Boundedness of Allocations

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1 Allocations
Recall that in an Arrow–Debreu description, an economy is summarized by a list
\[ E = ((X_i, \succ_i)_{i=1}^m, (Y_j)_{j=1}^n, \omega), \]
and an allocation for the economy \( E \) is a list
\[ (x^1, \ldots, x^m, y^1, \ldots, y^n) \]
satisfying
\[ x^i \in X_i \quad i = 1, \ldots, m \]
\[ y^j \in Y_j \quad j = 1, \ldots, n \]
\[ \sum_{i=1}^m x^i = \omega + \sum_{j=1}^n y^j. \]

2 Boundedness

Theorem 1 Let the economy \( E = ((X_i, \succ_i)_{i=1}^m, (Y_j)_{j=1}^n, \omega) \) satisfy:

1. For each \( i = 1, \ldots, m \), \( X_i \) is closed, convex and bounded from below.
2. For each \( j = 1, \ldots, n \), \( Y_j \) is closed, convex and \( 0 \in Y_j \).
3. \( AY \cap R^n_+ = \{0\} \). (This is implied by free disposability and the next assumption.)
4. \( Y \cap -Y = \{0\} \).

Then the set of allocations is compact.
If in addition we assume \( \omega \in \sum_{i=1}^m X_i \), then the set of allocations is nonempty.

Proof: (cf. Debreu [1, pp. 77–78]) This proof relies heavily on my handout on asymptotic cones. For convenience, I reproduce the statement of the results below (but not the proofs).

Let \( F \subset R^{(m+n)} \) denote the set of allocations. If \( \omega \in \sum_{i=1}^m X_i \), write \( \omega = \sum_{i=1}^m x^i \), where each \( x^i \in X_i \). Then \((x^1, \ldots, x^m, 0, \ldots, 0)\) clearly belongs to \( F \), so it is nonempty.

Let
\[ M = \left\{ (x_i), (y_j) \in (R^p)^{m+n} : \sum_{i=1}^m x_i - \sum_{j=1}^n y_j - \omega = 0 \right\}. \]
Then $F = (\prod_{i=1}^m X_i \times \prod_{j=1}^n Y_j) \cap M$.

Thus $F$ is clearly closed, being the intersection of two closed sets, so it is compact if it is bounded. By Lemma 3(1), to show boundedness it suffices to show that the asymptotic cone $AF = \{0\}$.

By Lemma 3(j) below,

$$AF \subset A \left( \prod_{i=1}^m X_i \times \prod_{j=1}^n Y_j \right) \cap AM.$$

Also by Lemma 3(e), $A \left( \prod_{i=1}^m X_i \times \prod_{j=1}^n Y_j \right) \subset \prod_{i=1}^m AX_i \times \prod_{j=1}^n AY_j$. Since each $X_i$ is bounded below there is some $b_i \in R^\ell$ such that $X_i \subset b_i + R^\ell_+$. Thus $AX_i \subset A(b_i + R^\ell_+) = AR^\ell_+ = R^\ell_+$. Also by Lemma 3(d,k), $AY_j \subset AY$. Again by Lemma 3(i), since $M - \omega$ is a closed cone, $AM = M - \omega$. Note that $M - \omega = \{(x,y) \in (R^\ell)^{m+n} : \sum_{i=1}^m x_i - \sum_{j=1}^n y_j = 0\}$. Thus we can show $AF = \{0\}$ if we can show that

$$\left( \prod_{i=1}^m R^\ell_+ \times \prod_{j=1}^n AY \right) \cap (M - \omega) = \{0\}.$$

In other words, we need to show that if $x_i \in R^\ell_+, i = 1, \ldots, m$, and $y_j \in AY, j = 1, \ldots, n$ and $\sum_{i=1}^m x_i - \sum_{j=1}^n y_j = 0$, then $x_1 = \ldots x_n = y_1 = \ldots = y_n = 0$. Now $\sum_{i=1}^m x_i \geq 0$, so that $\sum_{j=1}^n y_j \geq 0$ too. Since $AY$ is a closed cone (Lemma 3(g,i)), $\sum_{j=1}^n y_j \in AY$. Since $AY \cap R^\ell_+ = \{0\}$, $\sum_{i=1}^m x_i - \sum_{j=1}^n y_j = 0$ implies $\sum_{i=1}^m x_i = 0 = \sum_{j=1}^n y_j$. Now $x_i \geq 0$ and $\sum_{i=1}^m x_i = 0$ clearly imply that $x_i = 0, i = 1, \ldots, m$. Rewriting $\sum_{j=1}^n y_j = 0$ yields $y_k = -(\sum_{j\neq k} y_j)$. Both $y_k$ and this last sum belong to $Y$ as $AY \subset Y$ (again by Lemma 3(cc,h)). Thus $y_k \in Y \cap (-Y) \text{ so } y_k = 0$. This is true for all $k = 1, \ldots, n$.

3 Asymptotic cones

This section reproduces some of the statements of results from my handout on asymptotic cones. Consult it for proofs.

**Definition 2** Let $E \subset R^m$. The **asymptotic cone** of $E$, denoted $AE$ is the set of all possible limits $z$ of sequences of the form $\{\lambda_n x^n\}$, where each $x^n \in E$, each $\lambda_n > 0$, and $\lambda_n \to 0$. Let us call such a sequence a **defining sequence for** $z$.

The recession cone $0^+ F$ of a closed convex set $F$ is the set of all directions in which $F$ is unbounded, that is, $0^+ F = \{z : \forall x \in F \forall \alpha \geq 0 x + \alpha z \in F\}$. (See Rockafellar [2, Theorem 8.2].)

**Lemma 3** (a) $AE$ is indeed a cone.

(b) If $E \subset F$, then $AE \subset AF$.

(c) $A(E + x) = AE$ for any $x \in R^m$.

(cc) $0^+ E \subset AE$. 
(d) \( A E_1 \subset A(E_1 + E_2) \).

(e) \( A \prod_{i \in I} E_i \subset \prod_{i \in I} AE_i \).

(f) \( AE \) is closed.

(g) If \( E \) is convex, then \( AE \) is convex.

(h) If \( E \) is closed and convex, then \( AE = 0^+ E \). (The asymptotic cone really is a generalization of the recession cone.)

(i) If \( C \) is a cone, then \( AC = \overline{C} \).

(j) \( A \cap_{i \in I} E_i \subset \cap_{i \in I} AE_i \). The reverse inclusion need not hold.

(k) If \( E + F \) is convex, then \( AE + AF \subset A(E + F) \).

(l) A set \( E \subset \mathbb{R}^m \) is bounded if and only if \( AE = \{0\} \).

**Definition 4** Let \( C_1, \ldots, C_n \) be cones in \( \mathbb{R}^m \). We say that they are **positively semi-independent** if whenever \( x^i \in C_i \) for each \( i = 1, \ldots, n \),

\[
x_1 + \cdots + x_n = 0 \quad \implies \quad x^1 = \cdots = x^n = 0.
\]

Clearly, any subset of a set of semi-independent cones is also semi-independent.

**Proposition 5** Let \( E_i \subset \mathbb{R}^m \), \( i = 1, \ldots, n \), be closed and nonempty. If \( AE_i \), \( i = 1, \ldots, n \), are positively semi-independent, then \( \sum_{i=1}^n E_i \) is closed, and \( A \sum_{i=1}^n E_i \subset \sum_{i=1}^n AE_i \).

**References**
