,

$$w_k(p, A) = \sum_{j \in \{j' \in G | q_{j'k} > 0 \} \cap G^*_k} (p_j - r_{jk}) \cdot q_{jk} = \sum_{j \in G^*_k} (p_j - r_{jk}) \cdot q_{jk}. \quad (24)$$

Condition (24) holds because $\{j' \in G | q_{j'k} > 0 \} \cap G^*_k = \{j \in G | p_j - r_{jk} \geq 0\}$. To see that, let $j \in G_k^*$. Hence, there exists $\tilde{A} \in F^*$ such that $\tilde{A}_{ijk} > 0$, which implies, since $p$ and $\tilde{A}$ are compatible, $p_j - r_{jk} \geq 0$. Thus, the second inclusion holds. To prove the first one, assume $j \in \{j' \in G | q_{j'k} > 0 \} \cap \{p_{j'} - r_{j'k} > 0\}$. Then, since $p \in P^*$, by (E.S), $\sum_{i} A_{ijk} = q_{jk}$. Thus, $j \in G_k^*$. 

By Lemma 6, we can write the utilities of buyers and sellers as functions only of the equilibrium price vector $p$; namely, given $p \in P^*$, we write for each $i \in B$ and each $k \in S$,

$$u_i(p) = \gamma_i(p) \cdot d_i \quad (25)$$

and

$$w_k(p) = \sum_{j \in G_k^*} (p_j - r_{jk}) \cdot q_{jk}. \quad (26)$$

### 4.4 The Set of Restricted Equilibrium Price Vectors

We start this subsection with an example that illustrates two important facts. First, it shows that, in contrast with the assignment game, there are markets with two equilibrium price vectors $p, p' \in P^*$ with the property that $w_k(p') > w_k(p)$ for all $k \in S$ while $u_i(p') > u_i(p)$ for some $i \in B$ (the equivalence between statements (b) and (c) does not hold on $P^*$).
Second, it also shows that the (incomplete) binary relation $\geq$ on the set of vectors in $\mathbb{R}_+^n$ is not imbedded into the set of attainable equilibrium utilities (the equivalence between statements (a) and (b) does not hold on $P^*$). These two facts will have consequences for the lattice structures of the set(s) of (restricted) equilibrium price vectors and the sets of attainable equilibrium utilities that will be analyzed in Subsection 4.5.

**Example 2** Let $M = (B, G, S, V, d, R, Q)$ be a market where $B = \{b_1, b_2\}$, $G = \{g_1, g_2, g_3\}$, $S = \{s_1\}$, $V = \begin{pmatrix} 8 & 0 & 2 \\ 0 & 5 & 3 \end{pmatrix}$, $d = (2, 3)$, $R = \begin{pmatrix} 1 \\ 2 \\ 10 \end{pmatrix}$, and $Q = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$. It is easy to see that, for all $p_3, p'_3 \in [3, 10]$, $p = (5, 4, p_3)$ and $p' = (7, 2, p'_3)$ are two equilibrium price vectors of $M$ and $14 = w_1(p) > w_1(p') = 12$. Furthermore, $\gamma(p) = (3, 1)$ and $\gamma(p') = (1, 3)$. Then, $u_1(p) = 3 \cdot 2 = 6$, $u_2(p) = 1 \cdot 3 = 3$, $u_1(p') = 1 \cdot 2 = 2$, and $u_2(p') = 3 \cdot 3 = 9$. Thus, $w_1(p) > w_1(p')$ and $u_1(p) > u_1(p')$ and $u_2(p') > u_2(p)$. Moreover, observe that, for all $i \in \{1, 2\}$, $u_i(7, 2, p_3) = u_i(7, 2, p'_3)$ for all $3 \leq p_3 < p'_3 \leq 10$ but $p = (7, 2, p_3) < (7, 2, p'_3) = p'$. This is because no unit of good 3 is interchanged in any equilibria and hence, the equilibrium price vector $p = (7, 2, p_3)$ is equivalent (in terms of its induced demands and supplies) to the equilibrium price vector $p' = (7, 2, p'_3)$ as long as $3 \leq p_3 < p'_3 \leq 10$. \hfill \diamondsuit

In order to restore the interesting property that the (incomplete) binary relation $\geq$ on $\mathbb{R}_+^n$ reproduces itself in terms of buyers utilities (in the corresponding space) we have to eliminate an insubstantial multiplicity of equilibrium prices of the goods that are not interchanged at any equilibrium assignment. We do it by setting the prices of each non-interchanged good equal to the highest possible one (keeping the equilibrium property of the price vector).\footnote{The choice of the highest price is arbitrary. The important fact is to select, for each of these goods, just one of its potentially many equilibrium prices.} Formally, given an equilibrium price vector $p \in P^*$, define $\bar{p} = (\bar{p}_j)_{j \in G}$ as follows:

$$\bar{p}_j = \begin{cases} p_j & \text{if } j \in G^> \\ p_j^s & \text{if } j \notin G^> \end{cases}.$$  \hfill (27)

Proposition 4 below says that this distortion does not affect the equilibrium property of the original price vector.

**Proposition 4** Let $M$ be a market and let $p \in P^*$. Then, $\bar{p} \in P^*$.
Proof Let $A \in F^*$ be an optimal assignment of $M$. We will prove that $(\bar{p}, A)$ is an equilibrium of $M$ by showing that conditions (E.D) and (E.S) of Definition 3 are satisfied by $\bar{p}$ with respect to $A$.

(E.D) For every $i \in B$, $A(i) \in D_i(\bar{p})$.

Fix $i \in B$. Since $A$ is feasible, conditions (D.a) and (D.b) hold by Remark 3.

To show that condition (D.c) holds, assume $\nabla_i^\geq(\bar{p}) \neq \emptyset$. Then, there exists $j \in \nabla_i^\geq(\bar{p})$ such that $v_{ij} - \bar{p}_j > 0$. Since either $\bar{p}_j = p_j$ or $\bar{p}_j = p_j^S$ we have that either $0 < v_{ij} - \bar{p}_j = v_{ij} - p_j$ or $0 < v_{ij} - \bar{p}_j = v_{ij} - p_j^S$, which implies that either $\nabla_i^\geq(p) \neq \emptyset$ or $\nabla_i^\geq(p^S) \neq \emptyset$.

Since $p$ and $p^S$ are both compatible with $A$, $\sum_{jk} A_{ijk} = d_i$. Thus condition (D.c) holds for $\bar{p}$.

To show that condition (D.d) holds, let $(i, j) \in B \times G$ be such that $\sum_k A_{ijk} > 0$. Thus, $j \in G^>$. We have to show that $j \in \nabla_i^\geq(\bar{p})$. Since $p$ and $p^S$ are both compatible with $A$, $j \in \nabla_i^\geq(p) \cap \nabla_i^\geq(p^S)$. By definition of $\nabla_i^\geq(p)$,

$$v_{ij} - p_j \geq 0 \tag{28}$$

and

$$v_{ij} - p_j \geq v_{ij'} - p_{j'} \text{ for every } j' \in G. \tag{29}$$

By definition of $\nabla_i^\geq(p^S)$, $v_{ij} - p_j^S \geq 0$ and $v_{ij} - p_j^S \geq v_{ij'} - p_{j'}^S = v_{ij'} - \max_{p \in P^*} p_{j'}$ for every $j' \in G$. We next show that:

$$v_{ij} - \bar{p}_j \geq 0$$

and

$$v_{ij} - \bar{p}_j \geq v_{ij'} - \bar{p}_{j'} \text{ for every } j' \in G.$$

Since $j \in G^>$, $\bar{p}_j = p_j$. Thus, by (28), $v_{ij} - \bar{p}_j \geq 0$. We distinguish between the following two cases.

Case 1: $j' \in G^>$. Then, $\bar{p}_{j'} = p_{j'}$ and

$$v_{ij} - \bar{p}_j = v_{ij} - p_j \text{ by definition of } \bar{p}_j \geq v_{ij'} - p_{j'} \text{ by (29)} = v_{ij'} - \bar{p}_{j'} \text{ by definition of } \bar{p}_{j'}.$$

Hence, $v_{ij} - \bar{p}_j \geq v_{ij'} - \bar{p}_{j'}$ for every $j' \in G^>$.  

24
Hence, \( v_{ij} - \delta j \geq v_{ij'} - \delta j' \) for every \( j' \notin G^> \).

Thus, \( j \in \nabla_i^>(\delta) \).

(E.S) For every \( j \in G \), \( \sum_i A_{ijk} \in S_{jk}(\delta j) \).

Assume first that \( j \in G^> \). Then, \( \delta j = p_j \) and \( S_{jk}(\delta j) = S_{jk}(p_j) \). Since \( p \) and \( A \) are compatible, \( \sum_i A_{ijk} \in S_{jk}(\delta j) \). Thus, \( \sum_i A_{ijk} \in S_{jk}(\delta j) \).

Proposition 5 shows that the distortion in (27) coincides with the one produced in \( p \) by computing its associated price vector from its dual solution \((\gamma(p), \pi(p))\).

**Proposition 5** For every \( p \in P^* \), \( p^{(\gamma(p), \pi(p))} = \delta \).

**Proof** Let \( p \in P^* \) be given and let \( A^* \in F^* \) be any compatible assignment. By definition, for all \( j \in G \), \( \delta j = p_j^{(\gamma(p), \pi(p))} = \min_{k \in K | q_{jk} > 0} \{ \pi_{jk}(\delta p) + r_{jk} \} \).

Assume first that \( j \notin G^> \). Then, \( \sum_i A_{ijk} = 0 \). By (CS.2) of Theorem 3, \( \pi^*_{jk} = 0 \) for all \( k \in S \) and all \( \pi^*_{jk} \) such that there exists \( \gamma^* \) with the property that \( (\gamma^*, \pi^*) \in D^* \). Thus, by part (1.2) of Proposition 1, \( \pi_{jk}(p) = 0 \). Hence, \( \delta j = \min_{k \in K | q_{jk} > 0} r_{jk} \). By Corollary 1 and definition of \( p^* \), \( \delta j \leq p^S_{jk} \). To obtain a contradiction, assume \( \delta j < p^S_{jk} \). Then, there exists \( k \in K \) such that \( q_{jk} > 0 \) and \( r_{jk} < p^S_{jk} \). Since, by Proposition 3, \( p^S \in P^* \), (E.S) implies that \( \sum_i A_{ijk} = q_{jk} > 0 \), a contradiction.

Assume now that \( j \in G^> \). It is immediate to see that, for all \( p' \in P^* \),

\[
G^> \subseteq \bigcup_{i \in B} \nabla_i^>(p')
\]

holds. Next, we show that the following claim holds.

**Claim** Let \( p' \in P^* \) and \( (i, j) \in B \times G \) be such that \( j \in \nabla_i^>(p') \), then \( v_{ij} - p'_j = \gamma_i(p') \).

**Proof of Claim** Since \( j \in \nabla_i^>(p') \), \( v_{ij} - p'_j \geq 0 \) and for all \( j' \in G \), \( v_{ij} - p'_j \geq v_{ij'} - p'_j \). If \( v_{ij} - p'_j = 0 \), then \( v_{ij'} - p'_j \leq 0 \) for all \( j' \in G \). Thus, \( \gamma_i(p') = 0 = v_{ij} - p'_j \). If \( v_{ij} - p'_j > 0 \), then \( j \in \nabla_i^>(p') \). Thus, \( \gamma_i(p') = v_{ij} - p'_j \). \( \Diamond \)
By restriction (D.1) of the (DLP), for all \((\gamma, \pi) \in D^*\) and all \((i, j, k) \in B \times G \times S\),
\[ \gamma_i + \pi_{jk} \geq \tau_{ijk}. \]
Thus, by (3), for all \(i \in B\) and all \((j, k)\) such that \(q_{jk} > 0\), \(\gamma_i + \pi_{jk} \geq v_{ij} - r_{jk}\).
Hence,
\[ \pi_{jk} + r_{jk} \geq v_{ij} - \gamma_i. \] (31)
Since \(j \in G^>\), condition (30) implies that there exist \((i', k') \in B \times S\) such that \(q_{jk'} > 0\), \(A_{i'jk'} > 0\) and \(j \in \nabla_j^>(p')\). Thus, by (31) applied to \((\gamma(p'), \pi(p'))\) and \(i' \in B\), \(\pi_{jk}(p') + r_{jk} \geq v_{ij'} - \gamma(p')(p') = p'_j\). Thus,
\[ \min_{\{k \in K(q_{jk} > 0)\}} \{\pi_{jk}(p') + r_{jk}\} \geq p'_j. \] (32)
Moreover, by (CS.1) of Theorem 3, \(\gamma(p')(p') + \pi_{jk}(p') = \tau_{i'jk'} = v_{ij} - r_{jk}\). Thus, \(\pi_{jk}(p') + r_{jk} = v_{ij'} - \gamma(p')(p') = p'_j\). Thus, by (32), \(\min_{\{k \in K(q_{jk} > 0)\}} \{\pi_{jk}(p') + r_{jk}\} = p'_j\), which implies that \(p'_j = p_j\). Hence, \(p'_{\gamma(p), \pi(p)} = p\).

Given a market \(M\), we can define the set of restricted equilibrium price vectors \(P^{**}\) as those that are obtained from equilibrium prices vectors after setting the price of the goods that are not interchanged at any optimal assignment equal to their sellers-optimal equilibrium price. Namely,
\[ P^{**} = \{p \in P^* \mid p_j = p^S_j \text{ for every } j \notin G^>\}. \]

4.5 Lattices

Our objective in this subsection is to show that the sets \(P^*\) and \(P^{**}\) have a complete lattice structure with the natural order \(\geq\) on \(\mathbb{R}_+^n\) and that the structure of the set \(P^{**}\) translates into the set of utilities of buyers that are attainable at equilibrium. The fact that the lattice structure of the set of equilibrium price vectors is inherited in a dual way by the sets of equilibrium utilities of buyers and sellers is an important property because it says that there is a conflict of interests between the two sides of the market (and unanimity in each of the sides) with respect to two comparable equilibrium price vectors. In our more general setting \(P^{**}\) has partly this property but, as Example 2 in Subsection 4.4 shows, \(P^*\) does not; the set of utilities of buyers that are attainable at equilibrium inherit the structure (unanimity among buyers is preserved) while the duality, and its associated conflict of interest, is partly lost (it still holds that buyers unanimity implies sellers unanimity but the converse does not hold).
In order to state our results we introduce some notions to define a lattice in our setting. Let $X$ be a non-empty set. A partial order $\succeq$ on $X$ is a reflexive, transitive, and antisymmetric binary relation on $X$; that is, for all $x, y, z \in X$, $x \succeq x$, $x \succeq y \Rightarrow x = y$, and $x \succeq y$ and $y \succeq x \Rightarrow x = y$. Given a partial order $\succeq$ on $X$ and a subset $Y \subseteq X$, define the set of upper bounds of $Y$ as $\text{lub}_\succeq Y = \{x \in X \mid x \succeq y \text{ for all } y \in Y\}$ and the least upper bound of $Y$ as $\text{lub}_\succeq Y = \tilde{y}$, where $\tilde{y} \in \text{lub}_\succeq Y$ and, for all $y \in \text{lub}_\succeq Y$, $y \succeq \tilde{y}$.

Similarly, define the set of lower bounds of $Y$ as $\text{llb}_\succeq Y = \{x \in X \mid y \succeq x \text{ for all } y \in Y\}$ and the largest lower bound of $Y$ as $\text{llb}_\succeq Y = \tilde{y}$, where $\tilde{y} \in \text{llb}_\succeq Y$ and, for all $y \in \text{llb}_\succeq Y$, $y \succeq \tilde{y}$.

Given a partial order $\succeq$ on $X$, define the binary operations $\lor$ and $\land$ on $X$ as follows: for all $x, y \in X$, $x \lor y = \text{lub}_\succeq \{x, y\}$ and $x \land y = \text{llb}_\succeq \{x, y\}$. Observe that, in general, $\text{lub}_\succeq \{x, y\}$ and $\text{llb}_\succeq \{x, y\}$ may not exist; however, by the antisymmetry of $\succeq$, if they exist, they are unique.

**Definition 5** The four-tuple $(X, \succeq, \lor, \land)$ is a lattice if, for all $x, y \in X$, $\text{lub}_\succeq \{x, y\}$ and $\text{llb}_\succeq \{x, y\}$ exist. A set $X$ has a lattice structure if $(X, \succeq, \lor, \land)$ is a lattice for some $\succeq$, $\lor$, and $\land$.

A lattice $(X, \succeq, \lor, \land)$ is complete if for any subset $Y \subseteq X$, $\text{lub}_\succeq Y \in X$ and $\text{llb}_\succeq Y \in X$.

Observe that since the binary operations $\lor$ and $\land$ follow from the partial order $\succeq$ there is some redundancy in the notation of a lattice. However, it is useful (and common) to refer simultaneously to the partial order and to the two binary operations because there is an (equivalent) algebraic approach where, instead of starting from the partial order $\succeq$, one can start from two binary operations on $X$ as follows. A join $\lor$ and a meet $\land$ on $X$ are two idempotent, commutative, associative, and absorbing binary operations on $X$; that is, for all $x, y, z \in X$, $x \lor x = x$ and $x \land x = x$, $x \lor y = y \lor x$ and $x \land y = y \land x$, $x \lor (y \lor z) = (x \lor y) \lor z$ and $x \land (y \land z) = (x \land y) \land z$, and $x = (x \lor y) \land x$ and $x = (x \land y) \lor x$. Given a join $\lor$ and a meet $\land$ on $X$, define the partial orders $\geq^\lor$ and $\geq^\land$ on $X$ as follows: for all $x, y \in X$,

$$x \geq^\lor y \text{ if and only if } x = x \lor y$$

and

$$x \geq^\land y \text{ if and only if } y = y \land x.$$ 

Indeed, both approaches are equivalent in the sense that the partial orders $\geq^\lor$ and $\geq^\land$ obtained from $\lor$ and $\land$ are the same and coincide with $\succeq$ (i.e., the partial order from which $\lor$ and $\land$ are defined).

---

6See Birkhoff (1979) for a general description of lattice theory.

7See Grätzer (2003).
We will use this second approach to show that the set $P^*$ (as well as $P^{**}$) has a lattice structure with the following two natural join and meet. Let $p, p' \in P^*$. Define $p \lor p' \equiv \hat{p} = (\hat{p}_j)_{j \in G}$ and $p \land p' \equiv \check{p} = (\check{p}_j)_{j \in G}$ as follows. For each $j \in G$,

$$\hat{p}_j = \max\{p_j, p'_j\}$$

and

$$\check{p}_j = \min\{p_j, p'_j\}.$$

Now, we are ready to state and prove the main result of this subsection.

**Theorem 5**  
The four-tuple $(P^*, \geq, \lor, \land)$ is a complete lattice.

**Proof**  
To show that $(P^*, \geq, \lor, \land)$ is a lattice let $p, p' \in P^*$ and set $Z = \{p, p'\}$. Then, $p \lor p' = p^S(Z)$ and $p \land p' = p^B(Z)$. By Proposition 3, $p \lor p' \in P^*$ and $p \land p' \in P^*$. Moreover, it is immediate to check that $\lor$ and $\land$ are idempotent, commutative, associative, and absorbing binary operations on $P^*$. Thus, $(P^*, \geq, \lor, \land)$ is a lattice. To prove that it is complete, consider any $\emptyset \neq Z \subseteq P^*$. By definition, $\text{lub}_{\geq} Z = p^S(Z)$ and $\text{llb}_{\geq} Z = p^B(Z)$. By Proposition 3, $p^S(Z), p^B(Z) \in P^*$. Thus, $(P^*, \geq, \lor, \land)$ is a complete lattice. \[\square\]

**Remark 7**  
Similar arguments to those used in the proofs of Proposition 3 and Theorem 5 also show that the four-tuple $(P^{**}, \geq, \lor, \land)$ is a complete lattice.

Define the partial orders $\succeq_u$ and $\succeq_w$ on $P^*$ as follows: for any pair $p, p' \in P^*$,

$$p \succeq_u p' \text{ if and only if } u_i(p) \geq u_i(p') \text{ for every } i \in B$$

and

$$p \succeq_w p' \text{ if and only if } w_k(p) \geq w_k(p') \text{ for every } k \in S.$$

Example 2 has showed that we may have $p, p' \in P^*$ with the property that $p \neq p'$, but $u_i(p) = u_i(p')$ for all $i \in B$; i.e., the binary relation $\succeq_u$ is not a partial order on $P^*$ because it is not antisymmetric since $p \succeq_u p', p' \succeq_u p$ and $p \neq p'$. Hence, the lattice structure of the set $P^*$ with the binary relation $\succeq_u$ is not inherited by the set of utilities of buyers that are attainable at equilibrium. However, next proposition says that the partial order $\succeq$ on the set of restricted equilibrium price vectors translates into the set of utilities of the buyers that are attainable at equilibrium (i.e., the statements (a) and (b) at the beginning of Section 4 are equivalent on $P^{**}$).
Proposition 6  Let \( p, p' \in P^* \) be two restricted equilibrium price vectors of market \( M \). Then,
\[
u_i(p) \geq u_i(p') \text{ for every } i \in B \text{ if and only if } p'_j \geq p_j \text{ for every } j \in G.
\]

Proof It follows from definition of \( P^* \) and Lemma 7 below.  \( \blacksquare \)

Lemma 7  Let \( p, p' \in P^* \) be two equilibrium price vectors of market \( M \). Then,
\[
u_i(p) \geq u_i(p') \text{ for every } i \in B \text{ if and only if } p'_j \geq p_j \text{ for every } j \in G^>.
\]

Proof of Lemma 7  Let \( p, p' \in P^* \).

\( \Rightarrow \) Assume \( u_i(p) \geq u_i(p') \) for every \( i \in B \). By (25), \( \gamma_i(p) \geq \gamma_i(p') \) for every \( i \in B \). By part (1.2) of Proposition 1, \( (\gamma(p), \pi(p)) \in D^* \) and \( (\gamma(p'), \pi(p')) \in D^* \). Assume \( j \in G^> \) and let \( k \in S \) be such that \( j \in G^> \). Then, there exist \( A \in F^* \) and \( i \in B \) such that \( A_{ijk} > 0 \). Thus, and since \( (p, A) \) and \( (p', A) \) are equilibria of \( M \), \( \sum_{i'} A_{i'jk} \in S_{jk}(p_j) \) and \( \sum_{i'} A_{i'jk} \in S_{jk}(p'_j) \) imply that
\[
p_j \geq r_{jk} \quad \text{and} \quad p'_j \geq r_{jk}.
\]

By condition (CS.1) of Theorem 3,
\[
\gamma_i(p) + \pi_{jk}(p) - \tau_{ijk} = 0 \quad \text{(34)}
\]
and
\[
\gamma_i(p') + \pi_{jk}(p') - \tau_{ijk} = 0. \quad \text{(35)}
\]

Thus,
\[
\gamma_i(p) + \pi_{jk}(p) = \gamma_i(p') + \pi_{jk}(p').
\]

Since \( \gamma_i(p) \geq \gamma_i(p') \) for every \( i \), \( \pi_{jk}(p') \geq \pi_{jk}(p) \) holds. By definition of \( \pi_{jk}(p') \) and \( \pi_{jk}(p) \), and since (33) holds, \( \pi_{jk}(p') = p'_j - r_{jk} \geq p_j - r_{jk} = \pi_{jk}(p) \). Thus, \( p'_j \geq p_j \).

\( \Leftarrow \) Assume \( p'_j \geq p_j \) for every \( j \in G^> \). Hence, for every \( i \in B \) and every \( j \in G^> \),
\[
v_{ij} - p_j \geq v_{ij} - p'_j. \quad \text{(36)}
\]

Fix \( i \in B \) and assume \( \nabla_i^>(p') \neq \emptyset \). Then, there exists \( j' \in G^> \) such that \( v_{ij'} - p'_{j'} > 0 \). By (36), \( v_{ij'} - p_{j'} > 0 \), which implies that \( \nabla_i^>(p) \neq \emptyset \). Hence, if \( \nabla_i^>(p') \neq \emptyset \) there exists \( j' \in G^> \) such that
\[
\gamma_i(p') = v_{ij'} - p'_{j'} \leq v_{ij'} - p_{j'} = \gamma_i(p).
\]
Thus, by (25), \( u_i(p) \geq u_i(p') \). Assume now that \( \nabla_i^>(p') = \emptyset \). Then, since by definition \( 0 \leq \gamma_i(p), \gamma_i(p') = 0 \leq \gamma_i(p) \). Hence, by (25), \( u_i(p) \geq u_i(p') \). Thus, for every \( i \in B \), \( u_i(p) \geq u_i(p') \). \( \square \)

Consider now the restriction of the partial order \( \succeq_u \) on the set \( P^{**} \) (a subset of \( P^* \)) and define the binary operations \( \lor_u \) and \( \land_u \) on \( P^{**} \) as the binary operations on \( P^* \) restricted to the set \( P^{**} \); namely, for all \( p, p' \in P^{**} \),

\[
p \lor_u p' \equiv \tilde{p} \text{ and } p \land_u p' \equiv \tilde{p}.
\]

**Theorem 6** The four-tuple \( (P^{**}, \succeq_u, \lor_u, \land_u) \) is a complete lattice.

**Proof** It follows from Theorem 5 and Proposition 6. \( \square \)

Next proposition shows that the conflict of interests between the two sides of the market on the set of equilibrium price vectors holds partially in our general model (statement (b) in the beginning of Section 4 implies statement (b) on \( P^* \)); namely, if buyers unanimously consider the equilibrium price vector \( p \) as being at least as good as equilibrium price vector price \( p_0 \) then all sellers consider \( p_0 \) as being at least as good as \( p \) (remember that Example 2 shows that the converse does not hold).

**Proposition 7** Let \( p, p' \in P^* \) be two equilibrium price vectors of market \( M \) such that \( u_i(p) \geq u_i(p') \) for all \( i \in B \). Then, \( w_k(p') \geq w_k(p) \) for all \( k \in S \).

**Proof** Let \( p, p' \in P^* \) and assume that \( u_i(p) \geq u_i(p') \) for every \( i \in B \). By Lemma 7, \( p_j' \geq p_j \) for every \( j \in G^> \). Fix \( k \in S \). Then, \( p_j' - r_{jk} \geq p_j - r_{jk} \) for every \( j \in G_k^> \). Thus, by (26), \( w_k(p') \geq w_k(p) \). \( \square \)

Proposition 8 states that utilities of the two extreme equilibrium price vectors are extreme and opposite.

**Proposition 8** Let \( M \) be a market. Then, for every \( p \in P^* \), the following properties hold.

(8.1) For every \( i \in B \), \( u_i(p^B) \geq u_i(p) \geq u_i(p^S) \).

(8.2) For every \( k \in S \), \( w_k(p^S) \geq w_k(p) \geq w_k(p^B) \).

**Proof** Consider any \( p \in P^* \). By their definitions, for all \( j \in G \), \( p_j^B \leq p_j \leq p_j^S \). In particular, these inequalities hold for all \( j \in G^> \). By Lemma 7, \( u_i(p^B) \geq u_i(p) \geq u_i(p^S) \) for all \( i \in B \). Thus, (8.1) holds. By Proposition 7, \( w_k(p^B) \leq w_k(p) \leq w_k(p^S) \) for all \( k \in S \). Thus, (8.2) holds. \( \square \)
Consider again Example 2. Take \( p = (3, 2, 10) \) and \( p' = (\frac{3}{2}, 3, 10) \) and observe that \( p, p' \in P^{**} \) and \( w_1(p) = w_1(p') = 12 \). Hence, \( p \succeq_w p' \), \( p' \succeq_w p \), and \( p \neq p' \). Thus, the binary relation \( \succeq_w \) is not a partial order on \( P^{**} \) because it is not antisymmetric. Hence, the set \( P^{**} \) does not have a lattice structure with the binary relation \( \succeq_w \) (and the induced binary operations \( \vee_w \) and \( \wedge_w \)).

5 Concluding Remarks

Before finishing the paper some remarks are in order. The first one is related with the dual lattice structure of the sets of equilibrium payoffs. Define the binary relation \( \succeq_B^{**} \) on the set of utilities of buyers that are attainable at (a restricted) equilibrium \( U^{**} = \{ u \in \mathbb{R}^m \mid \text{there exists } p \in P^{**} \text{ such that, for all } i \in B, u_i = u_i(p) \} \) (which coincides with \( U^* \)) as follows: for \( p, p' \in P^{**} \), \((u_i(p'))_{i \in B} \succeq_B^{**} (u_i(p))_{i \in B}\) if and only if \( u_i(p') \geq u_i(p) \) for all \( i \in B \). Proposition 6 implies that \( \succeq_B^{**} \) is equal to the natural binary relation \( \leq \) on \( P^{**} \) (the dual relation of \( \geq \)), which coincides with \( \succeq_u \). Since \( \leq \) is reflexive, antisymmetric, and transitive, the binary relations \( \succeq_B^{**} \) and \( \succeq_u \) are reflexive, antisymmetric, and transitive as well; hence, the set \( U^{**} \) has a lattice structure with the binary relation \( \succeq_B^{**} \). Moreover, define the binary relation \( \succeq_S^{**} \) on the set of utilities of sellers that are attainable at (a restricted) equilibrium \( W^{**} = \{ w \in \mathbb{R}^l \mid \text{there exists } p \in P^{**} \text{ such that, for all } k \in S, w_k = w_k(p) \} \) (which coincides with \( W^* \)) as follows: \((w_k(p'))_{k \in S} \succeq_S^{**} (w_k(p'))_{k \in S}\) if and only if \( w_k(p') \geq w_k(p) \) for all \( k \in S \). Thus, for all \( p, p' \in P^{**} \),

\[
(u_i(p'))_{i \in B} \succeq_B^{**} (u_i(p))_{i \in B} \iff u_i(p') \geq u_i(p) \text{ for all } i \in B \\
\iff p' \succeq_u p \\
\iff p \geq p' \\
\iff (p_j)_{j \in G} \succeq (p'_j)_{j \in G} \\
\iff w_k(p) \geq w_k(p') \text{ for all } k \in S \\
\iff (w_k(p))_{k \in S} \succeq_S^{**} (w_k(p'))_{k \in S}.
\]

Then, \( \succeq_S^{**} \) does not coincide with the natural binary relation \( \geq \) on \( P^{**} \). To see that, consider Example 2 and observe again that \( p = (3, 2, 10) \) and \( p' = (\frac{3}{2}, 3, 10) \) are restricted equilibrium price vectors of \( M \), and \( w_1(p) = 3 \cdot 2 + 2 \cdot 3 = 12 \) and \( w_1(p') = \frac{3}{2} \cdot 2 + 3 \cdot 3 = 12 \). Thus, \( w_1(p) = w_1(p') \) (\( w_1(p) \succeq_S w_1(p') \) and \( w_1(p') \succeq_S w_1(p) \)) but \( p \neq p' \). Hence, in contrast with the assignment game, the binary relation \( \succeq_S^{**} \) is not the dual of the binary relation \( \succeq_B^{**} \).
The second remark is related with the computational advantage of the linear programming approach. It allows to find the (essentially) unique optimal assignment \( A^* \) as the (essentially) unique solution of the (PLP). Moreover, and following Leonard (1983), it also allows to compute the two extreme equilibrium price vectors \( p^S \) and \( p^B \) as solutions of two linear programs. To find \( p^S \) we first compute \( (\gamma^S, \pi^S) \), the best dual solution from the point of view of the sellers, by letting \( T = T(A^*) \) and solving the following associated dual program: choose \((\gamma, \pi) \in \mathbb{R}^m \times \mathbb{R}^{n \times t} \) in order to

\[
\begin{align*}
\max & \quad \sum_{j,k} q_{jk} \cdot \pi_{jk} \\
\text{s. t.} & \quad \sum_i d_i \cdot \gamma_i + \sum_{j,k} q_{jk} \cdot \pi_{jk} = T \\
& \quad \gamma_i + \pi_{jk} \geq \tau_{ijk} \quad \text{for all } (i,j,k) \in B \times G \times S, \\
& \quad \gamma_i \geq 0 \quad \text{for all } i \in B, \\
& \quad \pi_{jk} \geq 0 \quad \text{for all } (j,k) \in G \times S, \\
& \quad \pi_{jk} = 0 \quad \text{if } q_{jk} = 0.
\end{align*}
\]

It is easy to show that \( (\gamma^S, \pi^S) \) is the unique solution of this linear program, which is among all solutions of the (DLP) the one with the highest entries in the matrix \( \pi \). Then, set \( p^S = p^{(\gamma^S, \pi^S)} \). Analogously, to find \( p^B \) we first compute \( (\gamma^B, \pi^B) \) by solving a symmetric linear program. Then, set \( p^B = p^{(\gamma^B, \pi^B)} \). Thus \( p^S \) and \( p^B \) are obtained by solving two dual linear programs. In addition, since the set of solutions of the (DLP) \( D^* \) is a convex and compact subset of \( \mathbb{R}^m \times \mathbb{R}^{n \times t} \), the set of equilibrium price vectors \( P^* \) is a convex and compact subset of \( \mathbb{R}^n_+ \).

Finally, we leave for future research the study of alternative cooperative notions like pairwise stability, core, or set-wise stability of the natural TU-game associated to our market and the analysis of their relationships with the set of competitive equilibria (Camiña (2006) and Sotomayor (2002, and 2007) perform parts of this analysis in their respective settings). We conjecture that the core and the set-wise stable set are non-empty, they are different sets, and the set of utilities attainable at equilibrium coincides with the set of set-wise stable utilities. We also leave for future research the characterization of the Bertrand equilibrium in which sellers set simultaneously prices of the goods they own. We conjecture that, under conditions guaranteeing that there is enough competition, the Bertrand equilibrium of this non-cooperative game is the buyers-optimal equilibrium price vector \( p^B \).
References


