Campaign Finance in a Coalition Formation Game

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Abstract

We study a spatial model of campaign finance in which potential candidates need a fixed amount of money from lobbies to enter an election. We show that the set of pure strategy Nash equilibria in which lobbies finance candidates whose policies they prefer among the set of entrants is exactly the set of Nash equilibria with weakly less than two entering candidates. If the total amount of money held by lobbies is finite, there exists some minimal distance between the two candidates’ positions.

1. Introduction

This paper provides an analysis of the strategic incentives of lobbies and other parties who fund political campaigns. Our specific interest is in understanding the extent to which money is used to “manipulate” elections. A commonly held intuition is that third-party candidates often enter an election to “steal” votes from a candidate they do not like, thus biasing an election in their favor. We examine the extent to which this behavior is present.

To do so, we introduce a multicandidate spatial model of campaign finance (see Downs [7]). Voting is over policies. As in other models, financing directly affects the outcome of an election. Our innovation is that money is required for

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a candidate to enter an election. Other studies assume money buys uninformed voters, as in Baron [2], or affects the accuracy of signals sent by candidates, as in Austen-Smith [1]. Other multicandidate models include [4, 5, 6, 9, 10, 13, 14, 15]

Our model is similar to the citizen-candidate models of Besley and Coate [4, 5, 6] and Osborne and Slivinski. [14] We modify the model so that candidates must be financed by private lobbies. While our model does not allow candidates to lie about their most preferred policy, lobbies need not finance their most preferred candidate. Indeed, lobbies may finance a candidate whose policy is their least preferred among the set of all candidates. The vast literature on strategy-proofness (briefly surveyed by Barbera [3]) suggests the possibility that lobbies will generically “manipulate” elections in this fashion.

The introduction of these strategic incentives opens the door for new qualitative results. We ask the question, “when do lobbies finance candidates whose policy they prefer to those of other candidates?” We obtain a striking result; this type of truthful financing is observed if and only if the number of entrants in the election is at most two. This provides an argument for the commonly held intuition that third-party candidates often campaign solely to “steal” votes from candidates they do not like (see Besley and Coate, Osborne and Slivinski). Most importantly, it gives an empirically meaningful and qualitative characterization of truthfully financed elections.

2. The model

Let $X \subset \mathbb{R}$ be convex. The set $X$ represents policy space. Let $L$ be a set of lobbies, where $L$ is at most countable. For all $l \in L$, let $\omega_l \geq 0$ be an endowment. Lobby $l$’s strategy space is $S_l \equiv X \times [0, \omega_l]$, with generic element $s_l = (p_l, m_l)$. Let $S \equiv \prod_L S_l$. The interpretation is that lobby $l$ pledges an amount of money $m_l$ to support a candidate whose policy is $p_l$. The strategy space reflects the features that no lobby can pledge more money than it has, and that a lobby can pledge money to only one policy. These assumptions are made for simplicity and are easily discarded.

To describe lobbies’ preferences, we introduce more notation. For all $x \in X$, $s \in S$ let

$$M_x(s) = \sum_{\{l: p_l = x\}} m_l$$

be the total amount of money pledged to position $x$. 

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Let $F > 0$ be a fixed entry-fee. If a candidate raises $F$, she becomes an “entrant” in the election. We assume that for all $l \in L$, $\omega_l < F$. This assumption is made to keep the analysis simple, and only means that no lobby can finance an entire campaign on its own. This fixed cost assumption is not new to the literature; see Besley and Coate [4, 5, 6], Osborne and Slivinski [14], and Weber [16], for example.\(^1\)

The election of candidates is non-strategic. There is some distribution of voter types, where a type is a single-peaked binary relation over $X$. All voters vote for the policy which maximizes their preference (sincere voting).\(^2\) The winner(s) of the election are the candidates with the highest vote share. If there are no entering candidates, the outcome of the election is some policy $x^{sq} \in X$ (call it the status quo).

For all $l \in L$, let $u_l : X \to \mathbb{R}$ be (weakly) concave. Each lobby incurs a “cost” when spending $m_l$, which is reflected by a function $\psi_l : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, satisfying $\psi_l(0) = 0$ and $\psi_l$ strictly increasing. The requirement that $\psi_l(0) = 0$ is a normalization.

Let

$$E(s) \equiv \{x : M_x(s) \geq F\}$$

be the set of entrants of the election. A policy $x$ is in $E$ if it raises enough funds to enter the election. Let the vote share of an entrant be denoted $V_x(s).$$^3$$^4$ Let

\[^1\]In these papers, the cost is typically not a monetary cost.

\[^2\]Formally, a binary relation $R$ over $X$ is single-peaked if and only if there exists some $x^* \in X$ such that for all $x \in X \setminus \{x\}$, $x^*Px$, if $x^* < y < z$, then $yPz$, and if $z < y < x^*$, then $yPz$. Let $\mathcal{R}$ be the set of all single-peaked binary relations over $X$. Let $\Sigma$ be a $\sigma$-algebra on $\mathcal{R}$. For all $x,y \in X$, $\Sigma$ must include all sets of the form

$$\{R : xRy\}.$$

Then, we are interested in a measure $\mu$ so that $(\mathcal{R}, \Sigma, \mu)$ is a measure space, and $\mu$ is a countably additive, non degenerate measure.

\[^3\]Formally, define the vote share $V_x(s)$ by

$$\mu \left( \{R : xPy \forall y \in E(s) \setminus \{x\} \} \right) + \mu \left( \{R : \exists y \in E(s) \setminus \{x\} \ s.t. xIy, xPz \forall z \in E(s) \setminus \{x,y\} \} / 2 \right).$$

\[^4\]We note that our results easily extend to a case of weighted simple majority. For a weighted simple majority, we define a $\Sigma$-measurable weighting function $w$ which is an element of $L^1(\mu)$.
the **winners** be given by

\[ W(s) = \arg \max_{x \in E(s)} V_x(s) . \]

The **outcome** of \( s \) is an even-chance lottery over \( W(s) \).

Thus, for all \( l \in L \), let

\[ U_l(s) = E[u_l(W(s))] - \psi_l(m_l) . \]

### 3. Results

#### 3.1. A characterization of one-entrant equilibria

Our first theorem provides a simple characterization of Nash equilibria of the election game which consist of a unique entrant.

**Theorem 1:** Let \( s \) be a pure-strategy Nash equilibrium. Then \( |E(s)| = 1 \) if and only if the following conditions are satisfied (where \( E(s) = \{x\} \)):

\[ \begin{align*}
   i) & \quad \sum_{\{l: p_l = x\}} m_l = F, \ m_l = 0 \text{ if } p_l \notin \{x\}, \\
   ii) & \quad m_l > 0, p_l = x, \text{ implies } u_l(x) > u_l(x^{sq}), \\
   iii) & \quad \text{for all } l, \ \psi_l(m_l) \leq |u_l(x) - u_l(x^{sq})|. 
\end{align*} \]

**Proof:** Let \( s \) be a pure-strategy Nash equilibrium such that \( |E(s)| = 1 \), where \( E(s) = \{x\} \).

To verify \( i \), suppose by means of contradiction that it is false. Suppose that \( \sum_{\{l: p_l = x\}} m_l > F \). (We know it cannot be less than \( F \) as \( x \) is an entrant).

Let \( \varepsilon > 0 \) satisfy \( \varepsilon < \sum_{\{l: p_l = x\}} m_l - F \). Let \( l \in L \) satisfy \( p_l = x, m_l > 0 \). Let \( s'_l = (x_1, m_l - \varepsilon) \). Thus, \( W(s'_l, s_{-l}) = \{x\} \), so that \( U_l(s'_l, s_{-l}) > U_l(s) \), contradicting the fact that \( s \) is a Nash equilibrium. Let \( l \in L \) satisfy \( m_l > 0 \) and \( p_l \notin \{x\} \). Let \( s'_l = (p_l, 0) \). Then \( W(s'_l, s_{-l}) = \{x\} \), so that \( U_l(s'_l, s_{-l}) > U_l(s) \), contradicting the fact that \( s \) is a Nash equilibrium.

To verify \( ii \) and \( iii \), let \( l \in L \) satisfy \( m_l > 0, p_l = x \). Let \( s'_l = (x, 0) \). By \( i \), conclude \( W(s'_l, s_{-l}) = \{x^{sq}\} \). As \( s \) is a Nash equilibrium, conclude

\[ u_l(x) - \psi_l(m_l) \geq u_l(x^{sq}) . \]

Then weighted vote share is defined simply as the vote share induced by the measure

\[ \mu_w(A) = \int_A w(P) \ d\mu(P) . \]
so that
\[ \psi_l (m_l) \leq u_l (x) - u_l (x^q) . \]
As \( \psi_l (m_l) > 0 \), \( u_l (x) > u_l (x^q) \). Thus,
\[ \psi_l (m_l) \leq |u_l (x) - u_l (x^q)| . \]
If \( m_l = 0 \), the inequality is trivially satisfied.

It is trivial to verify that if the conditions are satisfied, nobody wishes to deviate. \( \blacksquare \)

3.2. Truthful financing and characterizations of two-entrant equilibria

Our next theorem concerns conditions under which a Nash equilibrium of the election game consists of only two entrants. The necessary and sufficient conditions we derive may be stated informally as follows.

Condition \( i \) states that the entrants tie in the vote. This is not an unusual result in these types of models. It is an artifact of the game being perfect-information.

Condition \( ii \) states that any financed position gets exactly enough to enter the election and no more; it also states that any lobby not financing an entering position finances no position at all. The first of these statements reflects the nature of the fact that once a policy becomes an entrant, more money is not beneficial. The second statement follows as money offered to any position cannot be returned back to the lobby if the position does not enter the election.

Condition \( iii \) has more content than the first two. We refer to it later as “truthful financing.” It states that any lobby financing a policy must finance a policy that it prefers (among the set of financed policies).

Condition \( iv \) is a simple condition which effectively places an upper bound on the amount of money that lobbies spend, as a function of the entrants. Read another way, it implies that as the entrants become more and more distinct, lobbies spend more and more money.

We remark that conditions \( ii \) and \( iii \) are stated for \( x_1 \); a symmetric statement holds for \( x_2 \).

**Theorem 2:** Let \( s \) be a pure-strategy Nash equilibrium. Then \( |E (s)| = 2 \) if and only if the following conditions are satisfied (where \( E (s) = \{x_1, x_2\} \)):

\( i \) \( x_1 \) and \( x_2 \) tie in the election
\( \sum_{\{l:p_l=x_1\}} m_l = F, \quad m_l = 0 \text{ if } p_l \notin \{x_1, x_2\} \)

\( m_l > 0, p_l = x_1, \text{ implies } u_l(x_1) > u_l(x_2) \)

\( \text{for all } l, \psi_l(m_l) \leq \frac{|u_l(x_1) - u_l(x_2)|}{2}. \)

**Proof:** Let \( x \) be a pure-strategy Nash equilibrium. Suppose that \( |E(s) = 2| \).

To verify \textit{i)}, suppose the statement is false. Without loss of generality, suppose \( W(s) = \{x_2\} \). Let \( l \in L \) satisfy \( p_l = x_1, m_l > 0 \). By deviating to \( s_l' = (x_1, 0) \), we have \( W(s_l', s_{-l}) = \{x_2\} \), so that \( U_l(s_l', s_{-l}) > U_l(s) \), contradicting the fact that \( s \) is a Nash equilibrium.

To verify \textit{ii)}, suppose by means of contradiction that it is false. Suppose that \( \sum_{\{l:p_l=x_1\}} m_l > F \). (We know it cannot be less than \( F \) as \( x_1 \) is an entrant). Let \( \varepsilon > 0 \) satisfy \( \varepsilon < \sum_{\{l:p_l=x_1\}} m_l - F \). Let \( l \in L \) satisfy \( p_l = x_1, m_l > 0 \). Let \( s_l' = (x_1, m_l - \varepsilon) \). Thus, \( W(s_l', s_{-l}) = W(s) \), so that \( U_l(s_l', s_{-l}) > U_l(s) \), contradicting the fact that \( s \) is a Nash equilibrium. Next, suppose by means of contradiction that there exists \( l \in L \) such that \( m_l > 0 \) and \( p_l \notin \{x_1, x_2\} \). Let \( s_l' = (p_l, 0) \). Then \( W(s_l', s_{-l}) = W(s) \), so that \( U_l(s_l', s_{-l}) > U_l(s) \), contradicting the fact that \( s \) is a Nash equilibrium.

We verify \textit{iii)} and \textit{iv)}. Let \( l \in L \) satisfy \( m_l > 0, p_l = x_1 \). Let \( s_l' = (x_1, 0) \). By \textit{ii)}, \( W(s_l', s_{-l}) = \{x_2\} \). As \( s \) is a Nash equilibrium,

\[
\frac{u_l(x_1) + u_l(x_2)}{2} - \psi_l(m_l) \geq u_l(x_2),
\]

so that

\[
\psi_l(m_l) \leq \frac{u_l(x_1) - u_l(x_2)}{2}.
\]

As \( \psi_l(m_l) > 0 \), conclude \( u_l(x_1) > u_l(x_2) \). Thus,

\[
\psi_l(m_l) \leq \frac{|u_l(x_1) - u_l(x_2)|}{2}.
\]

If \( m_l = 0 \), the inequality is trivially satisfied.

It is trivial to verify that if the conditions are satisfied, nobody wishes to deviate.

**Remark:** The characterizations of one and two entrant equilibria seem to have little in common in a general sense. However, the truthful financing concept alluded to in part \textit{iii)} of Theorem 2 is trivially satisfied for one entrant equilibria. To
make this precise, we say a pure strategy Nash equilibrium \( s \) of the above game satisfies \textit{truthful financing} if for all \( l \in L \) and all \( m_l > 0 \), for all \( x_j \in E(s) \),
\[
  u_l(p_l) \geq u_l(x_j).
\]

The preceding theorems establish that if \( |E(s)| \leq 2 \), \( s \) satisfies truthful financing. Unfortunately, truthful financing does not extend to \textit{all} pure-strategy Nash equilibria \( s \). We provide a three-entrant example.

\textbf{Example 1:} Let \( \mu \) be some distribution of voters. Let \( s \) be a pure-strategy Nash equilibrium such that \( |E(s)| = 3 \). Suppose that \( |W(s)| = 3 \). Label the policies of the entrants \( \{x_1, x_2, x_3\} \) (where \( x_1 < x_2 < x_3 \)). It is clear to see that \( s \) cannot satisfy the truthful financing property. To see this, suppose, without loss of generality, that \( x_2 \leq \frac{x_1 + x_3}{2} \). Suppose a lobby \( l \) finances \( x_1 \), and that \( u_l(x_1) \geq u_l(x_2) > u_l(x_3) \). Thus,
\[
  (u_l(x_1) + u_l(x_2) + u_l(x_3))/3 - \psi_l(m_l) \geq u_l(x_2).
\]
The left hand side is the payoff to lobby \( l \) under strategy profile \( s \); the right hand side is what lobby \( l \) would receive by taking back all of its money, \( m_l \). The inequality thus implies
\[
  \frac{u_l(x_1) + u_l(x_3)}{2} > u_l(x_2).
\]

However, by concavity of \( u_l \), we know that
\[
  u_l\left( \frac{x_1 + x_3}{2} \right) \geq \frac{u_l(x_1) + u_l(x_3)}{2} > u_l(x_2).
\]

Moreover, \( x_2 \) is a convex combination of \( \frac{x_1 + x_3}{2} \) and \( x_1 \) (as \( x_1 < x_2 \leq \frac{x_1 + x_3}{2} \)), say, with weights \( \lambda \) and \( 1 - \lambda \). Concavity of \( u_l \) thus implies
\[
  u_l(x_2) = u \left( \lambda x_1 + (1 - \lambda) \left( \frac{x_1 + x_3}{2} \right) \right) \geq \lambda u_l(x_1) + (1 - \lambda) u_l\left( \frac{x_1 + x_3}{2} \right) > u_l(x_2),
\]
a contradiction.

This example is somewhat disturbing, as we would hope that \textit{any} equilibrium would satisfy the truthful financing property. We find that the set of truthfully financed Nash equilibria are completely characterized as the set of Nash equilibria with weakly less than two entrants. That is, there are \textit{no} equilibria with more than two entrants in which some lobby does not “lie” about its’ preferred candidate.

We proceed with the theorem.
Theorem 3: Let \( s \) be a pure-strategy Nash equilibrium that satisfies truthful financing. Then \( |E(s)| \leq 2 \).

Proof: Let \( s \) be a pure-strategy Nash equilibrium that satisfies truthful financing. Suppose, by means of contradiction, that \( |E(s)| > 2 \). Without loss of generality, suppose \( |E(s)| < +\infty \), (and hence has maximal and minimal elements).\(^5\)

Thus, \( E(s) = \{x_1, \ldots, x_m\} \), where \( x_1 < x_2 < \ldots < x_m \) and \( m > 2 \). By arguments similar to those used in the first two theorems, we conclude that for all \( x_j \in E(s), M_{x_j} = F \).

We verify two cases.

Case i) Either \( x_1 \in W(s) \) or \( x_m \in W(s) \), but not both.

Suppose that \( x_1 \notin W(s) \). Let \( l \in L \) satisfy \( p_l = x_1, m_l > 0 \). As \( s \) is a Nash equilibrium, \( U_l(s) \geq U_l((x_1,0), s_{-l}) \). Clearly, \( x_1 \notin E(((x_1,0), s_{-l})) \). By single-peakness of voters’ preferences, any voter who votes for \( x_1 \) under \( s \) must prefer \( x_2 \) to all of the other remaining policies. Therefore, one of the three following possibilities is true: \( a) \ x_2 \) ties under profile \( s \), and is the sole winner under profile \( ((x_1,0), s_{-l}) \), \( b) \ x_2 \) loses under profile \( s \), and is the sole winner under profile \( ((x_1,0), s_{-l}) \), or \( c) \ x_2 \) loses under profile \( s \), and ties under profile \( ((x_1,0), s_{-l}) \). Consider case \( a) \). As \( s \) is a Nash equilibrium,

\[
\frac{\sum_{W(s)} u_l(x_j)}{|W(s)|} - \psi_l(m_l) \geq u_l(x_2),
\]

so that

\[
\frac{\sum_{W(s)} u_l(x_j)}{|W(s)|} > u_l(x_2).
\]

Rearranging terms leads to

\[
\frac{\sum_{W(s),x_2} u_l(x_j)}{|W(s)| - 1} > u_l(x_2).
\]

As \( x_1 \notin W(s), j \neq 2 \) implies \( x_j > x_2 \). Therefore, there exists \( x_j > x_2 \) such that \( u_l(x_j) > u_l(x_2) \), and by the concavity of \( u \), thus \( u_l(x_2) > u_l(x_1) \), contradicting the truthful financing hypothesis.

\(^5\)To see why this is without loss of generality, suppose \( |E(s)| = +\infty \). Let \( \{x_i\}_{i=1}^\infty \subset E(s) \) be a sequence such that for all \( i, j, x_i \neq x_j \). Then \( V_{x_i}(s) \to 0 \) as \( i \to +\infty \) (due to countable additivity of \( \mu \)). Thus, there exists \( K \) large enough so that for all \( l \in L \) such that \( p_l = x_K, W((p_l,0), s_{-l}) = W(s) \). Therefore, \( u_l((p_l,0), s_{-l}) > u_l(s) \).
The second case similarly obtains
\[ \frac{\sum_{W(s)} u_l(x_j)}{|W(s)|} > u_l(x_2). \]

Moreover, in this case, \( j \in W(s) \) implies \( x_j > x_2 \). We reach the same conclusion as in case 1.

The last case similarly obtains
\[ \frac{\sum_{W(s)} u_l(x_j)}{|W(s)|} > \frac{\sum_{W(s) \cup \{x_2\}} u_l(x_j)}{|W(s)| + 1}. \]

In this case, rearranging terms obtains:
\[ \frac{\sum_{W(s)} u_l(x_j)}{|W(s)|} > u_l(x_2). \]

Again, we reach the same conclusion. The case in which \( x_m \notin W(s) \) is symmetric.

Case ii) \( x_1, x_m \in W(s) \).

Let \( l \in L \) satisfy \( p_l = x_1, m_l > 0 \). Clearly, \( W((x_1, 0), s-1) = \{x_2\} \). This follows as for all \( x_k \in E(s), V_{x_k} > 0 \). As \( s \) is a Nash equilibrium, calculation shows that
\[ \frac{\sum_{W(s)} u_l(x_j)}{|W(s)|} - \psi_l(m_l) \geq u_l(x_2), \]

so that if \( x_2 \in W(s) \),
\[ \frac{\sum_{W(s) \setminus \{x_2\}} u_l(x_j)}{|W(s)| - 1} > u_l(x_2). \]

If \( x_2 \notin W(s) \)
\[ \frac{\sum_{W(s)} u_l(x_j)}{|W(s)|} > u_l(x_2). \]

As \( s \) satisfies truthful financing, and \( u_l \) is concave, as \( p_l = x_1, u_l(x_1) \geq u_l(x_2) \geq \ldots \geq u_l(x_m) \). Moreover, \( u_l \) is nonconstant by the inequalities listed above, so that \( u_l(x_2) > \ldots > u_l(x_m) \). In particular, these inequalities imply
\[ \frac{u_l(x_1) + u_l(x_m)}{2} > u_l(x_2). \]
By the concavity of $u_l$,

$$\frac{x_1 + x_m}{2} < x_2.$$  

The same logic can be shown to imply that,

$$\frac{x_1 + x_m}{2} > x_{m-1}.$$  

Conclude

$$x_{m-1} < x_2,$$  

a contradiction.

As either case leads to a contradiction, our supposition must be incorrect, so that if $s$ satisfies truthful financing, then $|E(s)| \leq 2$. ■

Any time there are three or more candidates, financing must be “strategic” in equilibrium. Moreover, there is no possibility of “strategic” financing in a two candidate equilibrium. Thus, the theorem can be viewed as helping to confirm the intuition that third party candidates campaign solely to steal votes from candidates from major parties. The parallels between Theorem 3 and the Gibbard-Satterthwaite Theorem (for example, [3]) are clear.

We remark that this theorem is a qualitative characterization of two-candidate equilibria, as opposed to the quantitative results obtained in Besley and Coate, [4], and Osborne and Slivinski, [14]. Moreover, the theorem does not reflect a simple non-existence result, as in Feddersen, [10].

3.3. Policy divergence

We now present a result which states that if each $\psi_l$, $u_l$ are continuous and total endowments are finite, then there is a minimal degree of separation between the two policies which is always present in a two-candidate equilibrium. That is, there is some constant $k > 0$ such that, if $(x_1, x_2)$ are the financed candidates, they are a distance of at least $k$ apart. If such a theorem were not true, then although two entrants’ policies could never be the same in equilibrium (indeed, this is a feature of the model), they could be “almost” the same.

**Theorem 4:** Suppose that for all $l \in L$, $\psi_l$ and $u_l$ are continuous and that $\sum_{l \in L} \omega_l < +\infty$. Then there exists $k > 0$ such that for all distributions $\mu$ of voter types, if $s$ is a pure-strategy Nash equilibrium and $|E(s)| = 2$, where $E(s) = \{x_1, x_2\}$, then $|x_1 - x_2| \geq k$.  

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This theorem is general in the sense that $k$ is independent of the distribution of voter types. The proof of the theorem is relegated to an Appendix, as it is fairly lengthy. However, we provide the following example as an aid to the intuition.

**Example 2:** Suppose that $X = \mathbb{R}$, and that voters have symmetric, single-peaked utility functions whose peaks have a Gaussian distribution with mean at zero. (This assumption implies a unique median; moreover, any equilibrium with two agents must be symmetric about the median). Let $n \geq 2$ be some integer. Let $\omega > 0$. For all $n$, let $F(n) = 2\omega$. For all $n$, let $L(n)$ be a set of lobbies such that $|L(n)| = 2n$ and for all $l \in L_n$, $\omega_l = \omega$. Suppose there are $n$ agents $l \in L(n)$ such that $u_l(x) = x$ (rightists), and $n$ agents $l \in L(n)$ such that $u_l(x) = -x$ (leftists). Each agent has the same cost function: $\psi_l(c) = c$. For varying values of $n$, what is the minimal $\varepsilon$ such that there exists a policy $x$ such that an equilibrium of type $(x - \varepsilon, x + \varepsilon)$ can be supported? According to Theorem 2, in any such equilibrium, the maximal amount any leftist lobby is willing to pay for $x - \varepsilon$, satisfies

$$m = \psi(m) \leq \min\{\varepsilon, \omega\}.$$  

A rightist lobby will never finance $x - \varepsilon$, again according to Theorem 2. Therefore, the minimal $\varepsilon$ which can be supported obtains when all $n$ leftists finance $x - \varepsilon$ with as much as they are willing, and when all $n$ rightists finance $x + \varepsilon$ with as much as they are willing. Hence, for $\varepsilon$ to be supported in some Nash equilibrium, $n\varepsilon \geq F$. Therefore, $\varepsilon \geq \frac{F}{n}$. So, for all $n$, any $\varepsilon \in \left[\frac{F}{n}, +\infty\right)$ can be supported. Note that $\bigcup_{n=2}^{\infty} \left[\frac{F}{n}, \infty\right) = (0, +\infty)$. Thus, if $n = +\infty$, there is no minimal distance. This possibility entails as total wealth is not finite in the limit.

The preceding example highlights the distinction between economies with finite wealth, and economies with infinite wealth. It is of interest as it establishes that when wealth is infinite, the equilibrium correspondence is not closed. We remark, however, that this result entails as total wealth is infinite; not because population is infinite. Indeed, even if population is infinite, yet total wealth is finite; the theorem holds.

**4. Existence of equilibrium**

It is trivial to see that a pure-strategy Nash equilibrium of this game always exists; namely, a Nash equilibrium in which no candidates are financed. This follows
directly from the assumption that for all $l$, $\omega_l < F$. A more interesting question is when does a two-entrant equilibrium exist? We provide the following theorem, stated without proof.

**Theorem 5:** If there $x_1, x_2 \in X$ such that $\mu(\{R : x_1Px_2\}) = \mu(\{R : x_2Px_1\})$, and

$$
\sum_{\{l:u_l(x_1)>u_l(x_2)\}} \min \left\{ \omega_l, \psi_l^{-1} \left( \frac{u_l(x_1) - u_l(x_2)}{2} \right) \right\} \geq F,
$$

$$
\sum_{\{l:u_l(x_2)>u_l(x_1)\}} \min \left\{ \omega_l, \psi_l^{-1} \left( \frac{u_l(x_2) - u_l(x_1)}{2} \right) \right\} \geq F,
$$

then there exists a pure-strategy Nash equilibrium $s$ such that $|E(s)| = 2$.

Of course, the two entrant Nash equilibrium alluded to in the previous theorem consists of $E(s) = \{x_1, x_2\}$.

### 5. Conclusion

Despite the shortcomings of the model, we feel that it provides an interesting insight into the nature of sincerity of political campaigns. Again, this insight is useful as it provides qualitative results which are very general (do not rely on symmetry of preferences, we allow lobbies to spend as much as they want). One may view our main characterization as a statement on “political parties,” where groups of lobbies financing a particular candidate are interpreted as a party. The characterization emphasizes the suspicious nature of third parties.

We remark that a three party system *can* exist in our model, in contrast to other models, such as Feddersen [9], and Palfrey [15]. Feddersen considers a model of strategic voting (although he also discusses how to interpret his model in terms of campaign finance); whereas our model follows the assumption of sincere voting (yet strategic financing). Palfrey’s is entirely different as he considers two “established parties” and a third party deciding whether or not to enter. In any case, our characterization is of interest as Duverger’s Law [8] certainly is not always empirically satisfied.

We may also interpret the main theorem as implying that a three party system may not be normatively appealing. Instead of being a way of eliminating partisan politics and political scandal, such a system may encourage deceitful behavior.
6. Appendix: Proof of policy divergence

Say \( x \in X \) supports \( \varepsilon \in \mathbb{R}_+ \) if there exists a vector \( z \in \prod_l [0, \omega_l] \) such that

\[
\sum_{\{l : u_l(x-\varepsilon) > u_l(x+\varepsilon)\}} z_l = F, \quad \sum_{\{l : u_l(x-\varepsilon) < u_l(x+\varepsilon)\}} z_l = F
\]

\[\psi_l (z_l) \leq \frac{|u_l(x-\varepsilon) - u_l(x+\varepsilon)|}{2} .\]

Define the correspondence \( g : X \rightarrow \mathbb{R}_+ \) by

\[g(x) \equiv \{ \varepsilon : x \text{ supports } \varepsilon \} .\]

We now prove a series of lemmas, leading to the proof of the theorem.

**Lemma 1:** The function

\[h_-(x, \varepsilon) \equiv \sum_{\{l : u_l(x-\varepsilon) > u_l(x+\varepsilon)\}} \min \left\{ \omega_l, \psi_l^{-1} \left( \frac{u_l(x-\varepsilon) - u_l(x+\varepsilon) - \frac{1}{2}}{2} \right) \right\} \]

is increasing and continuous in \( x \) for fixed \( \varepsilon \), where \( \psi_l^{-1}(s) = \infty \) for any \( s \) not in the range of \( \psi_l \).

**Proof:** To see this, simply note that as \( x \) increases, the set

\[\{ l : u_l(x-\varepsilon) > u_l(x+\varepsilon) \}\]

increases with respect to inclusion, due to weak concavity of \( u_l \) for all \( l \). Moreover, the quantity \( u_l(x-\varepsilon) - u_l(x+\varepsilon) \) is increasing, again due to weak concavity. Therefore, as \( \psi_l \) is increasing (and hence \( \psi_l^{-1} \) increasing), we establish that \( h_-(x, \varepsilon) \) is increasing. We see that \( h_-(x, \varepsilon) \) is continuous in \( x \) as \( u_l \) and \( \psi_l^{-1} \) are each continuous.

\[\blacksquare\]

**Lemma 2:** \( h_-(x, \varepsilon) \geq F \) if and only if there exists

\[z \in \prod_{\{l : u_l(x-\varepsilon) > u_l(x+\varepsilon)\}} [0, \omega_l] \]

such that \( \sum_{\{l : u_l(x-\varepsilon) > u_l(x+\varepsilon)\}} z_l = F \) and \( \psi_l (z_l) \leq \frac{u_l(x-\varepsilon) - u_l(x+\varepsilon)}{2} .\)
Lastly, for all \( fi \) can be shown to be decreasing and continuous, by a symmetric argument. The \( u \) are monotonic, in that

\[
\text{Suppose now that the second condition is satis} \quad \text{fied. Note that that monotonocity of } \psi \text{ implies } \psi' \text{ is the intersection of two monotonic correspondences.}
\]

\[
F = \sum_{\{l:u_l(x-\varepsilon)\leq u_l(x+\varepsilon)\}} \omega_l \psi^{-1}_l \left( \frac{u_l(x-\varepsilon) - u_l(x+\varepsilon)}{2} \right)
\]

which is simply \( h_-(x, \varepsilon) \). Therefore, the result is proved.

The function

\[
h_+(x, \varepsilon) \equiv \sum_{\{l:u_l(x+\varepsilon)\leq u_l(x-\varepsilon)\}} \omega_l \psi^{-1}_l \left( \frac{u_l(x+\varepsilon) - u_l(x-\varepsilon)}{2} \right)
\]

can be shown to be decreasing and continuous, by a symmetric argument. The first lemma implies that the correspondences \( g_- , g_+ \), defined by

\[
g_- (x) \equiv \{ \varepsilon : h_- (x, \varepsilon) \geq F \}
\]

\[
g_+ (x) \equiv \{ \varepsilon : h_+ (x, \varepsilon) \geq F \}
\]

are monotonic, in that \( x < x' \) implies \( g_-(x) \subset g_-(x') \), and \( g_+(x') \subset g_+(x) \). Lastly, for all \( x \), \( g(x) = g_-(x) \cap g_+(x) \). This follows from the second lemma. Thus, \( g \) is the intersection of two monotonic correspondences.
Lemma 3: For all \( x \), \( g(x) \), \( g_+(x) \), \( g_-(x) \) are closed (possibly empty) sets not containing zero.

Proof: We show \( g \) is closed; the cases for \( g_-, g_+ \) follow similarly. Let \( \{\varepsilon^n\} \subset g(x) \) be a convergent sequence with limit \( \varepsilon \). Then, there exists a corresponding sequence \( \{z^n\} \subset \prod_l [0, \omega_l] \) allowing each \( \varepsilon^n \) to be supported by \( x \). Now, as each set \([0, \omega_l]\) is compact in \( \mathbb{R} \), Tychonoff’s theorem implies that the set \( \prod_l [0, \omega_l] \) is compact in the product topology on \( \mathbb{R}^L \). Hence, there exists a subsequence of \( z^n \) which converges to some point \( z \in \prod_l [0, \omega_l] \). For this reason, we assume without loss of generality that \( \{z^n\} \) converges to some \( z \in \prod_l [0, \omega_l] \). Moreover, \( \{z^n\} \) converges to \( z \) if and only if for all \( i \in \mathbb{N} \), \( \{z^n\} \) converges to \( z \).

As \( \psi_l \) and \( u_l \) are continuous for all \( l \), the inequality
\[
\psi_l(z^n_l) \leq \frac{|u_l(x - \varepsilon^n) - u_l(x + \varepsilon^n)|}{2}
\]
implies
\[
\psi_l(z_l) \leq \frac{|u_l(x - \varepsilon) - u_l(x + \varepsilon)|}{2}.
\]

Next, note that
\[
\sum_{\{l: u_l(x - \varepsilon^n) > u_l(x + \varepsilon^n)\}} z^n_l \rightarrow \sum_{\{l: u_l(x - \varepsilon) > u_l(x + \varepsilon)\}} z_l.
\]

To see this, define the sequence \( \{z^{n*}_l\} \subset \prod_l [0, \omega_l] \) as
\[
z^{n*}_l \equiv \begin{cases} 
  z^n_l & \text{if } u_l(x - \varepsilon^n) > u_l(x + \varepsilon^n) \\
  0 & \text{otherwise}
\end{cases}.
\]

Note that
\[
\sum_l z^{n*}_l = \sum_{\{l: u_l(x - \varepsilon^n) > u_l(x + \varepsilon^n)\}} z^n_l
\]
and that \( z^{n*}_l \) converges to a vector \( z^* \) which is \( z \) on \( \{l: u_l(x - \varepsilon) > u_l(x + \varepsilon)\} \) and zero otherwise. This follows from continuity of \( u_l \); for all \( l \) such that \( u_l(x - \varepsilon) > u_l(x + \varepsilon) \), there exists some \( m \) such that for all \( n > m \), \( u_l(x - \varepsilon^n) > u_l(x + \varepsilon^n) \). Moreover, if \( u_l(x - \varepsilon) \leq u_l(x + \varepsilon) \), then if \( u_l(x - \varepsilon^n) > u_l(x + \varepsilon^n) \), we know by \( \psi_l(z^n_l) \leq \frac{|u_l(x - \varepsilon^n) - u_l(x + \varepsilon^n)|}{2} \) that \( z^n_l \rightarrow 0 \).
Now, recall that $z_l^* \leq \omega_l$ and that $\sum_l \omega_l < \infty$. The Lebesgue dominated convergence theorem implies

$$\sum_l z_l^* \to \sum_l z_l^*$$

so that

$$\sum_{l : u_l(x-\varepsilon) > u_l(x+\varepsilon)} z_l = F.$$

A similar argument holds for $\{l : u_l(x+\varepsilon) > u_l(x-\varepsilon)\}$. Therefore, $z$ allows $\varepsilon$ to be supported by $x$, so that $\varepsilon \in g(x)$. It is clear that $0 \notin g(x)$. Otherwise, the $z$ which allows $0$ to be supported by $x$ must satisfy $\psi_l(z_l) = 0$ for all $l$. But this would require that $z_l = 0$ for all $l$, so that the summation constraint is not met.

The case for $g_+$ and $g_-$ follow in similar ways, using the characterization given by the second lemma.

We now complete the proof of the theorem.

Let $f$ be defined as:

$$f(x) \equiv \min_{\varepsilon \in g(x)} \varepsilon,$$

where we take $\min \emptyset = \infty$. Similarly, let

$$f_-(x) \equiv \min_{\varepsilon \in g_-(x)} \varepsilon$$

and

$$f_+(x) \equiv \min_{\varepsilon \in g_+(x)} \varepsilon.$$

Note that these functions are all well-defined, $f_-(x)$ is increasing, and $f_+(x)$ is decreasing. Moreover, $f(x) = \max \{f_-(x), f_+(x)\}$. Therefore, there are four possible cases.

Case i) For all $x \in X$, $f(x) = f_+(x)$. In this case, we know that $0 \leq f_-(x) \leq f(x)$. Therefore, let $x^*$ be arbitrary. We know that $f_-(x^*) > 0$. If $x < x^*$, we have $f(x) \geq f(x^*) \geq f_-(x^*)$. If $x > x^*$, we have $f(x) \geq f_-(x) \geq f_-(x^*)$. Therefore, for all $x$, $f(x) \geq f_-(x^*) > 0$. In this case, set $k = 2f_-(x^*)$.

Case ii) For all $x \in X$, $f(x) = f_-(x)$. This case is symmetric to Case i).

Case iii) There exists $x^*$ such that

$$f(x) = \begin{cases} f_+(x) & x < x^* \\ f_-(x) & x \geq x^* \end{cases}$$

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Note that if \( x < x^* \), then \( f (x) = f_+(x) \geq f_+(x^*) \). If \( x \geq x^* \), \( f (x) = f_-(x) \geq f_-(x^*) \geq f_+(x^*) \). Therefore, for all \( x \), \( f (x) \geq f_+(x^*) > 0 \). In this case, set \( k = 2f_+(x^*) \).

Case iv) There exists \( x^* \) such that

\[
f(x) = \begin{cases} 
  f_+(x) & x \leq x^* \\
  f_-(x) & x > x^* 
\end{cases}
\]

This case is symmetric to Case iii).

Finally, suppose \((x_1, x_2)\) are the entrants in a Nash equilibrium. Suppose that \( x_2 > x_1 \), without loss of generality. Then it is clear that \( \frac{x_1 - x_2}{2} \) supports \( \frac{x_1 - x_2}{2} \); this follows by the characterization of two-entrant Nash equilibrium. Hence by the above lemmas, \( \frac{x_1 - x_2}{2} \in g(\frac{x_1 + x_2}{2}) \), so that \( \frac{x_1 - x_2}{2} \geq f(\frac{x_1 + x_2}{2}) \geq k/2 \). We then have \( x_1 - x_2 \geq k \), completing the theorem.

References


