Relative Extinction of Heterogeneous Agents*

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Abstract

In all the existing literature on survival in heterogeneous economies, the rate at which an agent vanishes in the long run relative to another agent can be characterized by the difference of the so-called survival indices, where each survival index only depends on the preferences of the corresponding agent and the properties of the aggregate endowment. In particular, one agent experiences extinction relative to another (that is, the wealth ratio of the two agents goes to zero) if and only if she has a smaller survival index. We consider a simple continuous-time model of the Merton-Black-Scholes type and show that the survival index is more complex if there are more than two agents in the economy. In fact, the following phenomenon may take place: even if agent i experiences extinction relative to agent j, adding a third agent k to the economy may reverse the situation and force the agent j to experience extinction relative to agent i. We also calculate the rate of convergence.

Keywords: equilibrium; heterogeneous agents; survival; extinction

JEL classification: D53, G11, G12

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1 Introduction

It has become a conventional wisdom that, with no intermediate consumption, the agent with logarithmic utility of terminal wealth will have the highest wealth growth rate and thus will eventually dominate in the long run. See, e.g., Rubinstein (1991), Blume and Easley (1992), Evstigneev et al. (2006). In this paper we address the following question: how quickly does this extinction happen and which agents experience extinction earlier than the others?

In all the existing literature on survival in heterogeneous economies, the rate at which an agent vanishes in the long run relative to another agent can be characterized by the difference of the so-called survival indices, where each survival index only depends on the preferences of the corresponding agent and the properties of the aggregate endowment. In particular, one agent experiences extinction relative to another (that is, the wealth quotient of the two agents goes to zero) if and only if she has a smaller survival index, irrespective of other properties of the agents in the economy. See, e.g., Sandroni (2000), Blume and Easley (2006), and Yan (2009). We show that with more than two agents in the economy the survival index of two agents may depend on a third agent, and the following can happen: even if agent i experiences extinction relative to agent j, in a given economy, introducing a new agent may reverse it, so that the agent j will experience extinction relative to agent i.

We work in a continuous time economy, populated by K agents with heterogeneous risk aversions, maximizing CRRA utility from terminal wealth. There is a single stock in the economy, whose terminal dividend is log-normally distributed and all agents are initially endowed with shares of the stock. We show that, as the horizon increases, the agent with risk aversion closest to one will, indeed, eventually dominate and own the whole economy. Furthermore, we explicitly determine the rate at which all other agents die out in the long run. Using this result, we show the above mentioned phenomenon, that, when the risk aversions of two agents i and j are on different sides of one, introducing a third agent may reverse the order of their survival indices. The results of the paper can be extended to a more general setup, as described in Remark 2.1.

2 Setup and notation

2.1 The Model

We consider a standard setting similar to that of Wang (1996). The economy has a finite horizon and evolves in continuous time. Uncertainty is described by a one-dimensional, standard Brownian motion $B_t,\ t \in [0\,,\,T]$ on a complete probability space $(\Omega, \mathcal{F}_T, P)$, where $\mathcal{F}$ is the augmented filtration generated by B_t. There is a single share of a risky asset in the economy, the stock, which pays a terminal dividend payment

$$D \ = D_T = \ e^{\rho T \ + \ \sigma B_T}.$$

We also assume that a zero coupon bond with instantaneous constant risk free rate r is available in zero net supply.[1] The price of the stock at time t is denoted by S_t. The instantaneous drift and volatility of the stock price S_t are denoted by μ_t and σ_t respectively,

$$S_t^{-1}\, dS_t \ = \ \mu_t dt \ + \ \sigma_t dB_t.$$

There are K competitive agents in the economy, which we will also call funds, who behave rationally, and are heterogeneous in risk preferences. Fund k is initially endowed with ψ_k shares of the stock at time zero,

$$\sum_k \psi_k \ = \ 1.$$

Fund k chooses portfolio strategy π_{kt}, the portfolio weight in the risky asset, as to maximize the CRRA expected utility

$$E\left[\frac{W_{kT}^{1-\gamma_k}}{1-\gamma_k}\right]$$

of its final wealth W_{kT}, where the wealth W_{kt} of fund k evolves as

$$dW_{kt} \ = \ W_{kt}(rdt \ + \ \pi_{kt}\,(S_t^{-1}dS_t - rdt)).$$

Remark 2.1 All the results of the paper can be directly extended to the case of agents having heterogeneous beliefs on the expected return rate of the endowment (as, e.g., in Sandroni (2000), Blume and Easley (2006) and Yan (2009)), and to utility functions generalizing CRRA utilities. However, the analysis for the latter becomes much more technical, and the details are available from the authors upon request.

2.2 The Equilibrium

Definition 2.1 *We say that the market is in equilibrium if the funds behave optimally and both the risky asset market and the risk-free market clear.*

It is well known that the above financial market is complete, if the volatility process σ_t of the stock price is almost everywhere strictly positive.[2] When the market is complete, there exists a unique stochastic discount factor (SDF) $M = M_T$ such that the stock price is given by

$$S_t \ = \ e^{r(t-T)}\,\frac{E_t[MD]}{E_t[M]}.$$

[1] The assumption of constant r is introduced only for simplicity of exposition.

[2] This follows, for example, from the results of Anderson and Raimondo (2008).

Because of the market completeness, equilibrium allocation is Pareto-efficient and can be characterized as an Arrow-Debreu equilibrium. See, e.g. Duffie (1986), Wang (1996). Because the endowments are co-linear (all agents hold shares of the same single stock), the equilibrium is in fact unique, up to a multiplicative factor, and unique if we fix the risk-free rate. See, e.g., Dana (1995), Dana (2001).[3]

It is well known (see, for example, Cvitanić and Zapatero (2004)) that in this complete market setting the optimal terminal wealth is of the form

$$W_{kT} \;=\; (\lambda_k\, M)^{-b_k}$$

where

$$b_k \;=\; \gamma_k^{-1}$$

is the relative risk tolerance of agent k, and λ_k is determined via the *budget constraint*

$$E[(\lambda_k M)^{-b_k}\, M] \;=\; W_{k0} \;=\; \psi_k\, S_0 \;=\; \psi_k\, E[DM].$$

We formalize this in

Proposition 2.1 *The equilibrium allocation is given by*

$$W_{kT} \;=\; \frac{\psi_k\, E[DM]}{E[M^{1-b_k}]}\, M^{-b_k}$$

and equilibrium SDF M solves the equation

$$\sum_k \frac{\psi_k\, E[DM]}{E[M^{1-b_k}]}\, M^{-b_k} \;=\; D. \tag{1}$$

When risk aversion is homogeneous across agents, the equilibrium SDF is explicitly determined by $D^{-\gamma}/E[D^{-\gamma}]$. However, when risk aversion is heterogeneous, SDF is the solution to highly non-linear equation (1), and no explicit solution is possible, except for some very special values of risk aversion; see, for example, Wang (1996). In the lemma below, we establish bounds on the equilibrium SDF.

Lemma 2.1 *Let $\Gamma \geq 1$ be such that $\Gamma\, b_i > 1$ for all i and $\gamma \leq 1$ be such that $\gamma b_i \leq 1$ for all i. Then,*

$$\left(\sum_i D^{-\gamma_i/\gamma}\big(\psi_i\, E[DM]/E[M^{1-b_i}]\big)^{\gamma_i/\gamma}\right)^{\gamma} \;\leq\; M \;\leq\; \left(\sum_i D^{-\gamma_i/\Gamma}\big(\psi_i\, E[DM]/E[M^{1-b_i}]\big)^{\gamma_i/\Gamma}\right)^{\Gamma}. \tag{2}$$

[3] Since the endowment is neither bounded away from zero nor from infinity, some additional care is needed to show the existence of equilibrium. See, e.g., Dana (2001) and Malamud (2008a).

The quantity $D^{-\gamma_i}\big(\psi_i\, E[DM]/E[M^{1-b_i}]\big)^{\gamma_i}$ can be viewed as the "individual" SDF in an economy populated only by fund i. It is known that, when risk aversion is heterogeneous, the equilibrium SDF can be represented as a generalized weighted Hölder average of the "individual" SDFs (see, e.g., Malamud (2008a), Malamud (2008b), Jouini and Napp (2008), Shefrin (2005)). Lemma 2.1 shows that M can be estimated from both below and above by Hölder averages with different exponents γ and Γ.

The weights $\big(\psi_i\, E[DM]/E[M^{1-b_i}]\big)^{\gamma_i}$ are not directly helpful for getting good bounds for M. The following useful lemma allows us to obtain uniform bounds for these weights. It has a very clear economic meaning: the maximal utility of a fund is larger than the utility from simply consuming its endowment, and is smaller than the utility from consuming the aggregate endowment of the economy.

Lemma 2.2 *Let M be the equilibrium SDF. If $\gamma_i < 1$ then*

$$1 \;\leq\; \frac{E[DM]^{1-\gamma_i}\, E[M^{1-b_i}]^{\gamma_i}}{E[D^{1-\gamma_i}]} \;\leq\; \psi_i^{\gamma_i-1}.$$

If $\gamma_i > 1$ then

$$\psi_i^{\gamma_i-1} \leq \frac{E[DM]^{1-\gamma_i}\, E[M^{1-b_i}]^{\gamma_i}}{E[D^{1-\gamma_i}]} \;\leq\; 1.$$

Proof: The utility of fund i's optimal wealth is given by

$$\begin{aligned}\frac{1}{1-\gamma_i}\, E[W_{iT}^{1-\gamma_i}] \;&=\; \frac{1}{1-\gamma_i}\psi_i^{1-\gamma_i}\left(\frac{E[DM]}{E[M^{1-b_i}]}\right)^{1-\gamma_i} E[M^{1-b_i}] \\ &=\; \frac{1}{1-\gamma_i}\psi_i^{1-\gamma_i} E[DM]^{1-\gamma_i}\, E[M^{1-b_i}]^{\gamma_i}. \qquad (3)\end{aligned}$$

The utility from just consuming its endowment (the terminal dividend of its initial portfolio) is

$$\frac{1}{1-\gamma_i}\, E[(\psi_i D)^{1-\gamma_i}] \;=\; \frac{1}{1-\gamma_i}\psi_i^{1-\gamma_i}\, E[D^{1-\gamma_i}].$$

Furthermore, by definition, in equilibrium we must have $W_{iT} \;\leq\; D$ and therefore

$$\frac{1}{1-\gamma_i}\, E[(\psi_i D)^{1-\gamma_i}] \;\leq\; \frac{1}{1-\gamma_i}\, E[W_{iT}^{1-\gamma_i}] \;\leq\; \frac{1}{1-\gamma_i}\, E[D^{1-\gamma_i}]\,.$$

Multiplying both sides by $1-\gamma_i$ and using (3), we get the result. ∎

Lemmas 2.1 and 2.2 together will allow us to obtain good bounds on the ratio W_{iT}/W_{jT}.

3 Relative Extinction

Definition 3.2 *We say that a function $f(T)$ converges to zero almost at rate ρ as $T \to \infty$ if, for any $\epsilon > 0$ there exist constants $K_1(\epsilon)$, $K_2(\epsilon)$ such that*

$$K_1(\epsilon)\, e^{-(\rho+\epsilon)\,T} \;\leq\; f(T) \;\leq\; K_2(\epsilon)\, e^{-(\rho-\epsilon)\,T}$$

for all $T \geq 0$.

We will also need the following

Definition 3.3 *We say that agent* i *experiences extinction relative to agent* j *if*

$$\lim_{T\to\infty} \frac{W_{iT}}{W_{jT}} \;=\; 0$$

almost surely.

We start with the following natural result, confirming the intuition of Rubinstein (1991).

Theorem 3.1 *Suppose that there exists a unique agent* 0 *such that* [4]

$$(1-\gamma_0)^2 \;=\; \min_k\,(1-\gamma_k)^2.$$

Then,

$$\frac{W_{kT}}{W_{0T}} \;\to\; 0$$

for all $k \neq 0$ *and the convergence happens almost at rate* $(-s_k)$ *with*

$$s_k \;=\; b_k\,(-(1-\gamma_k)^2 + (1-\gamma_0)^2).$$

We call s_k the *survival index* of agent k. Writing down

$$\frac{W_{iT}}{W_{jT}} \;=\; \frac{W_{iT}/W_{0T}}{W_{jT}/W_{0T}},$$

Theorem 3.1 implies

Corollary 3.1 *An agent* i *experiences extinction relative to agent* j *if and only if*

$$s_i < s_j$$

and W_{iT}/W_{jT} *converges to zero almost at rate* $s_j - s_i$.

A direct calculation shows that

$$s_j\,-\,s_i \;=\; \frac{\gamma_i-\gamma_j}{\gamma_i\gamma_j}\left(\gamma_i\gamma_j\,-\,1\,+\,(1-\gamma_0)^2\right).$$

In particular, if $\gamma_i > \gamma_j > 1$, we get $s_j\,-\,s_i > 0$. Similarly, if $\gamma_i < \gamma_j < 1$, we have

$$(1-\gamma_0)^2 < (1-\gamma_i)^2 < 1-\gamma_i < 1-\gamma_i\gamma_j$$

and we again get $s_j > s_i$. Thus, if risk aversions of agents i and j are on the same side of one, the relative extinction does not depend on the presence of agent 0 – the agent further away from risk aversion of one gets extinct relative to the other agent. The next corollary shows that the situation may be different if risk aversions are on different sides of one.

[4]The assumption is true for generic values of risk aversion. On the other hand, if there are two agents equally distant from the log, they will both survive and share the economy in the long run.

Corollary 3.2 *Let $\gamma_j < 1$ and suppose that $\gamma_i > 2 - \gamma_j$ and $\gamma_i \gamma_j < 1$. Then agent i experiences extinction relative to agent j in the economy populated only by agents i and j. On the other hand, if*

$$\gamma_0 \in \left(1 - \sqrt{1 - \gamma_i \gamma_j}\,,\, 1 + \sqrt{1 - \gamma_i \gamma_j}\right) \tag{4}$$

then, on the contrary, agent j experiences extinction relative to agent i in any economy in which all agents $(i, j, 0)$ are present, irrespective of risk aversions of other agents.

The intuition behind Corollary 3.2 is as follows. Condition (4) requires that γ_0 be very close to 1. If agent 0 is very close to being logarithmic, survival indices of other agents in the economy will become small and he will quickly own the whole economy. His presence will then lead to dramatic changes in prices, and make the strategy of agent i "better" relative to that of agent j.

Appendix

A Proofs

Proof of Lemma 2.1: Let

$$z_i = \psi_i \, E[DM]/E[M^{1-b_i}]\,.$$

Then, the equilibrium equation is

$$\sum_i z_i \, M^{-b_i} = D.$$

Suppose that

$$M > \left(\sum_i D^{-\gamma_i/\Gamma} \, z_i^{\gamma_i/\Gamma}\right)^{\Gamma}.$$

Then,

$$\sum_i z_i \, M^{-b_i} \, D^{-1} < \sum_i z_i \left(\sum_i D^{-\gamma_i/\Gamma} \, z_i^{\gamma_i/\Gamma}\right)^{-\Gamma/\gamma_i} = \sum_i \left(\frac{D^{-\gamma_i/\Gamma} z_i^{\gamma_i/\Gamma}}{\sum_i D^{-\gamma_i/\Gamma} \, z_i^{\gamma_i/\Gamma}}\right)^{\Gamma/\gamma_i}. \tag{5}$$

Since $\Gamma > \gamma_i$ for all i, we get

$$\left(\frac{D^{-\gamma_i/\Gamma} z_i^{\gamma_i/\Gamma}}{\sum_i D^{-\gamma_i/\Gamma} \, z_i^{\gamma_i/\Gamma}}\right)^{\Gamma/\gamma_i} < \frac{D^{-\gamma_i/\Gamma} z_i^{\gamma_i/\Gamma}}{\sum_i D^{-\gamma_i/\Gamma} \, z_i^{\gamma_i/\Gamma}}$$

and therefore

$$\sum_i z_i \, M^{-b_i} \, D^{-1} < \sum_i \frac{D^{-\gamma_i/\Gamma} z_i^{\gamma_i/\Gamma}}{\sum_i D^{-\gamma_i/\Gamma} \, z_i^{\gamma_i/\Gamma}} = 1$$

which is a contradiction. The estimate from below follows by the same argument. ■

Lemma A.1 *There exist constants $K_1,\ K_2 > 0$ such that*

$$K_1 \sum_i \frac{E[DM]}{E[D^{1-\gamma_i}]}\, D^{-\gamma_i} \ \le\ M \ \le\ K_2 \sum_i \frac{E[DM]}{E[D^{1-\gamma_i}]}\, D^{-\gamma_i}.$$

Proof: By Lemma 2.1,

$$\frac{1}{n} \sum_i D^{-\gamma_i} \big(\psi_i\, E[DM]/E[M^{1-b_i}]\big)^{\gamma_i} \ \le\ M \ \le\ n \sum_i D^{-\gamma_i} \big(\psi_i\, E[DM]/E[M^{1-b_i}]\big)^{\gamma_i}. \quad (6)$$

By Lemma 2.2,

$$K_1 \frac{E[DM]}{E[D^{1-\gamma_i}]} \ \le\ \big(\psi_i\, E[DM]/E[M^{1-b_i}]\big)^{\gamma_i} \ \le\ K_2 \frac{E[DM]}{E[D^{1-\gamma_i}]}$$

for some $K_1,\ K_2 > 0$. The proof is complete. ■

We will also need the following

Lemma A.2 *For any $\alpha > 0$ and any $x_i > 0$,*

$$\min\{n^{\alpha-1}, 1\} \sum_i x_i^\alpha \ \le\ \left(\sum_i x_i\right)^\alpha \ \le\ \max\{n^{\alpha-1}, 1\} \sum_i x_i^\alpha.$$

Proof of Theorem 3.1. Let first $\gamma_i > \gamma_0$. By Lemma 2.2,

$$\hat{K}_2 \ \le\ \frac{W_{iT}/W_{0T}}{\frac{E[D^{1-\gamma_0}]^{b_0}}{E[D^{1-\gamma_i}]^{b_i}} E[DM]^{b_i-b_0} M^{b_0-b_i}} \ \le\ \hat{K}_1.$$

for some $\hat{K}_1,\ \hat{K}_2 > 0$. Combining Lemmas A.1 and A.2, we get that

$$\tilde{K}_2 \ \le\ \frac{M^{b_0-b_i}}{\sum_k \frac{E[DM]^{b_0-b_i}\, D^{(b_i-b_0)\gamma_k}}{E[D^{1-\gamma_k}]^{b_0-b_i}}} \ \le\ \tilde{K}_1.$$

and therefore W_{iT}/W_{0T} converges to zero almost at the same rate as

$$\sum_k \frac{E[D^{1-\gamma_0}]^{b_0}\, D^{(b_i-b_0)\gamma_k}}{E[D^{1-\gamma_i}]^{b_i} E[D^{1-\gamma_k}]^{b_0-b_i}}$$
$$= \sum_k e^{\frac{1}{2}\sigma^2 T \left((1-\gamma_0)^2 b_0 - (1-\gamma_i)^2 b_i - (1-\gamma_k)^2 (b_0-b_i) + 2\,\sigma^{-1}\gamma_k (b_i-b_0)\, B_T/T\right)}. \quad (7)$$

Since, by assumption, $b_i = \gamma_i^{-1} < \gamma_0^{-1} = b_0$,

$$\max_k \{(1-\gamma_0)^2 b_0 - (1-\gamma_i)^2 b_i - (1-\gamma_k)^2 (b_0-b_i)\} \ =\ (1-\gamma_0)^2 b_0 - (1-\gamma_i)^2 b_i - (1-\gamma_0)^2 (b_0-b_i) \ =\ s_i$$

and the claim follows since, by the strong law of large numbers for the Brownian motion, $B_T/T \to 0$ almost surely.

Finally, if $\gamma_i < \gamma_0$, we can repeat the same argument and show that $W_{0T}/W_{iT} \to \infty$ almost at rate $-s_i$, which is what had to be proved.

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