Production and Cost functions

Figure 8.—Milk Production Surface from Equation (15). Ability and Time Set at a Mean.

Outline

• Production function
• The Firm's problem
• Parallel with Consumer
• From Production function to cost functions

Figure 10.—Pork Costulator for Determining Least-Cost Rations
Production function-facts

• Its easy, its engineering data.
  – You go and figure out what happens when you vary the quantities of inputs.
  – You may get nothing
    • because some ingredient is necessary (strict complement)
  – You may get faster output, or slower...
  – Tabulate all that data and you can get a set of points that represent different ways of producing the same amount.
  – Run plant 24 hours a day (three shifts) or Increase number of assembly lines
  – Use your workers for their strength, give them more machines. (unload ships with cranes or with people)
Production and Cost functions

From facts to Theory

• When you measure them, you tend to find
• Isoquants are weakly convex
  – marginal diminishing returns in increasing one input.
• They can have decreasing, constant, or increasing returns.
  – Involves issues of indivisibility (example plant)
    • Returns increase as you come close to capacity
    • Then fall because some input is not adjusted.
Cobb-Douglas Isoquants
Substitutes

\[ \frac{\partial^2 f}{\partial K \partial L} < 0 \]

Complements

\[ \frac{\partial^2 f}{\partial K \partial L} > 0 \]
Firms vs consumer

• Consumer has
  – Utility function
  – Cost of goods
  – Faces budget constraint

• Consumer problem 1
  – Max U subj to PX≤Y

• Consumer problem 2
  – Min PX subj to U(X)≥U

• Firm has
  – Production function
  – Costs of inputs
  – Faces production constraint.

• Firm problem 1
  – Max F(X) subj to PX≤Y

• Firm problem 2
  – Min PX subj to F(X)≥Q

If you can solve 1 you can solve 2 if you can solve the consumer’s problem you can solve the firm’s.
More symmetry

- **Consumer**
  \[
  \frac{\partial U}{\partial x_1} = \frac{p_1}{p_2}
  \]
  Ratio of marginal utilities equal to the price ratio

- **Firm**
  \[
  \frac{\partial F}{\partial x_1} = \frac{p_1}{p_2}
  \]
  \[
  \frac{\partial F}{\partial x_2} = \frac{w}{r}
  \]
  Ratio of marginal product equal to the price ratio
Short run vs long run

• Marginal product is non-negative

\[
\frac{\partial f}{\partial k} > 0 \quad \frac{\partial^2 f}{(\partial k)^2} < 0.
\]

• More is better but there are marginal diminishing returns

• Some inputs (labor, raw materials) more readily changed than others (plant and equipment)...so there is a long run and a short run.

• Short run, take fixed assets and technology as given
  – choose the right mix of inputs to run that firm

• Long run chose your plant size, technology.
  – Suppose the price of gas increases for a taxi firm
    • What should it do?
Short Run Profit Maximization
(take \( p \) as given)

\[
\pi = pF(K, L) - rK - wL.
\]

\[
0 = \frac{\partial \pi}{\partial L} = p \frac{\partial F}{\partial L}(K, L^*) - w. \quad \bullet \text{ FOC}
\]

- Wage (price of input) is equal to the value of its marginal product

\[
0 \geq \frac{\partial^2 \pi}{(\partial L)^2} = p \frac{\partial^2 F}{(\partial L)^2}(K, L^*). \quad \bullet \text{ SOC}
\]

- Guaranteed if there are marginal decreasing returns
Graphical Depiction

- Slope zero at maximum
- Slope negative to right of maximum
- Slope positive to left of maximum
Short-run Effect of a Wage Increase

\[ \frac{\partial \pi}{\partial L} = p \frac{\partial F}{\partial L} (K, L^*) - w = 0. \]

I can differentiate the FOC with respect to w (the wage)

\[ 0 = p \frac{\partial^2 F}{(\partial L)^2} (K, L^*(w)) L^*(w) - 1, \]

\[ L^*(w) = \frac{1}{p \frac{\partial^2 F}{(\partial L)^2} (K, L^*(w))} \leq 0. \]

\[ p \frac{\partial^2 F}{(\partial L)^2} (K, L^*(w)) \]

\[ P>0, F''<0 \]
Aside: Revealed Preference

• Revealed preference is a powerful technique to prove comparative statics
• Works without assumptions about continuity or differentiability
• Suppose $w_1 < w_2$ are two wage levels
• The entrepreneur chooses $L_1$ when the wage is $w_1$ and $L_2$ when the wage is $w_2$
Revealed Preference Proof

Prefer \( L_1 \) to \( L_2 \) when wage = \( w_1 \)
\[ pf(K,L_1) - rK - w_1L_1 \geq pf(K,L_2) - rK - w_1L_2 \]

Prefer \( L_2 \) to \( L_1 \) when wage = \( w_2 \)
\[ pf(K,L_2) - rK - w_2L_2 \geq pf(K,L_1) - rK - w_2L_1. \]

Sum these two
\[ pf(K,L_1) - rK - w_1L_1 + pf(K,L_2) - rK - w_2L_2 \geq \]
\[ pf(K,L_1) - rK - w_2L_1 + pf(K,L_2) - rK - w_1L_2 \]
\[ -w_1L_1 - w_2L_2 \geq -w_2L_1 - w_1L_2 \]
Revealed Preference, Cont’d

\[-w_1L_1 - w_2L_2 \geq -w_2L_1 - w_1L_2\]

• But remember \( w_1 < w_2 \)

\[(w_1 - w_2)(L_2 - L_1) \geq 0.\]

• So the only way for above to be true is

• \( 0 \geq (L_2 - L_1) \iff L_1 \geq L_2. \)

• Revealed preference shows that profit maximization implies \( L \) falls as \( w \) rises.
Cost Minimization

• Profit maximization requires minimizing cost
• Cost minimization for fixed output

\[ c(y) = \text{Min } wL + rK \]

subject to

\[ f(K,L) = y \]
Cost Minimization, Continued

• Profit maximization:

\[
\text{max } py - (wL + rK) \text{ s.t. } f(K,L) = y
\]

• For given \( y \), this is equivalent to minimizing cost.

• Cost minimization equation:

\[
- \frac{\partial f}{\partial L} = \frac{dK}{dL} \bigg|_{f(K,L)=y} = - \frac{w}{r}
\]
Short run vs Long run

\[ F(K_1, K_2, L) = \beta K_1(K_2^\alpha L^\beta) \]

- Short run fix \( K_1 \) (Plant size)
- Long run chose \( K_1 \)
- Short run Min \( wL + rK_2 \) subj \( F(K_2, L) > Q \)
- \( G(L, K_2, \lambda) = wL + rK_2 + \lambda (Q - F(K_2, L)) \)

\[
\frac{dG}{dL} = w - \lambda \frac{dF}{dL} = w - \lambda \gamma K_1 \beta K_2^\alpha L^{\beta-1} = 0
\]

\[
\frac{dG}{dK_2} = w - \lambda \frac{dF}{dK_2} = r - \lambda \gamma K_1 \alpha K_2^{\alpha-1} L^{\beta} = 0
\]

\[
\frac{dG}{d\lambda} = Q - F(K_2, L) = 0
\]
• Ratio of the first two gives
\[
\frac{w}{r} = \frac{\lambda y K_1 \beta K_2^\alpha L^\beta - 1}{\lambda K_1 \alpha K_2^\alpha - 1 L^\beta} = \frac{\beta K_2^\alpha L^\beta - 1}{\alpha K_2^\alpha - 1 L^\beta} = \frac{(1 - \alpha) K_2}{\alpha L}
\]

• Ratio of MP equals price ratio, or
\[
L = \frac{r \beta}{w} K_2
\]

• Plug back into last FOC
\[
Q = y K_1 K_2^\alpha L^{1-\alpha} = y K_1 K_2^\alpha \left(\frac{r \beta}{wr \alpha}\right)^\beta K_2^\beta = y K_1 \left(\frac{r \beta}{wr \alpha}\right)^\beta K_2^{\alpha + \beta}
\]
\[
K_2 = Q^{\frac{1}{\alpha + \beta}} \left(\frac{1}{y K_1} \left(\frac{w \alpha}{r \beta}\right)^\beta\right)^\frac{1}{\alpha + \beta} \quad L = Q^{\frac{1}{\alpha + \beta}} \left(\frac{1}{y K_1} \left(\frac{r \beta}{w \alpha}\right)^\alpha\right)^\frac{1}{\alpha + \beta}
\]

A bit of a mess but in fact its linear in Q if \(\alpha + \beta = 1\); its also decreasing in \(K_1\)
On the left Cost Min; on the right Max Q
Long run

• First notice that L and \( K_2 \) are optimal solutions given any \( K_1 \) ...so we could just search out the optimal \( K_1 \) given L and \( K_2 \)

\[
\frac{dG}{dK_1} = r - \lambda \frac{dF}{dK_1} = r - \lambda \gamma (K_2^\alpha L^\beta) = 0
\]

\[
r = \lambda \gamma \left\{ \frac{1}{Q^{\alpha+\beta}} \left( \frac{1}{\gamma K_1} \left( \frac{w\alpha}{r\beta} \right)^\beta \right) \right\}^\alpha + \lambda \gamma \left\{ \frac{1}{Q^{\alpha+\beta}} \left( \frac{1}{\gamma K_1} \left( \frac{r\beta}{w\alpha} \right)^\alpha \right) \right\}^\beta = 0
\]

\[
r = \lambda \gamma \left\{ \left( \frac{Q}{\gamma K_1} \right)^{\alpha+\beta} \left( \frac{w\alpha}{r\beta} \right)^{\alpha+\beta} + \left( \frac{Q}{\gamma K_1} \right)^{\beta} \left( \frac{r\beta}{w\alpha} \right)^{\alpha+\beta} \right\}
\]

What a mess! Notice that the solution \((K_1^*)\) is going to be increasing in \( Q \) and \( \gamma \) and declining in \( r \)
When $\alpha+\beta=0.5$

\[ r = \lambda \gamma \left\{ \left( \frac{Q}{\gamma K_1} \right)^{0.5} \left( \frac{W}{r} \right)^{0.25} + \left( \frac{Q}{\gamma K_1} \right)^{0.5} \left( \frac{r}{W} \right)^{0.25} \right\} \]

\[ r = \lambda \gamma \left( \frac{Q}{\gamma K_1} \right)^{0.5} \left\{ \left( \frac{W}{r} \right)^{0.25} + \left( \frac{r}{W} \right)^{0.25} \right\} \]

\[ \left( \frac{Q}{\gamma K_1} \right)^{0.5} = \frac{r}{\lambda \gamma \left\{ \left( \frac{W}{r} \right)^{0.25} + \left( \frac{r}{W} \right)^{0.25} \right\} } \]

\[ K_1 = Q r \frac{\lambda^2}{r^2} \left\{ \left( \frac{W}{r} \right)^{0.25} + \left( \frac{r}{W} \right)^{0.25} \right\}^2 \]

• As the target quantity $Q$, increases you invest more in plant, same if crowding is a big deal ($\gamma$) or if capital is cheap.

• Note this does not depend on $\alpha+\beta=0.5$
Long run cost function is the min of all the possible cost functions.

Note Marginal cost is constant for each level of K1
From production function to costs

• Rather than look at a production function one can summarize the firm’s decision into a simple cost function.

• Note: that implies that we are tracing out the optimal input mix given prices, and technology.

• In our last example cost were increasing in a linear way for each level of K1. So each K1 corresponds to a different short run cost function.

• Key: Cost functions assume that input prices are stable.
Reminder

• For every production function and input price set, you can find the vector $X = \{x_1, \ldots, x_i, \ldots, x_n\}$ that minimizes the cost of producing a given level of output $Q$. The cost of producing $Q$ is therefore $PX$. Now one can do this for every $Q$ over the relevant range.

• The function that maps $Q$ into cost exists if the production function is convex. $C(Q)$

• Marginal cost is simply the derivative of the cost function with respect to quantity.
From production to Firm

• One possibility is for the Entrepreneur to solve one big problem
  – Max \( \pi = pF(x_1, \ldots, x_i, \ldots, x_n) - (p_1x_1 + \ldots + p_ix_i \ldots p_nx_n) \)

• Another takes two steps
  – (1) The engineers to do the cost minimization
    • Min \( p_1x_1 + \ldots + p_ix_i \ldots p_nx_n \) subj to \( F(x_1, \ldots, x_i, \ldots, x_n) < Q \)
  – (2) Marketing finds the quantity to produce
    • Max \( \pi = pQ - C(Q) \)
Short-run Costs

• Short-run total cost
• L varies, K does not
• Short-run marginal cost
  – Derivative of cost with respect to output
• Short-run average cost
  – average over output
  – infinite at zero, due to fixed costs
• Short-run average variable cost
  – average over output, omits fixed costs
Comparative Statics

• What happens to $L$ as $K$ rises?

\[ L^{*'}(K) = -\frac{\partial^2 F}{\partial K \partial L}(K, L^{*}(K)) \frac{\partial^2 F}{(\partial L)^2}(K, L^{*}(K)) \]

• Remember the lower part of the ratio has to be negative (condition for a max)

• Thus, $L$ rises if $L$ and $K$ are complements, and falls if substitutes
Conclusion and Wrap up

• 1) remember the symmetry of our two key decision problems (consumer and producer)
• 2) short run problem is exactly symmetric
• 3) long run problem involves choosing scale