# Table of Contents

## PREFACE ................................................................. 1

## Part I. The Setting: Markets, Models, Interest Rates, Utility Maximization, Risk

1. **FINANCIAL MARKETS** .............................................. 8
   1.1. Bonds ......................................................................... 9
   1.1.1. Types of Bonds
   1.1.2. Reasons for Trading Bonds
   1.1.3. Risk of Trading Bonds
   1.2. Stocks ....................................................................... 12
   1.2.1. How are Stocks different from Bonds
   1.2.2. Going Long or Short
   1.3. Derivatives ............................................................... 15
   1.3.1. Futures and Forwards
   1.3.2. Marking to Market
   1.3.3. Reasons for Trading Futures
   1.3.4. Options
   1.3.5. Calls and Puts
   1.3.6. Option Prices
   1.3.7. Reasons for Trading Options
   1.3.8. Swaps
   1.3.9. Mortgage Backed Securities; Callable Bonds
   1.4. Organization of Financial Markets ................................. 26
   1.4.1. Exchanges
   1.4.2. Market Index
   1.5. Margins ..................................................................... 28
   1.5.1. Trades that involve Margin Requirements
   1.6. Transaction Costs ....................................................... 32

## SUMMARY ................................................................. 31

## PROBLEMS ................................................................. 30

## FURTHER READINGS ...................................................... 34
2. INTEREST RATES .......................................................... 35
   2.1. Computation of Interest Rates .............................................. 35
       2.1.1. Simple versus Compound Interest; Annualized Rates
       2.1.2. Continuous Interest
   2.2. Present Value .......................................................... 39
       2.2.1. Present/Future Values of Cash Flows
       2.2.2. Bond Yield
       2.2.3. Price-Yield Curves
   2.3. Term Structure of Interest Rates and Forward Rates ................. 45
       2.3.1. Yield Curve
       2.3.2. Calculating Spot Rates; Rates Arbitrage
       2.3.3. Forward Rates
       2.3.4. Term Structure Theories
SUMMARY ................................................................. 52
PROBLEMS ................................................................. 53
FURTHER READINGS ...................................................... 55

3. MODELS OF SECURITIES PRICES IN FINANCIAL MARKETS ............. 56
   3.1. Single-Period Models .................................................. 57
       3.1.1. Asset Dynamics
       3.1.2. Portfolio and Wealth Processes
       3.1.3. Arrow-Debreu Securities
   3.2. Multi-Period Models .................................................. 61
       3.2.1. General Model Specifications
       3.2.2. Cox-Ross-Rubinstein Binomial Model
   3.3. Continuous-Time Models .............................................. 66
       3.2.1. Simple Facts about the Merton-Black-Scholes Model
       3.3.2. Brownian Motion Process
       3.3.3. Diffusion Processes, Stochastic Integrals
       3.3.4. Technical Properties of Stochastic Integrals*
       3.3.5. Itô's Rule
       3.3.6. Merton-Black-Scholes Model
       3.3.7. Wealth Process and Portfolio Process
   3.4. Modeling Interest Rates .............................................. 81
       3.4.1. Discrete-Time Models
       3.4.2. Continuous-Time Models
   3.5. Nominal Rates and Real Rates ....................................... 83
       3.5.1. Discrete-Time Models
3.5.2. Continuous-Time Models

3.6. Arbitrage and Market Completeness ......................................................... 85
  3.6.1. Notion of Arbitrage
  3.6.2. Arbitrage in Discrete-Time Models
  3.6.3. Arbitrage in Continuous-Time Models
  3.6.4. Notion of Complete Markets
  3.6.5. Complete Markets in Discrete-Time Models
  3.6.6. Complete Markets in Continuous-Time Models

3.7. Appendix ......................................................................................................... 96
  3.7.1. More Details for the Proof of Itô’s Formula
  3.7.2. Multi-Dimensional Itô’s Rule

SUMMARY ............................................................................................................. 99
PROBLEMS ........................................................................................................... 99
FURTHER READINGS ............................................................................................ 102

4. OPTIMAL CONSUMPTION/PORTFOLIO STRATEGIES ......................... 103
  4.1. Preference Relations and Utility Functions .................................................. 103
    4.1.1. Consumption
    4.1.2. Preferences
    4.1.3. Concept of Utility Functions
    4.1.4. Marginal Utility; Risk Aversion; Certainty Equivalent
    4.1.5. Utility Functions in Multi-Period Discrete-Time Models
    4.1.5. Utility Functions in Continuous-Time Models
  4.2. Discrete-Time Utility Maximization ............................................................. 113
    4.2.1. Single Period
    4.2.2. Multi-Period Utility Maximization: Dynamic Programming
    4.2.3. Optimal Portfolios in Merton-Black-Scholes Model
    4.2.4. Utility from Consumption
  4.3. Utility Maximization in Continuous Time .................................................... 122
    4.3.1. Hamilton-Jacobi-Bellman PDE
  4.4. Duality/Martingale Approach to Utility Maximization ............................... 127
    4.4.1. Martingale Approach in Single-Period Binomial Model
    4.4.2. Martingale Approach in Multi-Period Binomial Model
    4.4.3. Duality/Martingale Approach in Continuous Time*
  4.5. Transaction Costs .......................................................................................... 138
  4.6. Incomplete and Asymmetric Information .................................................... 138
    4.6.1 Single Period
    4.6.2. Incomplete Information in Continuous Time*
Part II: Pricing and Hedging of Securities

6. ARBITRAGE AND RISK-NEUTRAL PRICING ............................. 174
   6.1. Arbitrage Relationships for Call and Put Options; Put-Call Parity  .... 156
   6.2. Arbitrage Pricing of Forwards and Futures .......................... 160
       6.2.1. Forward Prices
       6.2.2. Futures Prices
       6.2.3. Futures on Commodities
   6.3. Risk-Neutral Pricing .................................................. 183
       6.3.1. Martingale Measures; Cox-Ross-Rubinstein (CRR) Model
       6.3.2. State Prices in Single-Period Models
       6.3.3. No Arbitrage and Risk-Neutral Probabilities
       6.3.4. Pricing by No Arbitrage
       6.3.5. Pricing by Risk-Neutral Expected Values
6.3.6. Martingale Measure for the Merton-Black-Scholes Model
6.3.7. Computing Expectations by Feynman-Kac PDE
6.3.8. Risk-Neutral Pricing in Continuous Time
6.3.9. Futures and Forwards Revisited*

6.4. Appendix ................................................................. 200
   6.4.1. No Arbitrage Implies Existence of a Risk-Neutral Probability*
   6.4.2 Completeness and Unique EMM*
   6.4.3 Another Proof of Theorem ??*
   6.4.4. Proof of Bayes Rule**

SUMMARY ................................................................. 206
PROBLEMS .............................................................. 207
FURTHER READINGS .................................................... 209

7. OPTION PRICING ......................................................... 211
   7.1. Option Pricing in the Binomial Model ...................... 211
      7.1.1. Backward Induction and Expectation Formula
      7.1.2. Black-Scholes Formula as a Limit of the Binomial Model Formula
   7.2. Option Pricing in the Merton-Black-Scholes Model .......... 216
      7.2.1. Black-Scholes Formula as Expected Value
      7.2.2. Black-Scholes Equation
      7.2.3. Black-Scholes Formula for the Call Option
      7.2.4. Implied Volatility
   7.3. Pricing American Options ..................................... 221
      7.3.1. Stopping Times and American Options
      7.3.2. Binomial Trees and American Options
      7.3.3. PDE’s and American Options
   7.4. Options on Dividend-Paying Securities ..................... 228
      7.4.1. Binomial Model
      7.4.2. Merton-Black-Scholes Model
   7.5. Other Types of Options ....................................... 233
      7.5.1. Currency Options
      7.5.2. Futures Options
      7.5.3. Exotic Options
   7.6. Pricing in the Presence of Several Random Variables .... 239
      7.6.1. Options on Two Risky Assets
      7.6.2. Quantos
      7.6.3. Stochastic Volatility with Complete Markets
      7.6.4. Stochastic Volatility with Incomplete Markets; Market Price of Risk
7.6.5. Utility Pricing in Incomplete Markets*

7.7. Merton’s Jump-Diffusion Model ................................................................. 251
7.8. Estimation of Variance and ARCH/GARCH Models ................................. 254
7.9. Appendix: Derivation of the Black-Scholes Formula .................................. 256

SUMMARY ........................................................................................................... 258
PROBLEMS .......................................................................................................... 259
FURTHER READINGS ......................................................................................... 264

8. FIXED INCOME MARKET MODELS AND DERIVATIVES ......................... 265
8.1. Discrete-Time Interest Rate Modeling ......................................................... 265
  8.1.1. Binomial Tree for the Interest Rate
  8.1.2. Black-Derman-Toy Model
  8.1.3. Ho-Lee Model
8.2. Interest Rate Models in Continuous Time ................................................. 276
  8.2.1. One-Factor Short Rate Models
  8.2.2. Bond Pricing in Affine Models
  8.2.3. HJM Forward Rate Models
  8.2.4. Change of Numeraire*
  8.2.5. Option Pricing with Random Interest Rate*
  8.2.6. BGM Market Model*
8.3. Swaps, Caps and Floors ............................................................................. 289
  8.3.1. Interest Rate Swaps and Swaptions
  8.3.2. Caplets, Caps and Floors
8.4. Credit/Default Risk ...................................................................................... 293

SUMMARY ........................................................................................................... 296
PROBLEMS .......................................................................................................... 296
FURTHER READINGS ......................................................................................... 299

9. HEDGING .......................................................................................................... 301
9.1. Hedging with Futures .................................................................................. 301
  9.1.1. Perfect Hedge
  9.1.2. Crosshedging; Basis Risk
  9.1.3. Rolling the Hedge Forward
  9.1.4. Quantity Uncertainty
9.2. Portfolios of Options as Trading Strategies .............................................. 305
  9.2.1. Covered Calls and Protective Puts
  9.2.2. Bull Spreads and Bear Spreads
  9.2.3. Butterfly Spreads
11.2.3. Variance Reduction Techniques
11.2.4. Simulation in a Continuous-Time Multi-Variable Model
11.2.5. Computation of Hedging Portfolios by Finite Differences
11.2.6. Retrieval of Volatility Method for Hedging and Utility Maximization*

11.3. Numerical Solutions of PDE’s; Finite Difference Methods ............... 358
   11.3.1. Implicit Finite Difference Method
   11.3.2. Explicit Finite Difference Method

SUMMARY ................................................................. 361
PROBLEMS ............................................................. 362
FURTHER READINGS ..................................................... 364

Part III: Equilibrium Models

12. EQUILIBRIUM FUNDAMENTALS ........................................... 367
   12.1. Concept of Equilibrium .............................................. 367
       12.1.1. Definition and Single-Period Case
       12.1.2. A Two-Period Example
       12.1.2. Continuous-Time Equilibrium
   12.2. Single-Agent and Multi-Agent Equilibrium ............................. 372
       12.2.1. Representative Agent
       12.2.2. Single-Period Aggregation
   12.3. Pure Exchange Equilibrium ............................................ 375
       12.3.1. Basic Idea and Single-Period Case
       12.3.2. Multi-Period Discrete-Time Model
       12.3.3. Continuous-Time Pure Exchange Equilibrium
   12.4. Existence of Equilibrium .............................................. 381
       12.4.1. Equilibrium Existence in Discrete Time
       12.4.2. Equilibrium Existence in Continuous Time
       12.4.3. Determining Market Parameters in Equilibrium

SUMMARY ................................................................. 388
PROBLEMS ............................................................. 389
FURTHER READINGS ..................................................... 390

13. CAPM ................................................................. 391
   13.1. Basic CAPM ......................................................... 391
PREFACE

Why we wrote the book: The subject of financial markets is fascinating to many people: to those who care about money and investments, to those who care about the well-being of the modern society, to those who like gambling, to those who like applications of mathematics, and so on. We, the authors of this book, care about many of these things (no, not the gambling), but what we care about most is teaching. The main reason for writing this book has been our belief that we can successfully teach the fundamentals of the economic and mathematical aspects of financial markets to almost everyone (again, we are not sure about gamblers). Why are we in this teaching business instead of following the path of many of our former students, the path of making money by pursuing a career in the financial industry? Well, they don’t have the pleasure of writing a book for the enthusiastic reader like yourself!

Prerequisites: This text is written in such a way that it can be used at different levels and for different groups of undergraduate and graduate students. After the first, introductory chapter, each chapter starts with sections on the single-period model, goes to multi-period models, and finishes with continuous-time models. The single-period and multi-period models require only basic calculus and an elementary introductory probability/statistics course. Those sections can be taught to third and fourth year undergraduate students in Economics, Business, and similar. They could be taught to Mathematics and Engineering students even at an earlier stage. In order to be able to read continuous-time sections it is helpful to have been exposed to an advanced undergraduate course in probability. Some material needed from such a probability course is briefly reviewed in the last chapter.

Who is it for: The book can also serve as an introductory text for graduate students in Finance, Financial Economics, Financial Engineering and Mathematical Finance. Some material from continuous-time sections is, indeed, usually considered to be graduate material. We try to explain much of that material in an intuitive way, while providing some of the proofs in appendices to the chapters. The book is not meant to compete with numerous excellent graduate books in Financial Mathematics and Financial Economics, which are typically written in a mathematically more formal way, using theorem-proof type of structure. Some of those more advanced books are mentioned in the references, and they present a natural next step in getting to know the subject on a more theoretical and advanced level.

Structure of the book: We have divided the book into three larger parts. Introductory Part I goes over the basic securities and financial market organization, the concept of interest rates, the main mathematical models, and ways to measure in a quantitative way the risk and the reward of trading in the market. Part II deals with option pricing and hedging, and similar material is present in virtually every recent book on financial markets. We choose
to emphasize the so-called martingale, probabilistic approach consistently throughout the book, as opposed to the differential equations approach, or other existing approaches. For example, the one proof of the Black-Scholes formula that we provide is done calculating the corresponding expected value. Part III is devoted to one of the favorite subjects of Financial Economics, the equilibrium approach to asset pricing. This part is often omitted from books in the field of financial mathematics, having less direct applications to option pricing and hedging. However, it is this theory that gives a qualitative insight into the behavior of market participants and how the prices are formed in the market.

**What can a course cover:** We have used parts of the material from the book for teaching various courses at the University of Southern California: undergraduate courses in Economics and Business, a Masters level course in Mathematical Finance, and option and investment courses for MBA students. For example, an undergraduate course for Economics/Business students that emphasizes option pricing could cover (in this order):

- the first three chapters without continuous-time sections; Chapter 10 on bond hedging could also be done immediately after Chapter 2 on interest rates.
- the first two chapters of Part II on no-arbitrage pricing and option pricing, without most of continuous-time sections, but including basic Black-Scholes theory.
- chapters on hedging in Part II, with or without continuous-time sections.
- Mean-Variance section in Chapter 5 on risk; Chapter 13 on CAPM could also be done immediately after that section.

If time remains, or if this is an undergraduate Economics course that emphasizes equilibrium/asset pricing as opposed to option pricing, or if this is a two-semester course, one could also cover:

- discrete-time sections in Chapters 4 on utility;

Courses aimed at more mathematically oriented students could go very quickly through the discrete-time sections, and instead spend more time on continuous-time sections. A one semester course would likely have to make a choice: to focus on no-arbitrage option pricing methods in Part II, or to focus on equilibrium models of Part III.

**Web Page for this book, Excel Files:**
The web page [http://math.usc.edu/~cvitanic/book.html](http://math.usc.edu/~cvitanic/book.html) will be regularly updated with material related to the book, such as corrections of typos. It also contains Microsoft Excel files, with names like “ch1.xls”. That particular file has all the figures from Chapter 1, and all the computations needed to produce them. We use Excel because we wanted the reader to be able to reproduce and modify all the figures in the book. A slight disadvantage of this is that our figures sometimes look less professional than if they had been done by a specialized drawing software. We use only basic features of Excel, except for Monte Carlo simulation for which we use the Visual Basic programming language, incorporated in Excel.
The readers are expected to learn the basic features of Excel on their own, if they are not already familiar with it. At a few places in the book we do give “Excel Tips” that point out what trickier commands have been used for creating a figure. Other, more mathematically oriented software may be more efficient for longer computations such as Monte Carlo, and we leave the choice of the software to be used with some of the homework problems to the instructor or the reader. In particular, we do not use any optimization software or differential equations software, even though the instructor could think of projects using those.

**Notation: superscript ∗.** Some of the sections and problems have a ∗ superscript. These are either more sophisticated in mathematical terms, or they require extensive use of computer software, or they are otherwise somewhat unusual and outside of the main thread of the book. They could be skipped, although we would suggest that the students do most of the problems that require use of computers.

**Notation: solved problems with superscript ˜.** The end-of-chapters problems that are solved in the Student’s Manual have a superscript ˜ as in ˜14, when problem 14 is solved in the manual.

**Notation: Greek letters.** We use many letters from the Greek alphabet, sometimes both lower and upper case, and we list them here with appropriate pronunciation:

\[
\begin{align*}
\alpha & (\text{alpha}), \beta & (\text{beta}), \gamma, \Gamma & (\text{gamma}), \delta, \Delta & (\text{delta}), \varepsilon & (\text{epsilon}), \\
\zeta & (\text{zeta}), \eta & (\text{eta}), \theta & (\text{theta}), \\
\lambda & (\text{lambda}), \mu & (\text{mu}), \xi & (\text{xi}), \pi, \Pi & (\text{pi}), \\
\omega, \Omega & (\text{omega}), \rho & (\text{rho}), \sigma, \Sigma & (\text{sigma}), \tau & (\text{tau}), \varphi & (\text{phi}).
\end{align*}
\]

**Acknowledgements:** First and foremost, we are immensely grateful to our families for the support they provided us while working on the book. We have received great help and support from the staff of our publisher MIT Press, and, in particular, we have enjoyed working with Elizabeth Murry, which helped us go through the writing and the production process in a smooth and efficient manner. The J.C.’s research and the writing of this book has been partially supported by the National Science Foundation grant DMS-00-99549. Some of the continuous-time sections in Parts I and II originated from the lecture notes prepared in summer of 2000 while J.C. was visiting University of the Witwatersrand in Johannesburg, and he is very thankful to his host David Rod Taylor, the director of the Mathematical Finance Programme at Wits. Numerous colleagues have made useful comments and suggestions including: Krzysztof Burdzy, Paul Dufresne, Neil Gretzky, Assad Jalali, Dmitry Kramkov, Ali Lazrak, Lionel Martellini, Adam Ostaszewski, Kaushik Ronnie Sircar, Costis Skiadas, Halil Mete Soner, Adam Speight, David Rod Taylor, Mihail Zervos. In particular, D. Kramkov provided us with proofs in the appendix of Chapter 6. Some material on continuous-time utility maximization with incomplete information is taken from a joint work with A. Lazrak and L. Martellini, and on continuous-time mean-variance optimization from a joint work with A. Lazrak. Moreover, the following students provided their comments and pointed out
errors in the working manuscript: Paula Guedes, Frank Denis Hiebsch, Chulhee Lee. Of course, we are solely responsible for any remaining errors.

A Prevailing Theme - Pricing by Expected Values: Before we start with the book’s material, we would like to give a quick illustration here in the preface of a connection between a price of a security and the optimal trading strategy of an investor investing in that security. We present it in a simple model, but this connection is present in most market models, and, in fact, the resulting pricing formula is of the form that will follow us through all three parts of this book. We will repeat this type of arguments later in more detail, and we present it this early here only to give the reader a general taste of what the book is about. The reader may want to skip the following derivation, and go directly to equation (0.3).

Consider a security $S$ with today’s price $S(0)$, and at a future time 1 its price $S(1)$ either has value $s^u$ with probability $p$, or value $s^d$ with probability $1 - p$. There is also a risk-free security that returns $1 + r$ dollars at time 1 for every dollar invested today. We assume that $s^d < (1+r)S(0) < s^u$. Suppose an investor has initial capital $x$, and has to decide how many shares $\delta$ of security $S$ to hold, while depositing the rest of his wealth in the bank account with interest rate $r$. In other words, his wealth $X(1)$ at time one is

$$X(1) = \delta S(1) + (x - \delta S(0))(1 + r).$$

The investor wants to maximize his expected utility

$$E[U(X(1))] = pU(X^u) + (1 - p)U(X^d),$$

where $U$ is a so-called utility function, while $X^u, X^d$ is his final wealth in the case $S(1) = s^u, S(1) = s^d$, respectively. Substituting for these values, taking the derivative with respect to $\delta$ and setting it equal to zero, we get

$$pU'(X^u)[s^u - S(0)(1 + r)] + (1 - p)U'(X^d)[s^d - S(0)(1 + r)] = 0. $$

The left-hand side can be written as $E[U'(X(1))(S(1) - S(0)(1 + r))]$, which, when made equal to zero implies, with arbitrary wealth $X$ replaced by optimal wealth $\hat{X}$,

$$S(0) = E \left[ \frac{U'(\hat{X}(1))}{E[U'(\hat{X}(1))]} \frac{S(1)}{1 + r} \right]. \quad (0.1)$$

If we denote

$$Z(1) := \frac{U'(\hat{X}(1))}{E[U'(\hat{X}(1))]}, \quad (0.2)$$

we see that the today’s price of our security $S$ is given by

$$S(0) = E \left[ Z(1) \frac{S(1)}{1 + r} \right]. \quad (0.3)$$
We will see that prices of most securities (with some exceptions, like American options) in the models of this book are of this form: the today’s price $S(0)$ is an expected value of the future price $S(1)$, multiplied (“discounted”) by a certain random factor. Effectively, we get the today’s price as a weighted average of the discounted future price, but with weights which depend on the outcomes of the random variable $Z(1)$. Moreover, in standard option pricing models (having a so-called completeness property) we will not need to use utility functions, since $Z(1)$ will be independent of the investor’s utility. The random variable $Z(1)$ is sometimes called change of measure, while the ratio $Z(1)/(1 + r)$ is called state-price density, stochastic discount factor, pricing kernel, or marginal rate of substitution, depending on the context and interpretation. There is another interpretation of this formula, using a new probability; hence the name “change of (probability) measure”. For example, if, as in our example above, $Z(1)$ takes two possible values $Z^{u}(1)$ and $Z^{d}(1)$ with probabilities $p, 1 − p$, respectively, we can define

$$p^{*} := pZ^{u}(1) , \quad 1 − p^{*} = (1 − p)Z^{d}(1) .$$

The values of $Z(1)$ are such that that $p^{*}$ is a probability, and we interpret $p^{*}$ and $1 − p^{*}$ as modified probabilities of the movements of asset $S$. Then, we can write (0.3) as

$$S(0) = E^{*} \left[ \frac{S(1)}{1 + r} \right] , \quad (0.4)$$

where $E^{*}$ denotes the expectation under the new probabilities, $p^{*}, 1 − p^{*}$. Thus, the price today is the expected value of the discounted future value, where the expected value is computed under a special, so-called risk-neutral probability, usually different from the real world probability.

**Final word:** We hope that we have aroused your interest about the subject of this book. If you turn out to be a very careful reader, we would be thankful if you could inform us of any remaining typos and errors that you find, by sending an e-mail to our current e-mail addresses. Enjoy the book!


E-mail addresses: cvitanic@math.usc.edu, zapatero@usc.edu