A Polyhedral Cone Counterexample

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Abstract

This is an example of a pointed generating convex cone in $\mathbb{R}^4$ with 5 extreme rays, but whose dual cone has 6 extreme rays (and vice-versa).

Recall that a ray in a vector space is the set of nonnegative scalar multiples of a single nonzero point. A cone is a nonempty subset $C$ of a vector space that is closed under multiplication by nonnegative scalars. A cone is trivial if it contains only 0. A nontrivial cone is the union of the rays generated by its nonzero points. A cone $C$ is generating if $C - C$ is the entire vector space, or equivalently if it spans the space. A convex cone is a cone that is a convex set. A set in a vector space is a convex cone if and only if it is closed under nonnegative linear combinations. A convex cone is pointed if it includes no lines. A ray $A$ is an extreme ray of the cone $C$ if it is a subset of $C$ and if points on $A$ cannot be written as a linear combination of linearly independent points in $C$, that is, if $x \in A$, $x = y + z$, $y, z \in C$ together imply that $y$ and $z$ are dependent. A finite cone is the convex cone generated by finitely many nonzero points. A finite cone has finitely many extreme rays, and a pointed finite cone is the convex hull of its extreme rays. Finally, the dual cone $C^*$ of a cone $C \subset \mathbb{R}^m$ is defined by

$$C^* = \{ p \in \mathbb{R}^m : p \cdot y \leq 0 \text{ for all } y \in C \}.$$ 

For a finite cone $C$ (actually any closed convex cone), $C^{**} = C$. We shall use the following characterization of extreme rays of $C^*$:

**Weyl’s Facet Lemma** Let $C$ be a finite cone in $\mathbb{R}^m$ generated by $a_1, \ldots, a_n$. Then a nonzero point $p \in C^* \subset \mathbb{R}^m$ is on an extreme ray of $C^*$ if and only if $\{ a_i : p \cdot a_i = 0 \}$ has rank $m - 1$.

See, e.g., D. Gale [8, Theorem 2.16, p. 65] for a proof of this result. (Warning: He omits the requirement that $p$ be nonzero from the statement, but not the proof.) Or see Theorems 11–12 in H. Weyl [13], which are stated in terms of facets of cones. Note that a consequence of this is that the dual cone of a finite cone is also a finite cone.
Example

Consider the finite convex cone $C$ in $\mathbb{R}^4$ generated by the set $A = \{a_1, \ldots, a_5\}$ where

$$a_n = \begin{bmatrix} 1 \\ n \\ n^2 \\ n^3 \end{bmatrix}.$$ 

Let $A$ be the $4 \times 5$ matrix $A$ with columns in $A$:

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \\ 1 & 8 & 27 & 64 & 125 \end{bmatrix}.$$ 

Then the cone $C$ is just

$$C = \{Ax : x \geq 0\}.$$ 

It is easy to verify that every subset of $\{a_1, \ldots, a_5\}$ of size four is linearly independent. Thus the cone $C$ spans $\mathbb{R}^4$, or in other words, it is generating. It is also easy to see that $C$ is pointed (that is, it contains no lines, only half-lines), as it is a subset of the nonnegative cone.

I claim that the dual cone $C^*$

$$C^* = \{p \in \mathbb{R}^4 : p \cdot y \leq 0 \text{ for all } y \in C\} = \{p \in \mathbb{R}^4 : p^t A \leq 0\}$$

is generated by the 6 points $p_1, \ldots, p_6$ that make up the 6 columns of the $4 \times 6$ matrix

$$P = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \\ -60 & -30 & -10 & 6 & 12 & 20 \\ 47 & 31 & 17 & -11 & -19 & -29 \\ -12 & -10 & -8 & 6 & 8 & 10 \\ 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix}.$$ 

That is, $C^* = \{Pz : z \geq 0\}$. Moreover, I claim that the cone $C$ has five extreme rays (generated by $a_1, \ldots, a_5$), and $C^*$ has six extreme rays (generated by $p_1, \ldots, p_6$).
Proof

The cone $C^*$ is the set of solutions $p$ to the system of inequalities

\[
p \cdot a_1 \leq 0 \\
\vdots \\
p \cdot a_5 \leq 0
\]

We shall use Weyl’s Lemma to find the extreme rays of $C^*$. In our example $m = 4$ and $n = 5$. We shall use the “brute force” approach and look at all subsets of $A = \{a_1, \ldots, a_5\}$ of rank 3. Since any four vectors belonging to $A$ are linearly independent, a subset of $A$ has rank 3 if and only if it has three elements. Fortunately there are only $\binom{5}{3} = 10$ of these subsets, so it is feasible to enumerate them by hand. Each subset $B$ of size three determines a one-dimensional subspace in $\mathbb{R}^4$ (a line) consisting of vectors orthogonal to each element of $B$ (the orthogonal complement of $B$). It is straightforward to solve for this subspace, and I have done so. Points $p_i$ taken from each of these ten lines are used for the columns of the $4 \times 10$ matrix

\[
\hat{P} = \begin{bmatrix}
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} \\
-60 & -30 & -10 & 6 & 12 & 20 & -40 & -24 & -15 & -8 \\
47 & 31 & 17 & -11 & -19 & -29 & 38 & 26 & 23 & 14 \\
-12 & -10 & -8 & 6 & 8 & 10 & -11 & -9 & -9 & -7 \\
1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

(Note that you have seen $p_1, \ldots, p_6$ before.) Now construct the $5 \times 10$ matrix whose elements are the inner products $p_j \cdot a_i$:

\[
A'\hat{P} = \begin{bmatrix}
a_1 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} \\
-24 & -8 & 0 & 0 & 0 & 0 & -12 & -6 & 0 & 0 \\
-6 & 0 & 0 & -2 & -6 & 0 & 0 & 3 & 0 \\
0 & 0 & -4 & 0 & 0 & -4 & 2 & 0 & 0 & -2 \\
0 & -2 & -6 & -6 & 0 & 0 & 0 & -3 & 0 \\
0 & 0 & 0 & -24 & -8 & 0 & 0 & 6 & 0 & 12
\end{bmatrix}
\]

For the first six columns, all the entries are nonpositive, so $p_1, \ldots, p_6$ each belong to $C^*$. However for columns 7 through 10, there are entries of both signs. This means that for $i = 7, \ldots, 10$, no nonzero multiple of $p_j$ belongs to $C^*$.
Further inspection shows that
\[
\begin{align*}
\{ a_i : p_1 \cdot a_i = 0 \} &= \{ a_3, a_4, a_5 \} \\
\{ a_i : p_2 \cdot a_i = 0 \} &= \{ a_2, a_3, a_5 \} \\
\{ a_i : p_3 \cdot a_i = 0 \} &= \{ a_1, a_2, a_5 \} \\
\{ a_i : p_4 \cdot a_i = 0 \} &= \{ a_1, a_2, a_3 \} \\
\{ a_i : p_5 \cdot a_i = 0 \} &= \{ a_1, a_3, a_4 \} \\
\{ a_i : p_6 \cdot a_i = 0 \} &= \{ a_1, a_4, a_5 \} \\
\{ a_i : p_7 \cdot a_i = 0 \} &= \{ a_2, a_4, a_5 \} \\
\{ a_i : p_8 \cdot a_i = 0 \} &= \{ a_2, a_3, a_4 \} \\
\{ a_i : p_9 \cdot a_i = 0 \} &= \{ a_1, a_3, a_5 \} \\
\{ a_i : p_{10} \cdot a_i = 0 \} &= \{ a_1, a_2, a_4 \}
\end{align*}
\]
This accounts for all subsets of \{a_1, \ldots, a_5\} of rank 3. So by Weyl’s Facet Lemma, it shows that \(C^*\) is generated by \(p_1, \ldots, p_6\), which lie on distinct extreme rays of \(C^*\).

As an aside, you should verify that
\[
\begin{align*}
\{ p_j : p_j \cdot a_1 = 0 \} &= \{ p_3, p_4, p_5, p_6 \} \text{ has rank 3} \\
\{ p_j : p_j \cdot a_2 = 0 \} &= \{ p_2, p_3, p_4 \} \text{ has rank 3} \\
\{ p_j : p_j \cdot a_3 = 0 \} &= \{ p_1, p_2, p_4, p_5 \} \text{ has rank 3} \\
\{ p_j : p_j \cdot a_4 = 0 \} &= \{ p_1, p_5, p_6 \} \text{ has rank 3} \\
\{ p_j : p_j \cdot a_5 = 0 \} &= \{ p_1, p_2, p_3, p_6 \} \text{ has rank 3,}
\end{align*}
\]
confirming that \(a_1, \ldots, a_5\) are on distinct extreme rays of \(C^{**} = C\).

Notes on the example

The points \(a_1, \ldots, a_5\) are multiples of five distinct nonzero points on the moment curve in \(R^4\). The moment curve is the set of points of the form \((t, t^2, t^3, t^4)\), for \(t \geq 0\). G. M. Ziegler [15, Example 0.6, pp. 10–13] describes a polytope based on the moment curve that suggested this example. I used T. Christof and A. Loebel’s computer program PORTA [3, 4] to compute the dual cone and the facets of \(C\). The program uses the Fourier–Motzkin Elimination Algorithm (see, e.g., G. M. Ziegler [15, § 1.2, pp. 32–39]) with extensions due to N. V. Chernikova [1, 2] to efficiently find the six extreme rays of \(C^*\). That left me with only four subsets of rank 3 to find the orthogonal complement by hand. After finding two by hand, I used Mathematica 5.0 to compute \(p_7, \ldots, p_{10}\) and all the inner products \(p_j \cdot a_i\), and its MatrixRank function to double check the ranks. Feel free to check any of these computations by hand.
References


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