Paul Samuelson:

What does it mean to say that consumers max. utility?
Revealed Preference

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What does it mean to say that consumers max. utility?

It means that their \textit{behavior} is \textit{as if} they maximize some utility.

\begin{itemize}
  \item behavior $\rightarrow$ observable data.
  \item utility $\rightarrow$ unobservable.
\end{itemize}
Revealed Preference

Characterize data for which *there is some* utility function that could rationalize the data.

- Definition is a test
- ... but it’s useless.
The nature of falsifiable theories

Popper’s theories:

All swans are white:  \( \forall s \, W(s) \)  
Falsifiable

There exists a black swan:  \( \exists s \, B(s) \)  
Not falsifiable
Example: Rev. Pref. problem

Axiomatize $R, P$ (revealed preference relations) for which:

$\exists \succeq$ (satisfying some properties) such that

$\forall x \forall y, x R y \rightarrow x \succeq y$ and $x P y \rightarrow x \succ y$
Revealed Preference

A test is an effective “positive axiomatization.”

- A universal description of rationalizable data.
- Should not refer to theoretical objects, but only to observables.
- An algorithm should decide in finite time if data passes test.
Our paper:

Gives a sufficient condition for theory to have an effective positive axiomatization.

- Explains classical rev. pref. theory
- New applications to multiple selves (collective dec. making), Nash eq., and barg.
Main result (informal statement)

If a theory has an effective universal axiomatization assuming its theoretical terms are observable, then it has an effective universal axiomatization that only talks about observables.
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If a theory has an effective universal axiomatization *assuming* its theoretical terms are observable, then it has an effective universal axiomatization that only talks about observables.

So:

- Pretend that we can observe theoretical terms.
- Axiomatize the theory using statements about theoretical terms.
- This can be “projected” onto observables as an effective axiomatization.
Example

Language:
Example

Language: $\mathcal{L} = (\succ)$.
Axioms:
Example

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Axioms:

- transitivity
- completeness
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Structure:
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Language: \( \mathcal{L} = (\succ) \).

Axioms:
- transitivity
- completeness

Structure:
- \((\mathbb{R}, \succ^R)\),
- \((\mathbb{N}, \succ^*)\), \(n \succ^* m\) iff \(n - m > 5\).
Example - 2

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Axioms:

▫ $\succeq$ satisfies trans. & cplet. (a weak order)
▫ $\succ$ strict part of $\succeq$
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Language: \( \mathcal{L} = (\succeq, \succ) \).

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Structure:

\( (\mathbb{R}, \geq^R, >^R) \),

\( (\mathbb{N}, \geq^*, >^*) \), \( \geq^* = \geq^\mathbb{N} \), \( n >^* m \) iff \( m - n > 5 \).
Game Theory

Nash bargaining: $\mathcal{L} = \langle F, \in \rangle$
Normal-form games: $\mathcal{L} = \langle S_1, \ldots, S_n, C, \in \rangle$
In general:

- A *language* $\mathcal{L}$ is a list of relation symbols.
- A *axiom* is a logical sentence in $\mathcal{L}$.
- *Universal axioms* are those with $\forall$ quantification at the beginning of the sentence.
- A *structure* is a set together with an interpretation of each symbol in $\mathcal{L}$.
Axioms

Axioms are statements made using symbols in the language. Examples of (first order) axioms using symbols $P, R, \succ, \succeq, O$.
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- Completeness: $\forall x \forall y (x \succeq y) \lor (y \succeq x)$
- Transitivity: $\forall x \forall y \forall z (x \succeq y) \land (y \succeq z) \rightarrow (x \succeq z)$
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- Nonsatiation: $\forall x \exists y (y \succ x)$
- Rationalization: $\forall x \forall y (x R y) \rightarrow (x \succeq y)$
- Optimality: $\forall x O(x) \leftrightarrow \forall y (x R y)$
Universal axioms are those with universal ($\forall$) quantification, coming at the beginning of the sentence:

- $\forall x \forall y (x \succeq y) \lor (y \succeq x)$ is universal.
- $\forall x \exists y (x \succ y)$ is not.
- $\forall x (O(x) \leftrightarrow \forall y (x R y))$ is not.
What is a theory?

Given a language $\mathcal{L}$. A *theory* is a class of structures of $\mathcal{L}$ that is closed under isomorphism.

Example: Language $\langle \succeq, \succ \rangle$ and theory of utility maximization.
Main Result

Two languages, $\mathcal{L}$ and $\mathcal{F}$, where $\mathcal{F} = (R_1, \ldots, R_N)$ and $\mathcal{L} = (R_1, \ldots, R_N, Q_1, \ldots, Q_K)$.

Symbols $R_i$ or $Q_i$ for $k$-ary relations.
Main Result

Two languages, \( \mathcal{L} \) and \( \mathcal{F} \), where

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Symbols \( R_i \) or \( Q_i \) for \( k \)-ary relations.

Choice example: \( \mathcal{F} = (R, P) \) and \( \mathcal{L} = (R, P, \preceq, \succ) \). Language \( \mathcal{F} \): observables, language \( \mathcal{L} \): (observables and unobservables).
Main Result

Recall $\mathcal{F} \subseteq \mathcal{L}$.

Let $T$ be an $\mathcal{L}$-theory.

$F(T)$ is the class of all $\mathcal{F}$-structures $(X, R_1^X, \ldots, R_N^X)$ for which there exist $Q_1^X, \ldots, Q_K^X$ s.t.

$$(X, R_1^X, \ldots, R_N^X, Q_1^X, \ldots, Q_K^X) \in T.$$ 

$F(T)$ is a projection of $\mathcal{L}$-theory $T$ onto language $\mathcal{F}$. 

Revealed Preference Example

Two languages $\mathcal{F} \subseteq \mathcal{L}$:
$\mathcal{F} = (R, P)$, and $\mathcal{L} = (R, P, \succ, \succeq)$.
Revealed Preference Example

Two languages $\mathcal{F} \subseteq \mathcal{L}$:
$\mathcal{F} = (R, P)$, and $\mathcal{L} = (R, P, \succ, \succeq)$.

$T$: $\mathcal{L}$-theory of structures $(X, R^X, P^X, \succeq^X, \succ^X)$ for which
1. $\succeq^X$ is a weak order
2. $\succ^X$ is its strict part
3. $R^X \subseteq \succeq^X$
4. $P^X \subseteq \succ^X$. 
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$F(T)$ is the $\mathcal{F}$-theory of all structures $(X, R^X, Q^X)$ for which there exists $\succ^X, \succeq^X$ for which 1-4 is satisfied.
Main result

For languages \( \mathcal{F} \subseteq \mathcal{L} \), and \( \mathcal{L} \)-axioms \( \Sigma \), the set of \( \mathcal{F} \)-consequences of \( \Sigma \) is the collection of all logical consequences of \( \Sigma \) involving only symbols from \( \mathcal{F} \).
Main result

For languages $\mathcal{F} \subseteq \mathcal{L}$, and $\mathcal{L}$-axioms $\Sigma$, the set of $\mathcal{F}$-consequences of $\Sigma$ is the collection of all logical consequences of $\Sigma$ involving only symbols from $\mathcal{F}$.

Theorem

Suppose that $T$ is a universally axiomatizable $\mathcal{L}$-theory, and that $\mathcal{F} \subseteq \mathcal{L}$. Then $F(T)$ is a universally axiomatizable $\mathcal{F}$-theory, and is axiomatized by the set of all universal $\mathcal{F}$-consequences of $T$. 
Example

Let $\mathcal{F} = (R, P)$, and let $\mathcal{L} = (R, P, \succ, \succeq)$.

$T$: $\mathcal{L}$-theory of structures $(X, R^X, P^X, \succeq^X, \succ^X)$ for which

1. $\succeq^X$ is a weak order
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3. $R^X \subseteq \succeq^X$
4. $P^X \subseteq \succ^X$.

Has axiomatization:

1. $\forall x \forall y (\succeq (x, y) \lor \succeq (y, x))$
2. $\forall x \forall y (\succ (x, y) \leftrightarrow (\succeq (x, y) \land \neg \succeq (y, x)))$
3. $\forall x \forall y \forall z (\succeq (x, y) \land \succeq (y, z)) \rightarrow \succeq (x, z)$
4. $\forall x \forall y (R(x, y) \rightarrow \succeq (x, y))$
5. $\forall x \forall y (P(x, y) \rightarrow \succ (x, y))$
The Strong Axiom of Revealed Preference: For every $k$,

$$\forall x_1 \ldots \forall x_k \neg \bigwedge_{i=1}^{k} \left( x_i \ Q_i \ x_{(i+1) \ mod \ k} \right)$$

where for all $i$, $Q_i \in \{R, P\}$, and for at least one $i \in \{1, \ldots, k\}$, $Q_i = P$. 
Remarks on proof.

Any $\mathcal{F}$-consequence of $T$ is satisfied by $F(T)$, and if an $\mathcal{F}$ axiom is true for $F(T)$, it is true for $T$ (and hence an $\mathcal{F}$ consequence).

Main difficulty is in establishing that $F(T)$ is axiomatizable. Need not necessarily be true.
Remarks on proof.

Proof relies on a result of Tarski, which characterizes universally axiomatizable theories. Also relies on some form of choice (Szpirajn’s theorem is a corollary of our result).

Important: In general, $F(T)$ need not be axiomatizable, even if $T$ is. Universality of $T$ is critical.
Recursive enumerability

A set of axioms is **recursively enumerable** if there is an algorithm for listing them out, one by one.

**Corollary**

*If $T$ is universally and recursively enumerably axiomatizable, then so is $F(T)$.***
Fagin’s Theorem

$F(T)$ is in class NP if there is a non-deterministic Turing machine which, in poly. time, given any $\mathcal{F}$-structure $\mathcal{M}$, tells us whether $\mathcal{M} \in F(T)$.

**Theorem**

*Suppose that $T$ is a finitely axiomatized $\mathcal{L}$ theory. Then $F(T)$ is in NP.*
Applications

- Multiple selves
- Revealed game theory
- Group preferences (Pareto relation, majority rule, etc)
- Choice theory
Multiple selves, or group preferences

Observe relation $R$. Hypothesize that $R$ is generated by given finite set of agents $N$ and given social choice rule $f$ (satisfying neutrality and IIA).

Agents hypothesized to have “rational” preferences.

This theory is universally and r.e. axiomatizable.
Special case: Pareto extension relation on $N$ agents

Observe $\tilde{P}$.

Hypothesize: \( \exists P \exists R_i \exists P_i \) such that:

1. \( \forall x \forall y, x P y \iff \bigwedge_{i \in N} x P_i y \)
2. \( \forall x \forall y, x \tilde{P} y \rightarrow x P y \)
3. \( R_i \) weak orders, and \( P_i \) its strict part.
   1. \( \forall x \forall y, x R_i y \lor y R_i x \)
   2. \( \forall y \forall y \forall z, x R_i y \land y R_i z \rightarrow x R_i z \)
   3. \( \forall x \forall y, x P_i y \iff x R_i y \land \neg y R_i x \)

Axiomatization is known for the case \( |N| = 2 \).
Special case: Pareto extension relation on $N$ agents

Observe $\tilde{P}$.

Hypothesize: $\exists P \exists R_i \exists P_i$ such that:

- $\forall x \forall y, x P y \leftrightarrow \bigwedge_{i \in N} x P_i y$
- $\forall x \forall y, x \tilde{P} y \to x P y$
- $R_i$ weak orders, and $P_i$ its strict part.
  - $\forall x \forall y, x R_i y \lor y R_i x$
  - $\forall y \forall y \forall z, x R_i y \land y R_i z \to x R_i z$
  - $\forall x \forall y, x P_i y \leftrightarrow x R_i y \land \neg y R_i x$

Axiomatization is known for the case $|N| = 2$. 
Related Literature

- Simon (1985) (and other papers by H. Simon)
- Boland
- Mongin
- Chambers-Echenique-Shmaya
- Brown-Kubler (and Brown-Matzkin)