Homework Policy:
Study You can study the homework on your own or with a group of fellow students. You should feel free to consult notes, text books and so forth.

The quiz will be available Wednesday at 5pm. Following the Honor code, you should find 20 minutes and do the quiz, by yourself and without using any notes. Paper and pen should be all you need. Then turn it in by Thursday 5pm. (drop off in box in front of Baxter 133). It will include one question from each section

The answers to the whole homework will be available Friday at 2pm.

Definitions
Please explain each term in three lines or less!

- Isoquant
  An isoquant is a contour line drawn through the set of points at which different combinations of two inputs yield a given quantity of output. The typical downward slope of the isoquants reflects the fact that adding one input while holding the other constant eventually leads to decreasing marginal output.

- Isocost
  An isocost line represents a combination of inputs which all cost the same amount, i.e., that hold the total expenditure on inputs constant. The slope of the isocost is the negative ratio of the input prices.

- Marginal rate of technical substitution
  Amount of one input needed to make up for a decrease in another input and hold output constant. More precisely, the marginal rate of technical substitution between inputs L and K, is the rate at which units of input L can be substituted for units of input K while keeping output constant.
- Economy of scale
  An economy of scale arises when an increase in output reduces average costs. Economies of scale can be accomplished because as production increases, the cost of producing each additional unit falls.

- Edgeworth box
  The edgeworth box is a graphical representation of the exchange problem faced by the parties in a two person, two good “exchange economy”. It allows a simple way of solving and visualizing their exchange problem.

- Pareto efficient allocation
  A Pareto efficient allocation is such that it is impossible to make one person better off without making the other (or, more generally, someone else) worse off. It is worth noting that a Pareto efficiency allocation does not necessarily imply a fair or socially desirable distribution of resources, since the concept makes no statement about equality or the overall well-being of a society.

- Contract curve
  Is the set of Pareto efficient allocations in an economy. Every point on this curve maximizes one person's utility given the other, and they are characterized by the tangencies in the isoquants.

- Price system
  A price system involves a specific price for trading good Y for good X, and vice versa, that is available to both parties in a two person, two good exchange economy.

**Word problems**

Please explain each question in a few sentences.

- State the First Welfare Theorem, and describe in words how the allocation would look like in the Edgeworth box for a two person, two good exchange economy.
  The First Welfare Theorem states that every competitive equilibrium is Pareto Efficient. Graphically, this implies that in an Edgeworth box, the equilibrium allocation must be on the contract curve.

- A firm’s marginal rate of technical substitution captures the firm’s willingness to trade capital for labor, but not the degree to which capital and labor are substitutes in supply. True or false?
  This statement is true. The MRTS is the absolute value of the slope of the isoquant, so it measures the amount of capital that the firm is willing to give up for one unit of labor, in order to maintain the same level of production. This is not the same thing as measuring the degree to which capital and labor are substitutes or complements.
• Assume that, in a two-person (1 and 2), two-good (x and y), exchange economy, allocation A, 
(xA1 , yA 1 , xA2 , yA 2 ), and allocation B, (xB1 , yB 1 , xB2 , yB2 ), are both Pareto efficient. 
Explain why, if consumer 1 prefers allocation A, then consumer 2 must prefer allocation B. 
If consumer 2 did not prefer allocation B – i.e., she preferred A or would be indifferent between 
A and B - then by moving from B to A, 2 would make consumer 1 strictly better off without 
hurting herself. That would mean that allocation B is not Pareto efficient. However, we know 
that B is Pareto efficient. Thus, consumer 2 must prefer allocation B. 
Another way to see this is by drawing an Edgeworth Box plotting allocation B and the tangent 
indifference curves of consumer 1 and consumer 2. Because allocation A is preferred by 
consumer 1, it must lie above consumer 1’s indifference curve. However, all of those points lie 
on a lower indifference curve for consumer 2.

• A consumer in an exchange economy will always be better off at an allocation which is Pareto 
efficient than at an allocation which is not Pareto efficient. True or false? 
False. A Pareto-efficient allocation is not necessarily an allocation that everyone will prefer over 
non-Pareto efficient allocations.

• In a two-person, two-good exchange economy, explain why both consumers’ marginal rates of 
substitution are equal at every point on the contract curve. 
In an Edgeworth box diagram, the contract curve is the set of points where the indifference 
curves of the two individuals are tangent. We know that the marginal rate of substitution is 
equal to the (negative) slope of the indifference curves. Also, when two curves are tangent at a 
point, their slopes are equal at that point. Thus, by defining the contract curve as a set of 
indifference curve tangencies, the marginal rates of substitution between the two goods are 
equal for the two individuals given we assume convex indifference curves.

**Technical problems**

• Assume a two person, two good exchange economy. 
  Mr. Blue’s utility function for goods x and y is given by: \( u_B = x^{0.7} y^{0.3} \)  
  Mr. Red’s utility function for goods x and y is given by: \( u_R = x^{0.9} y^{0.1} \).

Blue has 20 ounces of x and 10 ounces of y. Red has 15 ounces of x and 15 ounces of y. 

a) Calculate the marginal rate of substitution (MRS) of y for x for both consumers at the 
endowment point.

The marginal rate of substitution (MRS) of y for x is the amount of y a consumer is willing to 
lose to obtain an extra unit of x: \( MRS = \frac{\partial u / \partial x}{\partial u / \partial y} \).
For Blue, the MRS is then: 

\[ MRS_B = \frac{0.7x^{-0.3}y^{0.3}}{0.3x^{0.7}y^{-0.7}} = \frac{7y}{3x}. \]

Since, at the endowment, \( x_B = 20, y_B = 10 \), we have that 

\[ MRS_B = \frac{7 \times 10}{3 \times 20} = -\frac{7}{6}. \]

Analogously, in the case of Mr. Red, the marginal rate of substitution of y for x at the endowment is: 

\[ MRS_R = -9 \frac{y}{x} = -9\frac{1}{1} = -9. \]

b) Draw an Edgeworth box showing the endowment and indifference curves of the consumers.

![Edgeworth Box](image)

c) Assume that Blue and Red can trade at a market price as price-takers. Assume that x is the numeraire (i.e., \( p_x = 1 \)), what is the price of y? What is the final allocation of x and y?

From the consumer’s utility maximization problem, the condition \( \frac{\partial u}{\partial x} / \frac{\partial u}{\partial y} = \frac{p_x}{p_y} \) in our exercise yields:

Mr. Blue - 

\[ 0.7yp_y = 0.3xp_x \rightarrow 0.7(M - p_xx) = 0.3xp_x \rightarrow x^B = \frac{0.7M}{p_x} \]

Mr. Red - 

\[ 0.9yp_y = xp_x \rightarrow 0.9(M - p_xx) = xp_x \rightarrow x^B = \frac{0.9M}{p_x}. \]

Given the satisfaction of the budget constraint, the original endowments, and the fact that \( p_x = 1 \), the equations above become:
\[ x^B = \frac{0.7 \left(20 + 10p_y\right)}{1}, \quad x^R = \frac{0.9 \left(15 + 15p_y\right)}{1}. \]

Hence, the market equilibrium condition is:
\[ \frac{0.7 \left(20 + 10p_y\right)}{1} + \frac{0.9 \left(15 + 15p_y\right)}{1} = 35 \rightarrow 27.5 + 20.5p_y = 35, \]
\[ \rightarrow p_y = 0.366 \]

with \( x^B = \frac{0.7 \left(20 + 10 \times 0.366\right)}{1} = 16.56, \quad x^B = \frac{0.9 \left(15 + 15 \times 0.366\right)}{1} = 18.44 \).

Analogously, \( y^B = 19.39, \quad y^R = 5.61 \).

d) Show the trading in the diagram from part b.

![Diagram showing trading in part b.](image)

- Assume that a firm’s production function is given by \( Y = f(K, L) = 6K^{\frac{1}{3}}L^{\frac{1}{3}} \).

a) Derive the expression for the marginal rate of technical substitution (MRTS) for this production function.

Recall that the marginal rate of technical substitution is the rate at which units of input L can be substituted for units of input K while keeping output constant. As such, it is the (negative) of the slope of the isoquant.

We can derive the marginal rate of technical substitution using two alternative - but equivalent - procedures:

a.1) Given the firm’s production function, we can write the isoquant equation as:
\[ K = \left( \frac{Y}{6L^{1/3}} \right)^3 = \left( \frac{Y}{6} \right)^3 L^{-1}. \]

Thus, \[ \frac{dK}{dL} = \left( \frac{Y}{6} \right)^3 L^{-2}. \] Substituting \[ 6K^{1/3}L^{1/3} \] for \( Y \) in this equation, we get \[ \frac{dK}{dL} = \left( \frac{6K^{1/3}L^{1/3}}{6} \right)^3 L^{-2} = \frac{K}{L}, \] and thus \( \text{MRST} = \frac{dK}{dL} = \frac{K}{L}. \)

a.2) Similarly, from the definition of MRTS, it can be calculated as the ratio of marginal products of the inputs \( K \) and \( L \). That is, \( \text{MRTS} = \frac{MPL}{MPK} \), where \( MPL = \frac{\partial f(K, L)}{\partial L} \), \( MPK = \frac{\partial f(K, L)}{\partial K} \).

In the case of the production function \( f(K, L) = 6K^{1/3}L^{2/3} \), \( MPL = 2K^{1/3}L^{-2/3} \), \( MPK = 2K^{-2/3}L^{1/3} \).

Hence, \( \text{MRTS} = \frac{2K^{1/3}L^{-2/3}}{2K^{-2/3}L^{1/3}} = \frac{K}{L}. \)

Of course, the equivalence between a.1 and a.2 is evident when noting that, from \( Y = f(K, L) = 6K^{1/3}L^{2/3} \), we have:
\[
dY = \frac{\partial f(K, L)}{\partial K} dK + \frac{\partial f(K, L)}{\partial L} dL = 0 \rightarrow \frac{dK}{dL} = \frac{\frac{\partial f(K, L)}{\partial L}}{\frac{\partial f(K, L)}{\partial K}}.\]

b) Let the wage rate \( w \) be $4 and the rental rate of capital \( r \) be $16. Assuming that the firm is producing 54 units of output, what will be the cost-minimizing combination of labor and capital used by the firm? What will be the firm’s total cost of producing 54 units of output?

The firm is seeking to minimize its total costs, \( TC = wL + rK \), subject to \( Q = 6L^{1/3}K^{1/3} \). That is, the firm’s problem can be written as:

\[
\min TC = wL + rK \quad \text{s.t.} \quad Q = 6L^{1/3}K^{1/3},
\]

so that the Lagrangian function is \( \ell = wL + rK + \lambda [Q - 6L^{1/3}K^{1/3}] \).

The first-order conditions are
\[ \frac{\partial \ell}{\partial L} = 0 \Rightarrow w - 2\lambda L^{\frac{2}{3}} K^{\frac{1}{3}} = 0 \]

\[ \frac{\partial \ell}{\partial K} = 0 \Rightarrow r - 2\lambda K^{\frac{2}{3}} L^{\frac{1}{3}} = 0 \]

\[ \frac{\partial \ell}{\partial \lambda} = 0 \Rightarrow Q = 6L^{\frac{2}{3}} K^{\frac{1}{3}} \]

Solving 1 and 2 for \( \lambda \), and setting them equal yields:

\[ \lambda = \frac{w}{2L^{\frac{2}{3}} K^{\frac{1}{3}}} = \frac{r}{2K^{\frac{2}{3}} L^{\frac{1}{3}}} \Rightarrow \frac{w}{r} = \frac{2L^{\frac{2}{3}} K^{\frac{1}{3}}}{2K^{\frac{2}{3}} L^{\frac{1}{3}}} \Rightarrow \frac{w}{r} = \frac{K}{L} \Rightarrow K = \frac{wL}{r} \]

Substituting into 3 yields

\[ Q = 6L^{\frac{1}{3}} \left( \frac{wL}{r} \right)^{\frac{1}{3}} \rightarrow \left( \frac{Q}{6} \right)^{\frac{1}{3}} \left( \frac{r}{w} \right)^{\frac{1}{3}} = L' \], which is the input demand for labor.

Analogously, the input demand for capital is

\[ K = \frac{wL}{r} \rightarrow K^* = \frac{w}{r} \left( \frac{Q}{6} \right)^{\frac{1}{3}} \left( \frac{r}{w} \right)^{\frac{1}{3}} = \left( \frac{Q}{6} \right)^{\frac{1}{3}} \left( \frac{w}{r} \right)^{\frac{1}{3}} \]

Hence, given \( Q = 100, r=40, w=10 \), we have

\[ L^* = \left( \frac{54}{6} \right)^{\frac{1}{3}} \left( \frac{16}{4} \right) = 27 \cdot 2 = 54 \], \( K^* = \left( \frac{54}{6} \right)^{\frac{1}{3}} \left( \frac{4}{16} \right) = 27 \cdot 0.5 = 13.5 \), with total production costs:

\[ TC = wL + rK = 4(54) + 16(13.5) = $432. \]

\[ \frac{dK}{dL} = -\left( \frac{6K^{\frac{1}{3}} L^{\frac{1}{3}}}{6} \right)^3 L^{-2} = -\frac{K}{L} \]

Note also that

\[ \frac{dK}{dL} = -\frac{K}{L} = -\frac{1}{4} = -\frac{w}{r} \], consistent with the fact that cost minimization requires a tangency between the isoquant and the isocost.

**c)** Derive the expressions for the short-run average cost of production. Does the firm have decreasing returns to scale?

The firm's short-run average cost of production is:
\[ SRAC = \frac{wL + rK}{Q} . \]

Substituting in the expressions for conditional factor demands derived in b), we obtain

\[
SRAC = \frac{wL + rK}{Q} = \frac{wQ}{6} \left( \frac{r}{w} \right)^{\frac{1}{2}} + r \left( \frac{Q}{6} \right)^{\frac{1}{2}} \left( \frac{w}{r} \right)^{\frac{1}{2}} + 2 \left( \frac{Q}{6} \right)^{\frac{3}{2}} \left( wr \right)^{\frac{1}{2}} = \frac{2Q^{1/2} (wr)^{1/2}}{6^{3/2}}
\]

Note, then, that SRAC increase as output increases, implying decreasing returns to scale.