SS201c—due in two weeks.

1. The Borda score for an individual agent is a utility representation of her preference. We know that any strictly increasing transformation of a utility representation for an individual obtains another, equally valid utility representation.

   a) Construct a society and preference profile \( \theta \). Show that \( b_i(\cdot, \theta) \) is a utility representation for \( R_i(\theta) \).

   b) Next, \( \sum_{i \in N} b_i(\cdot, \theta) \) gives a utility representation for the social preference given by Borda’s rule. For all \( i \), let \( \gamma_i : \mathbb{R} \rightarrow \mathbb{R} \) be a strictly increasing function. Show that \( \sum_{i \in N} \gamma_i(b_i(\cdot, \theta)) \) does not necessarily represent the social preference given by Borda’s rule. This shows us that adding preference representations can rank alternatives differently depending on which representations we use. (also read Arrow, p. 31-33)

   c) Does b) mean that utility is comparable across agents? (i.e. Is there any meaning to statements of the form: “Agent \( i \) is “as well off as agent \( j \) with alternative \( x \)”)

2. An alternative definition of the Borda score: For a given profile, denote \( \beta(x, \theta) = \sum_{y \in X \setminus \{x\}} \left[ |P(x, y|\theta)| - |P(y, x|\theta)| \right] \). Show that \( \beta(x, \theta) = \sum_{i \in N} B_i(x, \theta) \). Interpret.

3. AS&B, #2.4 (Sen’s impossibility of a Paretian liberal).

4. Demonstrate the independence of the axioms used in May’s Theorem. (for each axiom, construct a rule that satisfies the remaining axioms but which is not simple majority).

5. Demonstrate the independence of the axioms used in Arrow’s Theorem.

6. Prove that the two definitions of an oligarchy are equivalent. (as defined below)

   a) \( G \) is an oligarchy if \( x R(\theta) y \) if and only if \( G \subset P(x, y|\theta) \).

   b) \( G \) is an oligarchy if \( G \subset P(x, y|\theta) \) implies \( x P(\theta) y \), and for all \( i \in G \), \( x R_i(\theta) y \) implies \( x R(\theta) y \).

7. Suppose that preferences are weak orders, so that \( R_N(\Theta) = W(X)^N \). We will consider a version of Arrow’s Theorem. Strengthen the weak Pareto condition to the strong Pareto condition. This states that if for all \( i \in N \), \( x R_i(\theta) y \), then \( x R(\theta) y \), and additionally, if there also exists some \( j \in N \) such that \( x P_j(\theta) y \), then \( x P(\theta) y \).

   Now, we define a serial dictatorship as follows. There exists a linear order of the agents, say \( \preceq \). Then \( x P(\theta) y \) if and only if there exists some \( i \in N \) such that \( x P_i(\theta) y \), and for all \( j \prec i \), \( x I_j(\theta) y \). Thus, a serial dictatorship is a
rule for which the agents are ordered, and the comparison between $x$ and $y$ is made according to the preference of the highest ranked non-indifferent individual (unless all agents are indifferent between $x$ and $y$, so that $x$ is socially indifferent to $y$).

How would you go about showing that the serial dictatorships are the only rules satisfying Arrow’s axioms on the domain of weak orders? You do not have to provide a proof, but at least describe a construction for such a proof.

8. What is the analogous extension of Gibbard’s theorem to weak orders (when strengthening weak Pareto to strong Pareto)? State the definition formally. You do not have to provide any proofs, but explain carefully what led you to your conclusion.