Preference and Demand Examples

KC Border
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These notes show how to use the Lagrange–Karush–Kuhn–Tucker multiplier theorems to solve the problem of maximizing utility subject to a budget constraint:

$$\max_{x} u(x) \text{ subject to the constraints } m - p \cdot x \geq 0, \ x \geq 0,$$

where $p \gg 0$ and $m > 0$.

The basic steps are:

1. If possible, identify which constraints will bind, so the Karush–Kuhn–Tucker first-order conditions can be replaced by simple Lagrange conditions.

2. Use the first order conditions to express the optimal expenditures $p_i x_i$ in terms of the Lagrange multiplier $\lambda$.

3. Substitute these expenditures into the budget constraint to solve for $\lambda$.

4. Substitute $\lambda$ back into the expressions for $x_i$ to obtain the demand function.

Next we evaluate the utility at the demand function $x^*(p, m)$ to obtain the indirect utility function

$$v(p, m) = u(x^*(p, m)).$$

Then we use direct computation to verify Roy’s Law:

$$-\frac{\partial v(p, m)}{\partial m} = x^*_i(p, m).$$

Next I typically invert the indirect utility to get an expression for the Hicksian expenditure function $e(p, v)$, which is the optimal value function for the constrained minimization problem

$$\min_{x} p \cdot x \text{ subject to the constraints } u(x) \geq v, \ x \geq 0.$$

The indirect utility and the expenditure function satisfy

$$v(p, e(p, v)) = v,$$

so we can compute the expenditure function by solving for $m$ in terms of $v$, and changing the symbol for $m$ to $e$ and the symbol for $v$ to $u$. (Note the distinction between the Roman letter vee, $v$, and the Greek letter ypsilon, $\upsilon$. [Ok, maybe this is not the best set of fonts to show the difference.])

Then we use the envelope theorem to calculate the Hicksian compensated demands $\hat{x}_i(p, v)$ via

$$\frac{\partial v(p, v)}{\partial m} = \hat{x}_i(p, v).$$
On writing Lagrangeans

Some of my students have asked whether to write Lagrangeans with a plus sign or a minus sign in front of the Lagrange multipliers, that is, whether to write

\[ f(x) + \lambda g(x) \quad \text{or} \quad f(x) - \lambda g(x). \]

In one sense, it doesn’t matter, since the only difference is the sign of the Lagrange multiplier, but in economic problems because of the way we use the envelope theorem, the Lagrange multiplier usually has an interpretation as a rate of exchange, or price, and I usually want those numbers to be positive. To do this I use the following rule of thumb:

- Write the constraint function \( g \) so that the constrain is \( g \geq 0 \) even if you know that it must bind, and could be replaced by \( g = 0 \).
- For a maximization problem use a plus sign, and for a minimization problem use a minus sign.

In all the examples I can think of off the top of my head this will result in the quantity of interest being \( \lambda \geq 0 \) instead having the quantity of interest be \( -\lambda \) where \( \lambda \leq 0 \).

1 Cobb–Douglas preferences I: Logarithmic form

\[ u_1(x_1, \ldots, x_n) = \sum_{i=1}^{n} \alpha_i \ln x_i \]

where \( \alpha_i > 0, i = 1, \ldots, n \), and \( \sum_{i=1}^{n} \alpha_i = 1 \).

Remark 1 By convention \( \ln 0 = -\infty \), a common practice in convex analysis. It is clear then that any optimal consumption must satisfy \( x \gg 0 \), so we may ignore the nonnegativity constraints, and treat the first order conditions as equalities. It is also clear that \( u \) is monotonic, so the budget constraint will bind.
Lagrangian:
\[ \sum_{i=1}^{n} \alpha_i \ln x_i + \lambda \left( m - \sum_{i=1}^{n} p_i x_i \right) \]

First order conditions, using the binding constraint \( m = \sum_{i=1}^{n} p_i x_i \):
\[ \frac{\partial L}{\partial x_i} = \frac{\alpha_i}{x_i^*} - \lambda^* p_i = 0 \quad i = 1, \ldots, n. \]

So
\[ \alpha_i = \lambda^* p_i x_i^* \quad i = 1, \ldots, n. \]

Summing over \( i \) yields
\[ 1 = \lambda^* m. \]
as \( \sum_{i=1}^{n} \alpha_i = 1 \), so (1) becomes
\[ p_i x_i^* = \alpha_i m, \]
that is, \( \alpha_i \) is the fraction of income spent on good \( i \), so the demand function is
\[ x_i^*(p, m) = \frac{\alpha_i}{p_i} m. \]

Thus the indirect utility function is
\[ v(p, m) = \sum_{i=1}^{n} \alpha_i \ln \left( \frac{\alpha_i}{p_i} m \right) = \ln m - \sum_{i=1}^{n} \alpha_i \ln p_i + \sum_{i=1}^{n} \alpha_i \ln \alpha_i. \] 

(You might be tempted to write this as \( \ln m + \sum_{i=1}^{n} \alpha_i \ln \left( \frac{\alpha_i}{p_i} \right) \), which is more compact, but it makes it harder to read the derivatives.) The envelope theorem assures us that the partial derivatives of \( v \) are just the partial derivatives of the Lagrangean, so it must be that \( \lambda^* = \partial v / \partial m \), the marginal utility of money, which differentiation shows us is \( 1/m \), which we derived in the line after (1).

Verify Roy’s Law using the partials computed from (2):
\[ -\frac{\partial v}{\partial p_i} = -\frac{\alpha_i}{p_i} = \frac{\alpha_i}{m} = x_i^*(p, m). \]

Recall that the expenditure function \( e \) gives the level of income \( m \) needed to achieve a given level of utility \( v \). It therefore satisfies \( v(p, e(p, v)) = v \), so we can compute the expenditure function by solving (2) for \( m \) in terms of \( v \), and changing the symbol for \( m \) to \( e \) and the symbol for \( v \) to \( u \). So rewrite (2) to get
\[ v = \ln e - \sum_{i=1}^{n} \alpha_i \ln p_i + \sum_{i=1}^{n} \alpha_i \ln \alpha_i, \]
rearranging gives
\[ \ln e = v + \sum_{i=1}^{n} \alpha_i \ln p_i - \sum_{i=1}^{n} \alpha_i \ln \alpha_i, \]
so exponentiating gives
\[ e(p, v) = \exp(v) \prod_{i=1}^{n} \frac{p_i^{\alpha_i}}{\alpha_i}. \]

The Hicksian compensated demands are the derivatives of the expenditure function, so
\[ \hat{x}_j(p, v) = \frac{\alpha_j}{p_j} \exp(v) \prod_{i=1}^{n} \frac{p_i^{\alpha_i}}{\alpha_i}. \]

## 2 Cobb–Douglas preferences II: Multiplicative form

\[ u_{II}(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i^{\alpha_i} \]
where \( \alpha_i > 0, i = 1, \ldots, n, \) and \( \sum_{i=1}^{n} \alpha_i = 1. \)

**Remark 2** This functional form is gotten by transforming the previous utility by the increasing transformation \( u_{II} = \exp(u_I), \) so the demand should be the same, but the indirect utility and expenditure functions will be transformed. Note that in this formulation \( u \) is zero if any \( x_i \) is zero, so at any optimum we must have \( x \gg 0, \) and as before we may ignore the nonnegativity constraints, and treat the first order conditions as equalities. It is also clear that \( u \) is monotonic, so the budget constraint will bind.

Lagrangean:
\[ \prod_{i=1}^{n} x_i^{\alpha_i} + \lambda \left( m - \sum_{i=1}^{n} p_i x_i \right) \]
First order conditions, using the binding constraint \( m = \sum_{i=1}^{n} p_i x_i: \)
\[ \frac{\partial L}{\partial x_i} = \alpha_i \frac{\prod_{i=1}^{n} x_i^{\alpha_i}}{x_i^*} - \lambda^* p_i = 0 \quad i = 1, \ldots, n. \]
So letting \( u^* = \prod_{i=1}^{n} x_i^{\alpha_i}, \)
\[ \alpha_i u^* = \lambda^* p_i x_i^* \quad i = 1, \ldots, n. \quad (3) \]
Summing over \( i \) yields
\[ u^* = \lambda^* m. \]
as \( \sum_{i=1}^{n} \alpha_i = 1, \) so (3) becomes
\[ \alpha_i \lambda^* m = \lambda^* p_i x_i^*, \]
or
\[ p_i x_i^* = \alpha_i m, \]
that is, \( \alpha_i \) is the fraction of income spent on good \( i \), so the demand function is

\[
x_i^*(p, m) = \frac{\alpha_i m}{p_i}.
\]

Thus the indirect utility function is

\[
v(p, m) = m \prod_{i=1}^{n} \left( \frac{\alpha_i}{p_i} \right)^{\alpha_i}.
\]

Thus

\[
\frac{\partial v}{\partial p_j} = \alpha_j \left\{ \frac{m \prod_{i=1}^{n} \left( \frac{\alpha_i}{p_i} \right)^{\alpha_i}}{p_j} \right\} = \frac{-\alpha_j m \prod_{i=1}^{n} \left( \frac{\alpha_i}{p_i} \right)^{\alpha_i}}{p_j^2} = x_j^*(p, m).
\]

Use (4) to verify Roy’s Law:

\[
\frac{\partial v}{\partial m} = \frac{-\alpha_j m \prod_{i=1}^{n} \left( \frac{\alpha_i}{p_i} \right)^{\alpha_i}}{\prod_{i=1}^{n} \left( \frac{\alpha_i}{p_i} \right)^{\alpha_i}} = \frac{\alpha_j m \prod_{i=1}^{n} \left( \frac{\alpha_i}{p_i} \right)^{\alpha_i}}{p_j} = x_j^*(p, m).
\]

We can compute the expenditure function by solving (4) for \( m \) in terms of \( v \). Changing the symbol for \( m \) to \( e \) and the symbol for \( v \) to \( \upsilon \), rewrite (4) as

\[
\upsilon = e \prod_{i=1}^{n} \left( \frac{\alpha_i}{p_i} \right)^{\alpha_i}.
\]

Rearranging gives

\[
e(p, \upsilon) = \upsilon \prod_{i=1}^{n} \left( \frac{p_i}{\alpha_i} \right)^{\alpha_i}.
\]

If we wish, we can rewrite this as

\[
e(p, \upsilon) = \upsilon \frac{u(p)}{u'(\alpha)},
\]

where \( u = u_{II} \) is the Cobb–Douglas utility in multiplicative form, and \( \alpha = (\alpha_1, \ldots, \alpha_n) \).

So the Hicksian demands are given by

\[
\hat{x}_i(p, \upsilon) = \frac{\partial e(p, \upsilon)}{\partial p_i} = \frac{\alpha_i}{p_i} \exp(\upsilon) \prod_{j=1}^{n} p_j^{\alpha_j} \prod_{j=1}^{n} \alpha_j^{\alpha_j}.
\]
3 Logarithmic quasi-linear preferences

\[ u(y, x_1, \ldots, x_n) = y + \beta \sum_{i=1}^{n} \alpha_i \ln x_i \]

where \( \beta, \alpha_i > 0, i = 1, \ldots, n \), and \( \sum_{i=1}^{n} \alpha_i = 1 \).

Note that each indifference curve is a vertical translate of every other curve, and that each intersects the \( x \)-axis.

For reasons that will become clear, let us make \( y \) the numéraire \( (p_y = 1) \). Then the Lagrangean is

\[ y + \beta \sum_{i=1}^{n} \alpha_i \ln x_i + \lambda \left( m - y - \sum_{i=1}^{n} p_i x_i \right) \]

First order conditions, using the binding constraint \( m = y + \sum_{i=1}^{n} p_i x_i \):

\[ 1 - \lambda ^* \leq 0 \]

with \( \lambda ^* = 1 \) if \( y^* > 0 \), and

\[ \beta \frac{\alpha_i}{x_i^*} - \lambda ^* p_i = 0 \quad i = 1, \ldots, n. \]

So assuming \( y^* > 0 \), this gives

\[ p_i x_i^*(p, m) = \alpha_i \beta. \]

In other words, the amount spent on good \( i \) is independent of prices and income. Thus

\[ y^*(p, m) = m - \beta. \]

Note that this only works for \( m \geq \beta \), which corresponds to \( y^* \geq 0 \).

If \( m < \beta \), then \( \lambda ^* > 1 \), and the remaining first order conditions become

\[ \beta \frac{\alpha_i}{x_i^*} - \lambda ^* p_i = 0 \quad i = 1, \ldots, n, \]
which by the same reasoning as in problem (a) gives
\[ x_i^*(p, m) = \alpha_i \frac{m}{p_i}. \]

Putting this all together yields

\[
\begin{align*}
  y^*(p, m) & = \begin{cases} 
  m - \beta & m \geq \beta \\
  0 & m < \beta 
  \end{cases} \\
  x_i^*(p, m) & = \begin{cases} 
  \alpha_i \frac{\beta}{p_i} & m \geq \beta \\
  \alpha_i \frac{m}{p_i} & m < \beta 
  \end{cases}
\end{align*}
\]

This leads to the indirect utility

\[
v(p, m) = \begin{cases} 
  m - \beta + \beta (\ln \beta - \sum_{i=1}^n \alpha_i \ln p_i + \sum_{i=1}^n \alpha_i \ln \alpha_i) & m \geq \beta \\
  \beta (\ln m - \sum_{i=1}^n \alpha_i \ln p_i + \sum_{i=1}^n \alpha_i \ln \alpha_i) & m < \beta 
  \end{cases}
\]

(5)

Roy’s Law:

\[
- \frac{\partial v}{\partial p_i} = \begin{cases} 
  \frac{\alpha_i \beta}{p_i} & \text{for } m \geq \beta \\
  - \frac{\alpha_i \beta}{p_i} & \frac{\alpha_i m}{p_i} & \text{for } m < \beta 
  \end{cases} = x_i^*(p, m).
\]

Solving (5) for \( m = e \) in terms of \( v \) gives

\[
e(p, v) = v + \beta - \beta \left( \ln \beta - \sum_{i=1}^n \alpha_i \ln p_i + \sum_{i=1}^n \alpha_i \ln \alpha_i \right)
\]

for \( v \geq \beta (\ln \beta - \sum_{i=1}^n \alpha_i \ln p_i + \sum_{i=1}^n \alpha_i \ln \alpha_i) \), and

\[
e(p, v) = \exp(v/\beta) \frac{\prod_{i=1}^n p_i^{\alpha_i}}{\prod_{i=1}^n \alpha_i}
\]

otherwise.

So the Hicksian demands are given by
\[ \hat{x}_j(p, v) = \frac{\partial e(p, v)}{\partial p_j} = \begin{cases} \beta \alpha_j/p_j & \text{for } v \geq \beta \left( \ln \beta - \sum_{i=1}^n \alpha_i \ln p_i + \sum_{i=1}^n \alpha_i \ln \alpha_i \right) \\ \frac{\alpha_j}{p_j} \exp(v/\beta) \prod_{i=1}^n \frac{p_i^{\alpha_i}}{\alpha_i} & \text{otherwise.} \end{cases} \]

4 Linear preferences

\[ u(x_1, \ldots, x_n) = \sum_{i=1}^n \alpha_i x_i \]
where \( \alpha_i \geq 0, i = 1, \ldots, n \), and \( \sum_{i=1}^n \alpha_i = 1 \). (Hint: Remember Kuhn–Tucker.)

The Lagrange is
\[ \sum_{i=1}^n \alpha_i x_i + \lambda \left( m - \sum_{i=1}^n p_i x_i \right) \]

The first-order conditions are
\[ \alpha_i - \lambda^* p_i \leq 0 \quad i = 1, \ldots, n, \]
so \( \lambda^* = \max_i \frac{\alpha_i}{p_i} \), and \( \frac{\alpha_j}{p_j} < \lambda^* \) implies \( x_j^* = 0 \). So for now assume that \( i^* \) is the unique maximizer of \( \frac{\alpha_i}{p_i} \). Then

\[ x_j^*(p, m) = \begin{cases} \frac{m}{p_i^*} & j = i^* \\ 0 & \text{otherwise.} \end{cases} \]

When \( i^* \) is not unique, there is no unique solution, but convex combinations of the above are all valid demands. That is,
\[ x^*(p, m) = \text{convex hull of } \left\{ \frac{m e^j}{p_j} : \frac{\alpha_j}{p_j} \geq \frac{\alpha_i}{p_i}, i = 1, \ldots, n \right\}, \]
where \( e^j \) is the \( j^{th} \) unit coordinate vector.

The indirect utility is thus
\[ v(p, m) = \alpha_i^* x_i^* = \frac{\alpha_i^*}{p_i^*} = m \cdot \max_i \frac{\alpha_i}{p_i} = \frac{m}{\min_i \frac{p_i}{\alpha_i}}. \]

Roy’s Law:
\[
\frac{\partial v}{\partial p_i} = -\frac{m}{p_i} \frac{\alpha_i^*}{(p_i^*)^2} = \frac{m}{p_i^*} = x_{i^*}^*(p, m).
\]

\[
x_j^*(p, m) = \frac{\partial v}{\partial p_j} = -\frac{0}{\alpha_i^*} = 0 \quad j \neq i^*
\]

And the expenditure function satisfies

\[
e(p, v) = v \min_i \frac{p_i}{\alpha_i}
\]

So the Hicksian demands are given by

\[
\hat{x}_j(p, v) = \frac{\partial e(p, v)}{\partial p_j} = \begin{cases} \frac{v}{\alpha_j} & j = i^* \\ 0 & \text{otherwise} \end{cases}
\]

5 Leontieff fixed-proportion preferences

\[
u(x_1, \ldots, x_n) = \min \{\alpha_1 x_1, \ldots, \alpha_n x_n\}
\]

where \(\alpha_i \geq 0, i = 1, \ldots, n\). (Hint: Calculus is worthless here.)
It is easy to see that $\alpha_1 x_1^*(p, m) = \cdots = \alpha_n x_n^*(p, m)$, denote this common value by $c$. Then $p_i x_i^*(p, m) = c\frac{p_i}{\alpha_i}$, and summing over $i$ gives $m = c \sum_{i=1}^{n} \frac{p_i}{\alpha_i}$, so

$$x_i^*(p, m) = \frac{m}{\alpha_i \sum_{j=1}^{n} \frac{p_j}{\alpha_j}}$$

The indirect utility is then

$$v(p, m) = \frac{m}{\sum_{j=1}^{n} \frac{p_j}{\alpha_j}}.$$  \hfill (6)

Roy’s Law:

$$- \frac{\partial v}{\partial p_i} = -\frac{m}{\alpha_i} \left( \frac{\sum_{j=1}^{n} \frac{p_j}{\alpha_j}}{\sum_{j=1}^{n} \frac{p_j}{\alpha_j}} \right)^2 \frac{1}{\alpha_i \sum_{j=1}^{n} \frac{p_j}{\alpha_j}} = \frac{m}{\alpha_i \sum_{j=1}^{n} \frac{p_j}{\alpha_j}} = x_i^*(p, m).$$

And by (6) the expenditure function satisfies

$$e(p, v) = v \sum_{j=1}^{n} \frac{p_j}{\alpha_j}.$$  

So the Hicksian compensated demands are:

$$\hat{x}_i(p, v) = \frac{v}{\alpha_i}.$$