A characterization of combinatorial demand

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Montreal Nov 19, 2016
This paper

Literature on matching (e.g. Kelso-Crawford) and combinatorial auctions (e.g. Milgrom):

\[ D(p) = \arg\max \{ v(A) - \sum_{a \in A} p_a : A \subseteq X \} \]  

When is * true?
What is the behavioral content of the combined assumptions of rationality and quasilinearity?
Notation

- Let $X$ be a finite set (of items).
- Let $S$ be the set of all nonempty subsets of $2^X$.
- (so the empty set is not in $S$, but $\{\emptyset\}$ is).
- Identify $A \subseteq X$ with $1_A \in \mathbb{R}^X$.
- If $p \in \mathbb{R}^X$ then $\langle p, A \rangle = \sum_{x \in A} p_x$. 

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Combinatorial demand
Demand

A demand function is

\[ D : \mathbb{R}_+^X \rightarrow S \]

s.t. \( \exists \bar{p} \in \mathbb{R}_+^X \) with \( D(p) = \{\emptyset\} \) for all \( p \geq \bar{p} \).

(\( \bar{p} \) a choke price)
Demand

$D$ is \textit{quasilinear rationalizable} if

$$\exists \nu : 2^X \to \mathbb{R} \text{ s.t }$$

$$D(p) = \arg\max_{A \subseteq X} \nu(A) - \langle p, A \rangle$$
Suppose $D$ is QL-rationalizable

Let $A \in D(p)$ and $B \in D(q)$.

\[
v(A) - \langle p, A \rangle \geq v(B) - \langle p, B \rangle \\
v(B) - \langle q, B \rangle \geq v(A) - \langle q, A \rangle.
\]

Thus: $\langle p - q, A - B \rangle \leq 0$. 
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Thus: $\langle p - q, A - B \rangle \leq 0$.

The law of demand!
Demand

A demand function $D$

- satisfies the *law of demand* if for all $p, q \in \mathbb{R}_++^X$, and all $A \in D(p)$ and $B \in D(q)$,

  $$\langle p - q, A - B \rangle \leq 0;$$

- is *upper hemicontinuous* if, $\forall p \in \mathbb{R}_++^X$, $\exists$ nbd $V$ of $p$ s.t. $D(q) \subseteq D(p)$ when $q \in V$. 

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Combinatorial demand
Main result

**Theorem**

A demand function is quasilinear rationalizable iff it is upper hemicontinuous and satisfies the law of demand.
Identification

**Theorem**

*For any quasilinear rationalizable* $D$, *there is a unique monotone* $\nu : 2^X \rightarrow \mathbb{R}$ *for which* $\nu(\emptyset) = 0$ *which rationalizes* $D$.

Utility is identified up to an additive constant.
Monotone rationalization

\( D \) is **monotone, concave, quasilinear rationalizable** (MCQ-rationalizable) if \( \exists \) a monotone, concave \( g : \mathbb{R}^X_+ \to \mathbb{R} \) s.t 
\[
v(A) = g(1_A), \text{ and}
\]
\[
D(p) = \text{argmax}\{v(A) - \langle p, A \rangle : A \subseteq X\}.
\]

**Corollary**

*If a demand function is quasilinear rationalizable, then it is MCQ-rationalizable.*
Proof ideas

\[ D(p) = \arg\max_{A \subseteq X} \nu(A) - \langle p, A \rangle \]

If \( A \in D(p) \) then we want \( p \) to be the “gradient of \( \nu \) at \( A \).” Can recover \( \nu \) by “integrating” over \( p \).
Cyclic monotonicity

$D$ satisfies cyclic monotonicity if, for all $n$ (using summation mod $n$),

$$\sum_{i=1}^{n} \langle p_i, A_i - A_{i+1}\rangle \leq 0,$$

where $A_i \in D(p_i)$, for all sequences $\{p_i\}_{i=1}^{n}$. 
Cyclic monotonicity

Define:

\[ \nu(A) = \inf \langle p_1, A - A_1 \rangle + \ldots + \langle \bar{p}, A_k - \emptyset \rangle, \]

inf is taken over all finite seq. \((p_i, A_i)_{i=1}^k\) with \(A_i \in D(p_i)\).
Cyclic monotonicity

Define:

\[ v(A) = \inf \langle p_1, A - A_1 \rangle + \ldots + \langle \bar{p}, A_k - \emptyset \rangle, \]

inf is taken over all finite seq. \((p_i, A_i)_{i=1}^k\) with \(A_i \in D(p_i)\).

Observe, by CM,

\[- \{ \langle p_1, A - A_1 \rangle + \ldots + \langle \bar{p}, A_k - \emptyset \rangle \} + \langle p, A - \emptyset \rangle \leq 0.\]

So \(v(A)\) is well defined (and \(\geq 0\)).
Let $A \in D(p)$ and $B \subseteq X$ ($B \in D(R^X_{++})$ need a different arg. otherwise).
By defn. of $v$,
\[
v(B) \leq \langle p, B - A \rangle + v(A).
\]
Thus $v(A) - \langle p, A \rangle \geq v(B) - \langle p, B \rangle$. 


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Thus $v(A) - \langle p, A \rangle \geq v(B) - \langle p, B \rangle$.

Proof that if $A \in D(p)$ and $B \notin D(p)$ then $v(A) - \langle p, A \rangle > v(B) - \langle p, B \rangle$ requires more.
$D$ satisfies condition ♠ if
\[ \forall p \text{ and } B \not\in D(p) \exists A \in D(p) \text{ and } p' \text{ s.t.} \]

\[ A \in D(p') \text{ and } \langle p', A - B \rangle > \langle p, A - B \rangle. \]
Cyclic monotonicity

Lemma

If $D$ satisfies cyclic monotonicity, and condition ♠, then it is quasilinear rationalizable.

Based on ideas in Rochet/Rockafellar (but ♠ plays a technical role).
Lemma

A demand function satisfies cyclic monotonicity if it satisfies the law of demand.

Follows from recent results in mech. design (Lavi, Mu’alem, and Nisan; Saks and Yu; and Ashlagi, Braverman, Hassidim, and Monderer).
Related literature

- Rochet/Rockafeller
- Brown and Calsamiglia
- Sher and Kim
- Lavi, Mu’alem, and Nisan;
- Saks and Yu;
- Ashlagi, Braverman, Hassidim, and Monderer
Conclusions

- Quasilinear rational demand is a ubiquitous assumption.
- Our result is the first characterization in terms of observable behavior.
- Identification enables welfare analysis.
- New use for recent results in mech. design.